

The Coupling of Natural BEM and FEM for Three-Dimensional Exterior Problem with a Prolate Spherical Artificial Boundary *

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Abstract

In this paper, based on the natural boundary reduction method, we discuss a coupled natural BEM and FEM for three-dimensional exterior problems. We express the artificial boundary condition on the prolate spherical artificial boundary in a series form explicitly. In practical computation, we truncate this condition in finite terms. We discuss well-posedness about the variational problem of the coupled method. The error estimates are based on the mesh size, the terms after truncating the infinite series, and the location of the artificial boundary. Two numerical examples are presented to demonstrate the effectiveness and the error estimates.

Key words: Natural boundary reduction, Prolate spheroid, Finite element, Exterior harmonic problem, The coupled method.

1. Introduction

The standard procedure of the coupling of boundary element and finite element is described as follows. First, an artificial boundary is introduced to divide the original (unbounded) domain into two subregions, a bounded inner region and an unbounded outer one. Next, the original problem is reduced to an equivalent one in the bounded region. There are many ways to accomplish this reduction.

The natural boundary reduction method proposed by Feng and Yu [8] has advantages over the usual boundary reduction methods: the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form, so not only the analysis of the discrete problem is simplified but also the optimal error estimates and the numerical stability are restored (see [1,4,6,7]).

For three-dimensional exterior problems, a spherical surface (see [5],[9]) is usually selected as the artificial boundary. However, for an elongated cigar-shaped or ship-shaped obstacles, we use a prolate spheroid boundary as the artificial boundary very efficiently, since it leads to smaller computational domain. In this paper, based on the natural boundary reduction method, we discuss a coupled natural BEM and FEM for three-dimensional exterior problems with a prolate spherical artificial boundary. We express the exact artificial boundary condition on the prolate spheroid in a series form explicitly. We discuss well-posedness about the variational problem of the coupled method. In practical computation, we truncate this exact artificial condition in finite terms. Thus there exists a truncation error, which is often ignored in lots of previous papers, but it appears in

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[2,6]. Our error estimates are based on the mesh size, the terms after truncating the infinite series, and the location of the artificial boundary.

2. The natural boundary reduction and the coupled variational problem

Let Ω be a Cigar-shaped Lipschitz bounded domain and include the coordinate origin in \mathbb{R}^3 and $\Omega^c = \mathbb{R}^3 \setminus (\Omega \cup \partial\Omega)$. Assume that the given functions f and g satisfy $g \in H^{1/2}(\partial\Omega)$ and $f \in L^2(\Omega^c)$, $\text{supp}(f) \subset \Omega^c$. We consider the following exterior Dirichlet problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega^c, \\ u = g, & \text{on } \Gamma_0, \\ \text{some conditions at infinity.} \end{cases} \quad (2.1)$$

From [3], we know problem (2.1) is well-posed in $W_g^1(\Omega^c) = \{v \in W^1(\Omega^c) : v|_{\Gamma_0} = g\}$. The variational form of the boundary value problem (2.1) is: find $u \in W_g^1(\Omega^c)$ such that

$$D(u, v) = (f, v), \quad \forall v \in W_0^1(\Omega^c). \quad (2.2)$$

Let $\Gamma = \{(x, y, z) : \frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1, a > b > 0\}$ denote a prolate spheroid and $\Gamma \subset \Omega^c$. Then Γ divides Ω^c into two subregions, a bounded inner region Ω_1 and a unbounded outer region Ω_2 . $\Omega_1 \cap \Omega_2 = \emptyset, \overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega^c}, \text{supp}(f) \subset \Omega_1$.

To derive a coupled variational formula which is equivalent to (2.2), we define

$$D_k(u, v) = \int_{\Omega_k} \nabla u \cdot \nabla v dx dy dz, k = 1, 2.$$

Let $u \in W_g^1(\Omega^c)$ be the solution of (2.2) and $v \in W_0^1(\Omega^c)$. It is clear that

$$D(u, v) = D_1(u, v) + D_2(u, v). \quad (2.3)$$

From Green's formula on Ω_2 , we infer that

$$D_2(u, v) = \int_{\Gamma} \frac{\partial u(p)}{\partial \nu} \cdot v(p) dp, \quad (2.4)$$

where ν denotes the unit exterior normal vector on Γ (regarded as the inner boundary of Ω_2)

We consider the following exterior Dirichlet problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega_2, \\ u = u_1, & \text{on } \Gamma. \end{cases}$$

We introduce a prolate spheroidal system of coordinates (μ, θ, φ) , such that Γ coincides with the prolate spheroid $\mu = \mu_1$ and $\Omega_2 = \{(\mu, \theta, \varphi) : \mu > \mu_1 > 0, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$. Thus, the Cartesian coordinates (x, y, z) is related to the prolate spheroidal coordinates (μ, θ, φ) via

$$\begin{cases} x = f_0 \sinh \mu \sin \theta \cos \varphi, & \mu \geq \mu_1 > 0, \\ y = f_0 \sinh \mu \sin \theta \sin \varphi, & \theta \in [0, \pi], \\ z = f_0 \cosh \mu \cos \theta, & \varphi \in [0, 2\pi], \end{cases}$$

where $f_0 = \sqrt{a^2 - b^2}$, $a = f_0 \cosh \mu_1$, $b = f_0 \sinh \mu_1$.

Using natural boundary reduction [1,10], we obtain that Poisson integral formula and natural integral operator

$$u(\mu, \theta, \varphi) = \sum_{n=0}^{+\infty} \sum_{m=-n}^n \frac{Q_n^m(\cosh \mu)}{Q_n^m(\cosh \mu_1)} U_{nm} Y_{nm}(\theta, \varphi), \quad \mu \geq \mu_1 > 0 \quad (2.5)$$

and

$$\frac{\partial u}{\partial \nu} \Big|_{\Gamma} = - \frac{1}{f_0 \sqrt{\cosh^2 \mu_1 - \cos^2 \theta}} \sum_{n=0}^{+\infty} \sum_{m=-n}^n \frac{\frac{d}{d\mu} Q_n^m(\cosh \mu_1)}{Q_n^m(\cosh \mu_1)} U_{nm} Y_{nm}(\theta, \varphi) \doteq \mathcal{K}u_1. \quad (2.6)$$

where $U_{nm} = \int_0^\pi \int_0^{2\pi} u_1(\theta, \varphi) Y_{nm}^*(\theta, \varphi) \sin \theta d\theta d\varphi$, $Y_{nm}(\theta, \varphi)$ are spherical harmonic functions, Y_{nm}^* are the conjugate complex of Y_{nm} , and $Q_n^m(x) = (-1)^m (x^2 - 1)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_n(x)$, $x > 1$ are the second kind associated Legendre functions.

Let $\gamma v = v|_{\Gamma}$, $V_{nm} = \int_0^\pi \int_0^{2\pi} \gamma v Y_{nm}^*(\theta, \varphi) \sin \theta d\theta d\varphi$ and $\hat{D}(\gamma u, \gamma v) = \langle \mathcal{K}\gamma u, \gamma v \rangle_{\Gamma}$. Thus,

$$D_2(u, v) = \hat{D}(\gamma u, \gamma v) = -f_0 \sum_{n=0}^{+\infty} \sum_{m=-n}^n \frac{\frac{d}{d\mu} Q_n^m(\cosh \mu_1)}{Q_n^m(\cosh \mu_1)} \sinh \mu_1 V_{nm}^* U_{nm}. \quad (2.7)$$

Set

$$H_n^m(x_1) = - \frac{(x_1^2 - 1) \frac{d}{dx} Q_n^m(x_1)}{Q_n^m(x_1)} = - \frac{\frac{d}{d\mu} Q_n^m(\cosh \mu_1)}{Q_n^m(\cosh \mu_1)} \sinh \mu_1$$

where $x_1 = \cosh \mu_1$. Define

$$\hat{D}_N(\gamma u, \gamma v) = f_0 \sum_{n=0}^N \sum_{m=-n}^n H_n^m(x_1) V_{nm}^* U_{nm}. \quad (2.8)$$

We will need the Sobolev spaces $G_g^1(\Omega_1)$ which is defined as

$$G_g^1(\Omega_1) = \{v : v \in H^1(\Omega_1), v|_{\Gamma_0} = g\}.$$

Using (2.3) together with (2.7), we obtain the coupled natural BEM and FEM variational problem: find $u \in G_g^1(\Omega_1)$ such that

$$D_1(u, v) + \hat{D}(\gamma u, \gamma v) = \int_{\Omega_1} f \cdot v dx dy dz, \quad \forall v \in G_0^1(\Omega_1). \quad (2.9)$$

Lemma 2.1.^[10] Let n and m be both non-negative integer.

(1) If $0 \leq m \leq n$, then $\frac{(x^2 - 1)}{x} (n^2 + 1)^{\frac{1}{2}} < H_n^m(x) < \sqrt{2} (n^2 + 1)^{\frac{1}{2}} x$.

(2) If $0 \leq m \leq n$ and $1 < x_0 < x$, then $\left(\frac{x_0^2 - 1}{x^2 - 1}\right)^{\frac{n+1}{2}} \leq \frac{Q_n^m(x)}{Q_n^m(x_0)} \leq \left(\frac{x_0}{x}\right)^{n+1}$.

Lemma 2.2.^[10] $\hat{D}(\cdot, \cdot)$ and $\hat{D}_N(\cdot, \cdot)$ are both the symmetric, continuous and semi-positive bilinear form on $H^{\frac{1}{2}}(\Gamma)$, i.e.

$$0 \leq \hat{D}_N(\gamma v, \gamma v) \leq \hat{D}(\gamma v, \gamma v), \quad \forall v \in H^1(\Omega_2)$$

$$|\hat{D}(\gamma u, \gamma v)| \leq \sqrt{2} f_0 x_1 \|\gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \|\gamma u\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \forall v, u \in H^1(\Omega_2).$$

$$|\hat{D}_N(\gamma u, \gamma v)| \leq \sqrt{2} f_0 x_1 \|\gamma v\|_{H^{\frac{1}{2}}(\Gamma)} \|\gamma u\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \forall v, u \in H^1(\Omega_2),$$

where $x_1 = \cosh \mu_1$ and $H^{\frac{1}{2}}(\Gamma)$ is defined in [10] and its norm is equivalent to the usual definition on $H^{\frac{1}{2}}(\Gamma)$.

Theorem 2.1. The variational problem (2.9) has a unique solution $u \in G_g^1(\Omega_1)$.

Let

$$A_N(u, v) = D_1(u, v) + \hat{D}_N(\gamma u, \gamma v).$$

In fact, we compute the approximate variational problem: find $u^N \in G_g^1(\Omega_1)$ such that

$$A_N(u^N, v) = \int_{\Omega_1} f \cdot v dx dy dz, \quad \forall v \in G_0^1(\Omega_1). \quad (2.10)$$

According to Lemma 2.2 and Lax-Milgram Theorem, we know the variational problem (2.10) has a unique solution in $u^N \in G_g^1(\Omega_1)$.

Suppose that $\Gamma_1 = \{(\mu, \theta, \varphi) : \mu = \mu_0 < \mu_1, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$ and $\Gamma_1 \subset \Omega_1 \setminus \text{supp}(f)$.

Theorem 2.2. Suppose that u and u^N are the solution of problem (2.9) and problem (2.10), respectively, and $u|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$. Then there is a positive constant C independent of N , such that

$$\|u - u^N\|_{H^1(\Omega_1)} \leq \frac{C}{N+1} \left(\frac{x_0}{x_1} \right)^{N+1} \|u\|_{H^{\frac{3}{2}}(\Gamma_1)},$$

where $x_1 = \cosh \mu_1$, $x_0 = \cosh \mu_0$.

3. Discrete Variational Problem and Its Error Estimate

We make the finite element partitions in Ω_1 and suppose that $S^h(\Omega_1)$ is composed of continuous piecewise linear elements on Ω_1 , $S^h(\Omega_1) \subset H^1(\Omega_1)$ and $S_0^h(\Omega_1) = \{v^h : v^h \in S^h(\Omega_1), v^h|_{\Gamma_0} = 0\}$.

The discrete variational problem of (2.10) is as follows: find $u^{Nh} \in S^h(\Omega_1)$ and $u^{Nh}|_{\Gamma_0} = g$, such that

$$A_N(u^{Nh}, v) = \int_{\Omega_1} f \cdot v dx dy dz, \quad \forall v \in S_0^h(\Omega_1). \quad (3.1)$$

Problem (3.1) is well-posed. Now we discuss error estimates between the numerical solution u^{Nh} and the solution u of the original problem (2.1).

Theorem 3.1. Suppose that $u \in H^2(\Omega_1)$ and $u|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$. Then there is a positive constant C independent of h and N , such that

$$\|u - u^{Nh}\|_{H^1(\Omega_1)} \leq C \left(h \|u\|_{H^2(\Omega_1)} + \frac{1}{N+1} \left(\frac{x_0}{x_1} \right)^{N+1} \|u\|_{H^{\frac{3}{2}}(\Gamma_1)} \right). \quad (3.2)$$

Theorem 3.2. Suppose that $u \in H^2(\Omega_1)$ and $u|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$. Then there is a positive constant C independent of h and N , such that

$$\|u - u^{Nh}\|_{L^2(\Omega_1)} \leq C \left(h^2 \|u\|_{H^2(\Omega_1)} + \frac{1}{N+1} \left(\frac{x_0}{x_1} \right)^{N+1} \|u\|_{H^{\frac{3}{2}}(\Gamma_1)} \right). \quad (3.3)$$

4. Numerical Examples

We divide respectively the intervals $[0, 2\pi]$, $[0, \pi]$ and $[\mu_0, \mu_1]$ into N_2 , N_1 and N_3 equal-spaced parts and then make the corresponding finite element partitions in Ω_1 . In our practical computation, continuous piecewise linear elements are used.

Example 1. Let $f = 0$, $\Gamma_0 = \{(\mu_0, \theta, \varphi) : \mu_0 = 0.5, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$, and the artificial boundary $\Gamma = \{(\mu, \theta, \varphi) : \mu = 1, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$. The exact solution of problem (2.1) is

$$u = \frac{8\sqrt{2} \sinh \mu \sin \theta \cos \varphi (5 \cosh 2\mu \cos 2\theta + 3 \cosh 2\mu + 3 \cos 2\theta + 5)}{f_0^4 (\cosh 2\mu + \cos 2\theta)^{\frac{7}{2}}},$$

and $g = u|_{\Gamma_0}$, where $f_0 = 4$.

The error of approximate solution is as follows:

Table 4.1: The effect of the mesh parameters (N_1, N_2, N_3) for the solution u .

Mesh(N_1, N_2, N_3)	(4,8,1)	(8,16,2)	(16,32,4)	(32,64,8)
$\ u - u^{N_h}\ _{L^\infty(\Omega_1)}$	2.2938e-3	5.9829e-4	1.5128e-4	3.8020e-5
$\ u - u^{N_h}\ _{L^2(\Omega_1)}$	1.9658e-2	4.9272e-3	1.2166e-3	2.9615e-4
$\ u - u^{N_h}\ _{H^1(\Omega_1)}$	2.7862e-2	1.0669e-2	5.3343e-3	2.5402e-3

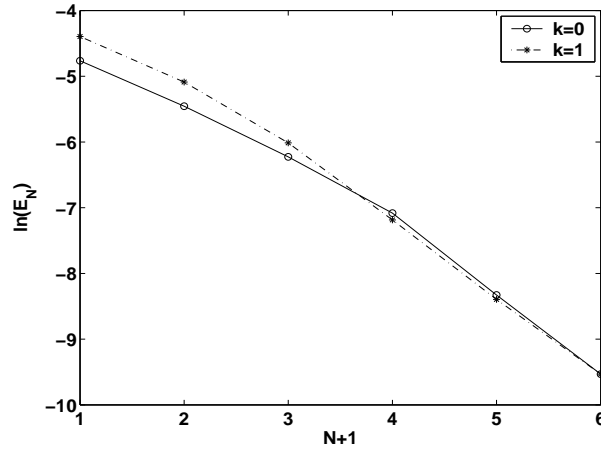


Fig. 4.1. The effect of the terms N for the solution u .

In Table 4.1, we give concerned error for large $N = 100$. The results show that the convergent rates of $\|u - u^{N_h}\|_{L^\infty(\Omega_1)}$, $\|u - u^{N_h}\|_{L^2(\Omega_1)}$ with respect to the mesh size h is 2, while the convergent rates of $\|u - u^{N_h}\|_{H^1(\Omega_1)}$ with respect to h is 1. Finally, we test the effect of the terms N for the solution u . Let u^{100h} denote the finite element approximate solution of the problem on the boundary Ω_1 with large N , where $N = 100$. In figure 4.1, we calculate $E_N = \|u^{N_h} - u^{100h}\|_{H^k(\Omega_1)}$ ($k = 0, 1$) on the mesh $(32 \times 64 \times 8)$ for different N . These results are consistent with theoretical analysis in Theorem 3.1 and 3.2.

Example 2. Let $f = 0$, $\Gamma_0 = \{(\mu, \theta, \varphi) : \mu = 0.5, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$, $g = \frac{\sqrt{2}}{f_0(1.5431 + \cos 2\theta)^{\frac{1}{2}}}$, and the artificial boundary $\Gamma = \{(\mu, \theta, \varphi) : \mu = 1, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$. Thus, the exact solution of problem (2.1) is

$$u = \frac{\sqrt{2}}{f_0(\cosh 2\mu + \cos 2\theta)^{\frac{1}{2}}},$$

where $f_0 = 4$.

Then the relationship between the errors of solution and each parameters are as follows:

Table 4.2: The effect of the mesh parameters (N_1, N_2, N_3) for solution u_1 and u .

Mesh(N_1, N_2, N_3)	(4,8,1)	(8,16,2)	(16,32,4)	(32,64,8)
$\ u - u^{Nh}\ _{L^\infty(\Omega_1)}$	2.3038e-2	5.9223e-3	1.4851e-3	3.6214e-4
$\ u - u^{Nh}\ _{L^2(\Omega_1)}$	1.2350e-1	3.1364e-2	7.8214e-3	1.9601e-3
$\ u - u^{Nh}\ _{H^1(\Omega_1)}$	1.8815e-1	7.1476e-2	3.0187e-2	1.4612e-2

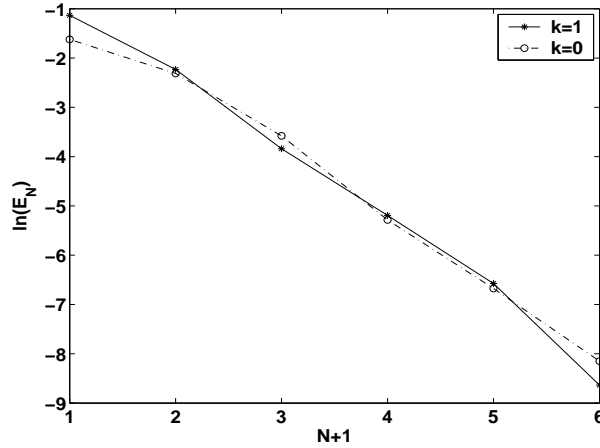


Fig. 4.2. The effect of the terms N for the solution u .

In Table 4.2 and Figure 4.2, the corresponding parameters selected are the same as those in Example 1. The numerical results and conclusions are similar to those in Example 1.

References

- [1] D.H. Yu, Natural Boundary Integral Method and Its Applications, Kluwer Academic Publishers, 2002.
- [2] Houde Han and Weizhu Bao, Error Estimates for the Finite Element Approximation of Problems in Unbounded Domains, SIAM J. NUMER. ANAL., **37**(2000), 1101-1119.
- [3] J.L. Zhu, Boundary Element Analysis of Elliptic Boundary Value Problems, Science Press, Beijing, 1992.
- [4] J.B. Keller, M.J. Grote, Exact Non-reflecting Boundary Conditions, J.Comp. Phys., **82**(1989), 172-192.
- [5] M.J. Grote, J.B. Keller, On Non-reflecting Boundary Conditions, J.Comp. Phys., **122**(1995), 231-243.
- [6] Yu De-hao, Approximation of boundary conditions at infinity for a harmonic equation, J.Comp. Math., **3**:3(1985), 219-227.
- [7] Yu De-hao, Coupling canonical boundary element method with FEM to solve harmonic problem over cracked domain, J.Comp. Math., **1**:3(1983), 195-202.
- [8] Feng Kang, Yu De-hao, Canonical integral equations of elliptic boundary value problems and their numerical solutions, Proc. of China-France Symposium on FEM (1982, Beijing), Science Press, Beijing, 1983, 211-252.
- [9] J.M. Wu, Natural Boundary Element Method and Domain Decomposition Algorithm for Three Dimensional Problems, PhD Thesis, The Institute of Computational Mathematics and Scientific/ Engineering Computing of the Chinese Academy of Sciences, 1999.
- [10] Huang Hong-ying, Yu De-hao, Natural boundary element method For three dimensional exterior harmonic problem with a inner prolate spheroid boundary ,submitted to J. Comp. Math.