Construction of Subdivision Surfaces by Fourth-Order Geometric Flows with G^1 Boundary Conditions *

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Abstract

In this paper, we present a method for constructing Loop's subdivision surface patches with given G^1 boundary conditions and a given topology of control polygon of the subdivision surface, using several fourth-order geometric partial differential equations. These equations are solved by a mixed finite element method in a functional space defined by the extended Loop's subdivision scheme. The method is flexible to the shape of the boundaries. There is no limitation on the number of boundary curves and on the topology of the control polygon. Several properties for the basis functions of the finite element space are developed.

Key words: Level-set method, Topolygy preserved.

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1 Introduction

A surface satisfing a geometric partial differential equation (PDE) is referred to as geometric PDE surface in this paper. Geometric PDE surfaces, such as minimal surfaces (see [15]), constant mean-curvature surfaces (see [8, 19]), Willmore surfaces (see [3, 10, 11, 22])) and minimal mean-curvature variation surfaces (see [26]), are important and preferred in the shape design and modeling because they share certain optimal properties. For instance, the minimal surfaces have minimal area, the Willmore surfaces have minimal total squared mean curvature and minimal mean-curvature variation surfaces have minimal total mean-curvature variation. Here the terminology *total* means the integration over the surfaces. Various type geometric PDE surfaces have been constructed in the literatures (see [24]). Most of them are discrete surfaces

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(triangular or quadrilateral control polygons), a few of them are continuous surfaces. Usually, the representation of the continuous surfaces are Bezier (see [9]), rational Bezier, B-spline (see [13, 14]) and NURB surfaces.

Obviously, Bezier surfaces, B-spline or NURB surfaces have to be three- or four-sided. This is a serious limitation for designing geometric PDE surface with arbitrary shaped boundaries. In this paper, our intension is to construct geometric PDE subdivision surfaces with piecewise B-spline curve boundary and normal condition. There is no limitation on the number of spline pieces. B-spline representation for curves and surfaces have been widely accepted in the CAD and industrial design. Using B-spline to represent surface boundary is preferable and acceptable. To represent a surface patch with any topology, subdivision surfaces are the best candidates, since there is no limitation on the topology of the control polygon. However, subdivision surfaces, such as Loop's subdivision surface and Catmull-Clark subdivision surface are traditionally closed, which cannot be used directly for serving our purpose.

For many surface modeling problems, such as the construction of the bodies of cars and aircrafts, machine parts and roofs, surfaces are usually constructed in a piecewise manner with fixed boundaries for each of the pieces. In such a case Loop's subdivision scheme could not be applied near the boundary of the control polygon. Therefore, an extension of the Loop's scheme to control polygon with boundaries are required. On this aspect, an excellent work has been done by Biermann et al [2] and that is just sufficient for the aim of constructing piecewise smooth surface.

In this paper we construct geometric PDE subdivision surface patches with given G^1 boundary conditions and a given topology of the control polygon using several fourth-order geometric partial differential equations. These equations are solved by a mixed finite element method in a functional space defined by the extended Loop's subdivision scheme. By the term *topology of the control polygon*, we mean the connection mode between the vertices of the control polygon.

Fourth-order geometric flows have been used to solve the discrete surface blending problem and the free-form surface fitting problem (see [4, 20, 21, 25]). In [20, 21], the surface diffusion flow has been used for fairing/smoothing meshes while satisfying the G^1 boundary conditions. The finite element method is used by Clarenz et al. [4] to solve the Willmore flow equation, based on a new variational formulation of the flow, for the discrete surface restoration.

Problem Description: Given an initial control polygon of a surface patch with some of the boundary control points fixed. The boundary curve is defined as piecewise cubic B-spline with the boundary control points as the B-spline control points and equal spaced knots for each piece. The fixed boundary control points are served as the end-points of the B-spline curves. Our goal is to construct a geometric PDE subdivision surface that interpolates the boundary curves and tangents and its control polygon has the same topology as the initial one.

2 Geometric PDEs and Their Weak-form Formulations

To construct smooth geometric PDE subdivision surface patches with G^1 boundary conditions, we use three fourth-order equations, namely surface diffusion flow (SDF), Willmore flow (WF) and quasi-surface flow (QSDF). To describe these equations precisely, we need to introduce a few notations, the details of them could be found in [24].

2.1 Notations

Let $\mathcal{S} := \{\mathbf{x}(u^1, u^2) \in \mathbb{R}^3 : (u^1, u^2) \in \mathscr{D} \subset \mathbb{R}^2\}$ be a parametric surface. For simplicity, we assume it is sufficiently smooth and orientable. Let $g_{\alpha\beta} = \langle \mathbf{x}_{u^\alpha}, \mathbf{x}_{u^\beta} \rangle$ and $b_{\alpha\beta} = \langle \mathbf{n}, \mathbf{x}_{u^\alpha u^\beta} \rangle$ be the coefficients of the first and the second fundamental forms of $\mathcal{S}$ with

$$\mathbf{x}_{u^\alpha} = \frac{\partial \mathbf{x}}{\partial u^\alpha}, \ \mathbf{x}_{u^\alpha u^\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}, \ \alpha, \beta = 1, 2,$$
$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|, \quad (u, v) := (u^1, u^2),$$

where $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ and $\cdot \times \cdot$ stand for the usual inner product, Euclidean norm and cross product in $\mathbb{R}^3$, respectively.

Curvatures. To introduce the notions of the mean curvature and the Gaussian curvature, we use the concept of *Weingarten map* or *shape operator* (see [6]). It is a self-adjoint linear map on the tangent space $T_{\mathbf{x}}\mathcal{S} := \text{span}\{\mathbf{x}_u, \mathbf{x}_v\}$. In matrix form, it could be represented by a 2×2 matrix $S = [\, b_{\alpha\beta} \,][\, g^{\alpha\beta} \,]$ with $[\, g^{\alpha\beta} \,] = [\, g_{\alpha\beta} \,]^{-1}$. The eigenvalues k_1 and k_2 of S are *principal curvatures* of $\mathcal{S}$ and their arithmetic average and product are the *mean curvature* H and the *Gaussian curvature* K, respectively. That is

$$H = \frac{k_1 + k_2}{2} = \frac{\text{tr}(S)}{2}, \qquad K = k_1 k_2 = \det(S).$$

Let $\mathbf{H} = H\mathbf{n}$. It is referred to as the mean curvature normal. Now we introduce a few geometric differential operators.

Tangential gradient opertor. Suppose $f \in C^1(\mathcal{S})$, where $C^1(\mathcal{S})$ stands for a function space consisting of C^1 smooth functions on $\mathcal{S}$, then the tangential gradient operator ∇_s acting on f is defined as

$$\nabla_s f \quad = \quad [\mathbf{x}_u, \ \mathbf{x}_v][\, g^{\alpha\beta} \,][f_u, \ f_v]^T \in \mathbb{R}^3. \tag{2.1}$$

For a vector-valued function $\mathbf{f} = [f_1, \cdots, f_k]^T \in C^1(\mathcal{S})^k$, its gradient is defined as $\nabla_s \mathbf{f} = [\nabla_s f_1, \cdots, \nabla_s f_k] \in \mathbb{R}^{3 \times k}$.

The third tangential operator. Let $f \in C^1(\mathcal{S})$. Then the third tangential operator $\oslash$ acting on f is defined as

$$\oslash f = [\mathbf{x}_u, \mathbf{x}_v][\, g^{\alpha\beta} \,]S[f_u, f_v]^{\mathrm{T}} \in \mathbb{R}^3.$$

Apart from these two tangential operators, there is another one, called the second tangential operator (see [24]). Since it is not involved in this work, we do not introduce it.

Divergence operator. Suppose $\mathbf{v}$ is a smooth vector field on surface $\mathcal{S}$, then the divergence operator div_s acting on $\mathbf{v}$ is defined as

$$\text{div}_s(\mathbf{v}) = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] \left[\sqrt{g} \, [\, g^{\alpha\beta} \,] \, [\mathbf{x}_u, \mathbf{x}_v]^T \mathbf{v} \right]. \tag{2.2}$$

Laplace-Beltrami operator. Let $f \in C^2(\mathcal{S})$. Then the Laplace-Beltrami operator (LBO) Δ_s acting on f is defined as (see [6], p. 83)

$$\Delta_s f = \mathrm{div}_s(\nabla_s f).$$

Obviously, Δ_s is a second order differential operator. It is well known that LBO relates to the mean curvature vector via the equation: $\Delta_s \mathbf{x} = 2\mathbf{H}$.

Theorem 2.1 (Green's formula for LBO) *Let $\mathcal{S}$ be an orientaable surface, Ω a subregion of $\mathcal{S}$ with a piecewise smooth boundary $\partial\Omega$. Let $\mathbf{n}_c \in T_{\mathbf{x}}\mathcal{S}$ ($\mathbf{x} \in \partial\Omega$) be the outward unit normal along the boundary $\partial\Omega$. Then for a given C^1 smooth vector field $\mathbf{v}$ on $\mathcal{S}$, we have*

$$\int_\Omega [\langle \mathbf{v}, \nabla_s f\rangle + f\,\mathrm{div}(\mathbf{v})]\mathrm{d}A = \int_{\partial\Omega} f\langle \mathbf{v}, \mathbf{n}_c\rangle \mathrm{d}s. \tag{2.3}$$

2.2 Used Geometric PDEs

For completeness, we describe briefly the equations used and their behaviors. More details on these equations can be found in [24].

Surface Diffusion Flow

$$\frac{\partial \mathbf{x}}{\partial t} = -2\Delta_s H\mathbf{n}. \tag{2.4}$$

This flow is introduced by Mullins in 1957 (see [16]), to describe the interface motion law of growing crystal. If $\mathcal{S}$ is a closed surface and A stands for its area and V stands for the enclosed volume, then by Green's formula we obtain (see [5, 18] for the change rates of the surface area and the enclosed volume of the evolved surface):

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = 2\int_{\mathcal{S}(t)} \Delta_s H H \mathrm{d}A = -2\int_{\mathcal{S}(t)} \|\nabla_s H\|^2 \mathrm{d}A \le 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\frac{2}{3}\int \mathrm{div}_s(\nabla_s H)\mathrm{d}A = \frac{2}{3}\int (\nabla_s H)^{\mathrm{T}}\nabla_s(1)\mathrm{d}A = 0.$$

Hence surface diffusion flow is volume preserving and area shrinking. The area shrinkage stops when H is a constant. It is easy to see that surfaces with constant mean curvature are the the steady solution of (2.4).

Willmore flow

$$\frac{\partial \mathbf{x}}{\partial t} = -2\left[\Delta_s H + 2H(H^2 - K)\right]\mathbf{n}. \tag{2.5}$$

Willmore flow is derived from minimizing total squared mean-curvature $\int_{\mathcal{S}} H^2 \mathrm{d}A$. Notice that a factor 2 is added to the original Willmore flow for comparability with other equations used. Since the constant factor could be absorbed by the parameter t in the equation, there is no influence on the behavior of the equation by adding this factor. There are sound published research papers that use this flow (see [3, 10, 11, 22]). There is no volume/area preserving/shrinking property for this flow. However, if the initial surface is a sphere, Willmore flow keeps the spherical shape unchanged. Moreover, surfaces with zero mean curvature are the the steady solution of (2.5). A torus with $R/r = \sqrt{2}$ is a steady solution of (2.5), where the torus is defined by rotate a circle with radius r along a another circle with radius R.

Quasi-surface diffusion flow

$$\frac{\partial \mathbf{x}}{\partial t} = -\Delta_s^2 \mathbf{x}. \tag{2.6}$$

This flow is introduced in [25], and is used in discrete surface design. It is well-known that tangent motion of a surface does not alter the surface shape (see [7]). Hence if we remove the tangential movement of (2.6), we obtain the following geometric flow

$$\frac{\partial \mathbf{x}}{\partial t} = -2\left[\Delta_s H - 2H(2H^2 - K)\right]\mathbf{n}. \tag{2.7}$$

If $\mathcal{S}$ is a closed surface, then it is easy to derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -2\int_{\mathcal{S}(t)} \left[\|\nabla_s H\|^2 + 2H^2(2H^2 - K)\right] \mathrm{d}A \leq 0.$$

Hence, Quasi-surface diffusion flow is area diminishing. Since $\|\nabla_s H\|^2 + 2H^2(2H^2 - K) = 0$ if and only if $H = 0$, the shrinkage stops when $H \equiv 0$. Hence, the solution surfaces of (2.6) approach to the minimal surface.

Note that these three flows share the same fourth-order term $-2\Delta_s H\mathbf{n}$, only the second order terms are different. However, their behaviors are quite different. This is the reason we choose to use them to achieving different effect in the shape design.

2.3 Mixed Variational Formulations

Now we present variational form formulations for these equations. The detailed derivations could be found in [24]. Let $\mathbf{y}(\mathbf{x}) = H(\mathbf{x})\mathbf{n}(\mathbf{x}) \in \mathbb{R}^3$ stand for the mean curvature normal. Then the mixed variational form of (2.4) is: Find $(\mathbf{x}, \mathbf{y}) \in H^2(\mathcal{S})^3 \times H^1(\mathcal{S})^3$ such that

$$\begin{cases} \displaystyle\int_{\mathcal{S}} \frac{\partial \mathbf{x}}{\partial t}\phi \,\mathrm{d}A + 2\int_{\mathcal{S}} \left[\phi \oslash \mathbf{y} - \mathbf{n}(\nabla_s \phi)^{\mathrm{T}} \nabla_s \mathbf{y}\right] \mathbf{n} \,\mathrm{d}A = \mathbf{0}, & \forall \phi \in H_0^1(\mathcal{S}), \\ \displaystyle\int_{\mathcal{S}} \mathbf{y}\psi \,\mathrm{d}A + \frac{1}{2}\int_{\mathcal{S}} (\nabla_s \mathbf{x})^{\mathrm{T}} \nabla_s \psi \,\mathrm{d}A - \frac{1}{2}\int_{\partial\mathcal{S}} \mathbf{n}_c \psi \,\mathrm{d}s = \mathbf{0}, & \forall \psi \in H^1(\mathcal{S}), \\ \mathcal{S}(0) = \mathcal{S}_0, \quad \partial\mathcal{S}(t) = \Gamma, \quad \mathbf{n}_c(\mathbf{x}) = \mathbf{n}_c^{(\Gamma)}(\mathbf{x}), \ \forall \mathbf{x} \in \Gamma, & \end{cases} \tag{2.8}$$

where $\mathbf{n}_c^{(\Gamma)}$ is the given co-normal on the boundary curve Γ. The the mixed variational form of (2.5) and (2.7) are similar. The differences occur only in the first equation. For Willmore flow

(2.5), the first equation of the mixed variational form is as follows.

$$\begin{aligned}\int_{\mathcal{S}} \frac{\partial \mathbf{x}}{\partial t}\phi \, \mathrm{d}A + 2\int_{\mathcal{S}} \left[\phi \oslash \mathbf{y} - \mathbf{n}(\nabla_s \phi)^{\mathrm{T}} \nabla_s \mathbf{y}\right] \mathbf{n} \, \mathrm{d}A \\ + 4\int_{\mathcal{S}} \mathbf{n}(H^2 - K)\phi \mathbf{n}^{\mathrm{T}} \mathbf{y} \, \mathrm{d}A = \mathbf{0}, \qquad \forall \phi \in H_0^1(\mathcal{S}),\end{aligned} \tag{2.9}$$

For the quasi-surface diffusion flow (2.7), the first equation of the mixed variational form is

$$\begin{aligned}\int_{\mathcal{S}} \frac{\partial \mathbf{x}}{\partial t}\phi \, \mathrm{d}A + 2\int_{\mathcal{S}} \left[\phi \oslash \mathbf{y} - \mathbf{n}(\nabla_s \phi)^{\mathrm{T}} \nabla_s \mathbf{y}\right] \mathbf{n} \, \mathrm{d}A \\ - 4\int_{\mathcal{S}} \mathbf{n}(2H^2 - K)\phi \mathbf{n}^{\mathrm{T}} \mathbf{y} \, \mathrm{d}A = \mathbf{0}, \qquad \forall \phi \in H_0^1(\mathcal{S}),\end{aligned} \tag{2.10}$$

In the next section, systems (2.8)–(2.10) are numerically solved using a mixed finite element method in a finite element space defined by the extended Loop subdivision scheme.

3 Subdivision Surfaces and Finite Element Space

We consider only the construction of Loop's subdivision surfaces for triangular control polygons. The idea for constructing Catmull-Clark surfaces for quadrilateral control polygons is similar. Note that the Loop's subdivision scheme is usable only for control polygons without boundary. Therefore, an extension of the subdivision scheme to control polygons with boundary is required. We use Biermann et al's extension (see [2]) to achieve our goal. For saving the space, we do not describe the well known Loop's subdivision scheme (see [12]) and Biermann et al's extension. The details of this extension could be found in [2].

3.1 Basis Functions and Their Properties

Now let us define the basis functions of the finite element function space, denoted as $V_{\mathcal{S}(t)}$. For each control point $\mathbf{x}_i$, including the corner control point and boundary control points, of a control polygon $\mathcal{S}_d$, we shall associate it with a basis function ϕ_i, where ϕ_i is defined as the limit of the extended Loop's subdivision scheme applying to the zero control values everywhere except at $\mathbf{x}_i$ where it is one.

The control polygon $\mathcal{S}_d$, as a piecewise linear surface, is served as the domain of the basis function ϕ_i. The mapping from $\mathcal{S}_d$ to ϕ_i is defined by a dual subdivision schemes. More precisely, when the extended Loop's subdivision scheme is applied to the control function values recursively, the linear subdivision scheme (each triangle is partitioned into four equal-sized sub-triangles) is applied to the control polygon correspondingly. The limit of the former is ϕ_i and that of later is $\mathcal{S}_d$ itself.

The basis functions share some properties with the well known B-spline basis. These properties are important in our finite element method. Now let us describe these properties.

1. *Positivity.* The weights of the extended subdivision rules are positive. Hence the basis function ϕ_i is nonnegative everywhere and positive around $\mathbf{x}_i$.

2. *Locality.* It is known that the limit value at a control point is a linear combination of the one-ring neighbor values. Hence, the limit value is zero at a control point if the control values on the one-ring neighbor control points are zeros. Therefore, the support of the basis function is within the two-ring neighborhood.

3. *Partition of Unity.* Since the all the subdivision rules have the properties that the weights are summed to one. Therefore, if we choose all the control values as one. The control values after one subdivision step are still one. This implies that $\sum_{i=1}^{m} \phi_i = 1$. This property is called partition of unity.

4. *Interpolatory Properties at the Boundary.* The extended subdivision rules on the boundary do not involve the interior control points. Hence the basis functions for the interior control points are zero at the boundary. This means that the given boundary curves are interpolated.

5. *Tangential Property.* Let $\mathbf{x}_i$ be a control point, with non of its one-ring neighbor control points is boundary control points. Then $\nabla_s \phi_i$ vanishes on the boundary. This fact could be observed by considering the eigen-decomposition of the control points. Let $\mathbf{p}^{(k)} \in \mathbb{R}^{(n+1)\times 3}$ be a vector consisting of one-ring neighbor control points of $\mathbf{x}_i^{(k)}$ at the subdivision level k, $S \in \mathbb{R}^{(n+1)\times(n+1)}$ the local subdivision matrix that convert $\mathbf{p}^{(k)}$ to $\mathbf{p}^{k+1}$, i.e.,

   $$\mathbf{p}^{(k+1)} = S\mathbf{p}^{(k)} = S^k\mathbf{p}^{(1)}, \quad k = 1, 2, \cdots .$$

   Here n stands for the valence of $\mathbf{x}_i^{(k)}$. Suppose $\mathbf{p}^{(1)}$ is decomposed into

   $$\mathbf{p}^{(1)} = \mathbf{e}_0\mathbf{a}_0^T + \mathbf{e}_1\mathbf{a}_1^T + \mathbf{e}_2\mathbf{a}_2^T + \cdots + \mathbf{e}_n\mathbf{a}_n^T, \quad \mathbf{a}_j \in \mathbb{R}^3,$$

   where $\mathbf{e}_0$, $\mathbf{e}_1$, $\cdots$, $\mathbf{e}_n$ are the eigenvectors of S. Here we assume that these eigenvectors are arranged in the order of non-increasing eigenvalues λ_j. Then

   $$\mathbf{p}^{(k+1)} = \lambda_0^k\mathbf{e}_0\mathbf{a}_0^T + \lambda_1^k\mathbf{e}_1\mathbf{a}_1^T + \lambda_2^k\mathbf{e}_2\mathbf{a}_2^T + \cdots + \lambda_n^n\mathbf{e}_n\mathbf{a}_n^T,$$

   where $\lambda_0 = 1$, $\lambda_1 = \lambda_2 < 1$. It is well-known that the limit position at the center is $\mathbf{a}_0$. The tangent direction at this point are $\mathbf{a}_1$ and $\mathbf{a}_2$, and $\mathbf{a}_j$ is given by $\mathbf{a}_j^T = \tilde{\mathbf{e}}_j^T\mathbf{p}^{(1)}$. $\tilde{\mathbf{e}}_j$ are the left eigenvectors of $\mathcal{S}$ with normalized condition $\tilde{\mathbf{e}}_j^T\mathbf{e}_j = 1$.

   The analysis above is valid for control function values.

6. *Linear independency.* As a set of basis functions, $\{\phi_i\}_{i=1}^m$ must be linearly independent. For Loop's subdivision scheme, this fact is implied by a result from [23] on the solvability of interpolation problem:

For the given function values $\{f_i\}_1^m$, find the control function values $\{g_i\}_1^m$ such that

$$\sum_{j=1}^{m} g_j \phi_j(\mathbf{v}_i) = f_i, \quad i = 1, \cdots, m, \tag{3.1}$$

where $\mathbf{v}_i$ is the limit position of the subdivision surface corresponding to the control point $\mathbf{x}_i$.

Theorem 3.2 *(see [23]) The interpolation problem (3.1) always has a unique solution.*

This theorem clearly implies that $\{\phi_i\}_{i=1}^m$ is linearly independent. For the extended Loop's subdivision scheme, this fact can be similarly proved. We do not focus the proof in this paper.

3.2 Spatial Discretizations

Suppose ϕ_i is a basis function of $V_{\mathcal{S}(t)}$ corresponding to control point $\mathbf{x}_i$, $i = 0, \cdots, m$. Assume $\mathbf{x}_1, \cdots, \mathbf{x}_{m_0}$ are the interior control points, and $\mathbf{x}_{m_0+1}, \cdots, \mathbf{x}_m$ are the boundary control points. Then $\mathbf{x}(t) \in \mathcal{S}(t)$ could be represented as

$$\mathbf{x}(t) = \sum_{j=1}^{m_0} \mathbf{x}_j(t)\phi_j + \sum_{j=m_0+1}^{m} \mathbf{x}_j(t)\phi_j, \quad \mathbf{x}_j(t) \in \mathbb{R}^3, \tag{3.2}$$

and therefore,

$$\nabla_s \mathbf{x}(t) = \sum_{j=1}^{m_0} \nabla_s \phi_j [\mathbf{x}_j(t)]^T + \sum_{j=m_0+1}^{m} \nabla_s \phi_j [\mathbf{x}_j(t)]^T, \quad \mathbf{x}_j(t) \in \mathbb{R}^{3\times 3}. \tag{3.3}$$

The mean curvature vector of the surface is represented approximately as

$$\mathbf{y}(t) = \sum_{j=1}^{m} \mathbf{y}_j(t)\phi_j, \quad \mathbf{y}_j \in \mathbb{R}^3, \quad \nabla_s \mathbf{y}(t) = \sum_{j=1}^{m} \nabla_s \phi_j [\mathbf{y}_j(t)]^T \in \mathbb{R}^{3\times 3}. \tag{3.4}$$

Since the boundary control points are fixed and the interior control points are to be determined, the coefficients $\mathbf{x}_j$ in the first term of (3.2) are unknowns, while the coefficients $\mathbf{x}_j$ in the second term are the given control points on the boundary. Furthermore, since the curvature on the surface boundary involves the unknown interior control points, hence all the coefficients in (3.4) are treated as unknowns.

Now let us discretize equations (2.8)–(2.10) in the finite space $V_{\mathcal{S}(t)}$. Since these equations are similar in form, we treat them together. Let $\mathcal{S}$ be the limit surface of the extended Loop's subdivision scheme for the control polygon $\mathcal{S}_d$. Substituting (3.2)–(3.4) into (2.8)–(2.10), and taking the test functions ϕ as $\phi_i (i = 0, \cdots, m_0)$, ψ and $\phi_i (i = 0, \cdots, m)$, and finally noting that $\frac{\partial \mathbf{x}_j(t)}{\partial t} = \mathbf{0}$ if $j > m_0$, we obtain the following matrix representations of (2.8)–(2.10):

$$\left\{ \begin{array}{lcl} M^{(1)}_{m_0} \frac{\partial X_{m_0}(t)}{\partial t} & + & L^{(1)}_m Y_m(t) = \mathbf{0}, \\ M^{(2)}_m Y_m(t) & + & L^{(2)}_m X_m(t) = B, \end{array} \right. \tag{3.5}$$

where

$$\begin{array}{lll} X_j(t) & = [\mathbf{x}^{\mathrm{T}}_0(t), \cdots, \mathbf{x}^{\mathrm{T}}_j(t)]^{\mathrm{T}} & \in \mathbb{R}^{3(j+1)}, \\ Y_m(t) & = [\mathbf{y}^{\mathrm{T}}_0(t), \cdots, \mathbf{y}^{\mathrm{T}}_m(t)]^{\mathrm{T}} & \in \mathbb{R}^{3(m+1)}, \end{array}$$

are matrices consisting of control points for the surface and curvature, respectively, and

$$\begin{array}{lll} B & = [\mathbf{b}^{\mathrm{T}}_0, \cdots, \mathbf{b}^{\mathrm{T}}_m]^{\mathrm{T}} & \in \mathbb{R}^{3(m+1)}, \\ M^{(1)}_{m_0} & = (m_{ij} \mathrm{I}_3)^{m_0, m_0}_{ij=0}, & M^{(2)}_m = (m_{ij} \mathrm{I}_3)^{m,m}_{ij=0}, \\ L^{(1)}_m & = \left(l^{(1)}_{ij} \right)^{m_0, m}_{ij=0}, & L^{(2)}_K = \left(l^{(2)}_{ij} \mathrm{I}_3 \right)^{m,K}_{ij=0}. \end{array}$$

are the coefficient matrices. The elements of these matrices are defined as follows:

$$\begin{array}{rcl} m_{ij} & = & \displaystyle\int_{\mathcal{S}} \phi_i \phi_j \; \mathrm{d}A, \\ l^{(1)}_{ij} & = & \left\{ \begin{array}{ll} l^{(s)}_{ij} & \text{for SDF}, \\ l^{(s)}_{ij} + 4 \displaystyle\int_{\mathcal{S}} \left[\mathbf{n}(H^2 - K)\phi_i \phi_j \right] \mathbf{n}^{\mathrm{T}} \mathrm{d}A & \text{for WF}, \\ l^{(s)}_{ij} - 4 \displaystyle\int_{\mathcal{S}} \left[\mathbf{n}(2H^2 - K)\phi_i \phi_j \right] \mathbf{n}^{\mathrm{T}} \mathrm{d}A & \text{for QSDF}, \end{array} \right. \\ l^{(2)}_{ij} & = & \dfrac{1}{2} \displaystyle\int_{\mathcal{S}} \left[(\nabla_s \phi_i)^{\mathrm{T}} \nabla_s \phi_j \right] \; \mathrm{d}A, \\ \mathbf{b}_i & = & \dfrac{1}{2} \displaystyle\int_{\Gamma} \mathbf{n}_c \phi_i \; \mathrm{d}s, \end{array} \tag{3.6}$$

with

$$l^{(s)}_{ij} = 2 \int_{\mathcal{S}} \left[\phi_i \oslash \phi_j - \mathbf{n}(\nabla_s \phi_i)^{\mathrm{T}} \nabla_s \phi_j \right] \mathbf{n}^{\mathrm{T}} \; \mathrm{d}A.$$

Moving the terms related to the known control points $\mathbf{x}_{m_0+1}, \cdots, \mathbf{x}_m$ in the second equation of (3.5) to the equations's right-hand side, we can rewrite (3.5) as

$$\left\{ \begin{array}{lcl} M^{(1)}_{m_0} \frac{\partial X_{m_0}(t)}{\partial t} & + & L^{(1)}_m Y_m(t) = \mathbf{0}, \\ M^{(2)}_m Y_m(t) & + & L^{(2)}_{m_0} X_{m_0}(t) = B^{(2)}. \end{array} \right. \tag{3.7}$$

Note that, matrices $M^{(1)}_{m_0}$ and $M^{(2)}_{m}$ are symmetric and positive definite. The integrals in defining the matrix elements are computed using Gaussian quadrature formulas over the domain triangles. The knots in the barycentric coordinate form and weights of the Gaussian quadrature formulas can be found in [1].

In the boundary integrals (3.6), $\mathbf{n}_c$ is the co-normal of the surface, it is infeasible to compute these co-normals from the previous approximation, since they do not satisfy the given boundary condition. The right way is to replace $\mathbf{n}_c$ with $\mathbf{n}_c^{(\Gamma)}$. That is

$$\mathbf{b}_i = \frac{1}{2}\int_\Gamma \mathbf{n}_c^{(\Gamma)}\phi_i \, \mathrm{d}s.$$

3.3 Temporal Direction Discretization

Suppose we have approximate solutions $X^{(k)}_{m_0} = X_{m_0}(t_k)$ and $Y^{(k)}_m = Y_m(t_k)$ at $t = t_k$. We want to obtain approximate solutions $X^{(k+1)}_{m_0}$ and $Y^{(k+1)}_m$ at $t = t_{k+1} = t_k + \tau^{(k)}$ using a forward Euler scheme. Specifically, we use the following approximation

$$\frac{X_{m_0}(t_{k+1}) - X_{m_0}(t_k)}{\tau^{(k)}} \approx \frac{\partial X_{m_0}}{\partial t}.$$

The matrices $M^{(1)}$, $M^{(2)}$, $L^{(1)}$ and $L^{(2)}$ in (3.7) are computed using the surface data at $t = t_k$. This yields a linear system with $X^{(k+1)}_{m_0}$ and $Y^{(k+1)}_m$ as unknowns:

$$\begin{bmatrix} M^{(1)}_{m_0} & \tau^{(k)}L^{(1)}_m \\ L^{(2)}_{m_0} & M^{(2)}_m \end{bmatrix} \begin{bmatrix} X^{(k+1)}_{m_0} \\ H^{(k+1)}_m \end{bmatrix} = \begin{bmatrix} \tau^{(k)}B^{(1)} + M^{(1)}_{m_0}X^{(k)}_{m_0} \\ B^{(2)} \end{bmatrix}$$

Though the matrices $M^{(1)}$ and $M^{(2)}$ are symmetric and positive definite, the total matrix is neither symmetric nor positive definite. However the coefficient matrix of this system is highly sparse, hence a stable iterative method for its solution is desirable. We use Saad's iterative method, namely GMRES (see [17]), to solve our sparse linear system. The numerical tests show that this iterative method works very well.

Remark 3.1 *In each step of the iteration, the matrices $M^{(1)}$, $M^{(2)}$, $L^{(1)}$ and $L^{(2)}$ need to be recomputed, since they depend on the evolved surface $\mathcal{S}(t)$. However, the basis function ϕ_i and their partial derivatives (up to the second orders) do not change in the evolution. Hence, function values and partial derivatives of each ϕ_i could be pre-computed at the knots of the Gaussian quadrature. Furthermore, the boundary terms $\mathbf{b}_i$ are fixed during the iterations, since both $\mathbf{n}_c^{(\Gamma)}$ and ϕ_i are fixed.*

4 Conclusions

Mesh subdivision technology can provide a simple and efficient method to construct surfaces with any topology structure, at the same time satisfy some smoothness requirement. In this paper, we construct geometric PDE Loop's subdivision surfaces, with given G^1 boundaries condition, using three fourth order geometric flows. A numerical solution method of the finite element based on the extended Loop's subdivision scheme is adopted. The geometric PDE surfaces can be efficiently constructed through our method, and our numerical method is also convergent. The results are satisfactory.

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