# Multilevel correction for collocation solutions of Volterra integral equations with proportional delays

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**Abstract.** In this paper we propose a convergence acceleration method for collocation solutions of the linear second-kind Volterra integral equations with proportional delay $qt$ $(0 < q < 1)$. This convergence acceleration method called multilevel correction method is based on a kind of hybrid mesh, which can be viewed as a combination between the geometric meshes and the uniform meshes. It will be shown that, when the collocation solutions are continuous piecewise polynomials whose degrees are less than or equal to $m$ $(m \leqslant 2)$, the global accuracy of $k$ level corrected approximation is $O(N^{-(2m(k+1)-\varepsilon)})$, where $N$ is the number of the nodes, and $\varepsilon$ is an arbitrary small positive number.

**Keywords.** delay integral equation, geometric mesh, collocation method, superconvergence, high order interpolation operator, multilevel correction, hybrid meshes.

**AMS subject classification.** 65R20, 34K06, 34K28

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## 1 Introduction

We consider a convergence acceleration method for collocation solution of the Volterra integral equation with vanishing variable delay

$$u(t) = f(t) + \int_0^t k_1(t,s,u(s))ds + \int_0^{\theta(t)} k_2(t,s,u(s))ds, \quad t \in J := [0,T], \tag{1.1}$$

where $\theta(t) := t - \tau(t) \geqslant 0$ is such that the continuous delay $\tau$ satisfies $\tau(0) = 0$. Equation (1.1) includes an important special case (see [7, 16]) where $\tau$ is the proportional delay $\tau(t) = (1-q)t$ with $0 < q < 1$, i.e.,

$$u(t) = f(t) + \int_0^t k_1(t,s,u(s))ds + \int_0^{qt} k_2(t,s,u(s))ds, \quad t \in J := [0,T]. \tag{1.2}$$

There are many literature to study delay functional equations frequently encountered in physical and biological processes, see, for example, [19]-[21], [26] and [28], [30] and [33]. The analysis to the second-kind Volterra integral equations with proportional delays dates back to the works in [38](pp. 92-101), [1] and [17]. Some more recent results on this subject can be found in [14], [16] and [32]. During the past decade, numerical methods for (1.1) or (1.2) has attracted wide attention of many researchers. Various numerical methods for (1.1) have been introduced such as quadrature method [3], iterated collocation method [6, 11], Euler-type method [31] and spectral method [2, 37]. Numerical methods for functional integral and integro-differential equations of Volterra type have been summarized in [8].

It is well known that Sloan iteration first proposed in [34] can greatly raise the convergence rate of projection-type solutions of compact operator equations. The convergence of Sloan iteration for integral equations with smooth kernel can be further improved by convergence acceleration methods such as extrapolation method and correction method (see, for example, [27] and [35]). Two kinds of multilevel correction methods for collocation solutions of Fredholm integro-differential equations and for discrete collocation solutions of the Volterra integral equations with constant delay were introduced in [24] and [25], respectively. For the multilevel correction method, the convergence rate of the multilevel corrected approximation is much higher than that of the original collocation approximation. This means that the multilevel correction method has significant advantages for reducing the cost of calculation and improving the computational efficiency. In the present paper, we try to develop a multilevel correction method for collocation approximation of the equation (1.2).

It is well known that for the classical Volterra integral equations ($k_2 \equiv 0$ in (1.2)) the iterated collocation solution associated with piecewise $(m-1)$st degree polynomial spline collocation based on a uniform mesh possesses the optimal superconvergence order $2m$ at the nodes of the mesh, provided that the collocation parameters are chosen as the $m$ Gauss points in $(0, 1)$. However, it has been shown in [7] and [36] that these superconvergence properties on uniform meshes do not carry over to equation (1.1) ($k_2 \not\equiv 0$). In fact, it can be seen from [7] and [36] that for this kind of delay integral equation the optimal (local) superconvergence order $p^*$ is at most $p^* = 2m - 1$. Fortunately, an important observation was fall to be escaped. It has been proved in [13] that a properly chosen *geometric meshes*, which is similar to the meshes introduced in [4, 23], can generate iterated collocation solutions possessing the almost optimal local superconvergence order $p^* = 2m - \varepsilon$ at all mesh points (see also [11]). It is certain that we can consider multilevel correction method for the collocation solutions based on such *geometric meshes*. In order to develop a multilevel correction method, we need to construct a *high order interpolation* operator which must be uniform bounded. However, we find that the high order interpolation operator defined on *geometric meshes* is not uniform bounded yet. Therefore, we need to make the change of the distribution of grid points.

In the present paper we introduce a kind of *hybrid mesh*, which can be viewed as a combination between the geometric meshes and the uniform meshes. We find that not only the *hybrid mesh* can generate collocation solutions possessing the almost optimal local superconvergence as *geometric meshes*, but also the high order interpolation operator defined on such *hybrid mesh* is uniform bounded. It will be shown that when the collocation solutions are continuous piecewise polynomials whose degrees are less than or equal to $m$ ($m \leqslant 2$), the global accuracy of $k$ time corrected approximation is $O(N^{-(2m(k+1)-\varepsilon)})$, where $N$ is the number of the nodes and $\varepsilon$ is an arbitrary small positive number.

The paper is organized as follows. In section 2, we describe the main result about multilevel correction for collocation solutions of the linear version of the equation (1.2). In section 3, we analyze the properties of the collocation method and the high order interpolation operator based on hybrid meshes. Then we prove a few of auxiliary results. In section 4, the proof of main results is given. In section 5, some numerical results are reported to confirm the theoretical result.

## 2 Main result

The theoretical analysis of the equation (1.2) will be carried out in the Banach space $C^n[a, b]$ of $n$ times differentiable and continuous functions being real-valued on $[a, b]$. When $y(t)$ is $k$ times differentiable, $y^{(k)}(t)$ coincides with the usual notion of derivative: $y^{(k)}(t) =$

$D_t^k y(t) = d^k y/dt^k$. This space is equipped with uniform norm

$$\|y\|_{n,\infty,[a,b]} = \sup_{\substack{a\leqslant t\leqslant b\\0\leqslant k\leqslant n}} |y^{(k)}(t)|, \quad \forall\, y \in C^n[a,b]. \tag{2.1}$$

Assume that the given function $f \in C^{2p+2}[0,T]$, $K_i \in C^{2p+2}(\Omega)(i = 1,2)$ where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t,s) :\ 0 \leqslant s \leqslant t \leqslant T\}$, $\Omega_2 := \{(t,s) :\ 0 \leqslant s \leqslant qt,\ t \in J\}$ and $p$ is a nonnegative integer (see also [7]). We consider the numerical methods for solving the linear version of the equation (1.2),

$$u(t) = f(t) + \int_0^t K_1(t,s)u(s)ds + \int_0^{qt} K_2(t,s)u(s)ds, \quad t \in J, \tag{2.2}$$

where $0 < q < 1$. It has been shown in [5] and [13] that the integral equation (2.2) has a unique solution $u \in C^{2p+2}[0,T]$.

## 2.1 The collocation method for solving Volterra integral equation

Let $\mathbf{N}$ denote the set of all positive integers. For any $N \in \mathbf{N}$, let $\tilde{J}_N$: $0 = t_0 < t_1 < \cdots < t_N = T$ denote a mesh (or partition) on the given interval $J$, and set $e_n := [t_{n-1},\ t_n]$, $h_n := t_n - t_{n-1}$ $(n = 1,\cdots,N)$, $h := \max_{1\leqslant n\leqslant N} h_n$. The finite-dimensional collocation space on the meshes $\tilde{J}_N$ is defined as

$$S_m^{(0)}(\tilde{J}_N) := \{v :\ v \in C(J),\ v\,|_{e_n} \in P_m(e_n)\ (n = 1,\cdots,N)\},$$

where $m \in \mathbf{N}$ satisfying $m \geqslant 1$ and $P_m(e_n)$ denotes the set of polynomials defined on $e_n$, whose degree is less than or equal to $m$.

The collocation method for solving Volterra integral equation (2.2) concentrates on looking for $\widetilde{u}_h \in S_m^{(0)}(\tilde{J}_N)$ satisfying

$$\widetilde{u}_h(t) = f(t) + \int_0^t K_1(t,s)\widetilde{u}_h(s)ds + \int_0^{qt} K_2(t,s)\widetilde{u}_h(s)ds,\ t \in \widetilde{X}_n(\ 1 \leqslant n \leqslant N), \tag{2.3}$$

where $\widetilde{X}_n := \{t_{n,j} := t_{n-1} + c_j h_n,\ 0 = c_1 < c_2 < \cdots < c_m < c_{m+1} = 1\}$ $(n = 1,\cdots,N)$. The set $\widetilde{X}(N) := \bigcup_{n=1}^N \widetilde{X}_n$ will be referred to as the set of collocation points, which is completely determined by the given mesh $\tilde{J}_N$ and the collocation parameters $\{c_j\}_{j=1}^{m+1}$.

## 2.2 Multilevel correction for collocation solution

In the subsection, we introduce a multilevel correction method. For convenience, we define operators $\widetilde{Q}_h$ and $\overline{\pi}$, which will be referred to repeatedly below. For any function $y \in C(J)$, we set

$$f_y(t) = y(t) - \int_0^t K_1(t,s)y(s)ds - \int_0^{qt} K_2(t,s)y(s)ds.$$

The sequence of collocation operators $\widetilde{Q}_h : C(J) \to S_m^{(0)}(\tilde{J}_N)$ is defined as $\widetilde{Q}_h y \in S_m^{(0)}(\tilde{J}_N)$, which is the unique solution of the discretic system:

$$\widetilde{Q}_h y(t) = f_y(t) + \int_0^t K_1(t,s)\widetilde{Q}_h y(s)ds + \int_0^{qt} K_2(t,s)\widetilde{Q}_h y(s)ds,\ t \in \widetilde{X}_n(\ 1 \leqslant n \leqslant N). \quad (2.4)$$

Using such definition we know that the collocation solution $\widetilde{u}_h$ defined by (2.3) can be written as $\widetilde{u}_h = \widetilde{Q}_h u$, with $u$ being the analytic solution of the equation (2.2).

A multilevel collocation method will involve a higher order interpolation operator $\overline{\pi}$. Let the collocation parameters are chosen as Lobatto points. We define $\overline{\pi}$ as the sequence of interpolation operators such that $\overline{\pi}y(t_n) = y(t_n)(n = 1, 2, \cdots, N)$ for any $y \in C(J)$ and $\overline{\pi}y(t)$ is a piecewise polynomial of higher order which is completely determined by $y(t_n)(n = 1, 2, \cdots, N)$. The detailed definition of $\overline{\pi}$ will be stated below.

Throughout this paper we let $C_r^i$ denote the combination number defined as usual, where $i$ is a non-negative integer and $r$ is a positive integer, which satisfy $i \leqslant r$.

Let $u$ denote the analytic solution of the equation (2.2), and define

$$\widetilde{u}_{h,k} = (-1)^k \sum_{j=0}^{k} (-1)^j C_{k+1}^j (\overline{\pi}\widetilde{Q}_h)^{k-j}\overline{\pi}\widetilde{u}_h = (-1)^k \sum_{j=0}^{k} (-1)^j C_{k+1}^j (\overline{\pi}\widetilde{Q}_h)^{k-j}\overline{\pi}\widetilde{Q}_h u.$$

In general the approximation $\widetilde{u}_{h,k}$ has a faster convergence than the original collocation solution $\widetilde{u}_h$. But the convergence rate of $\widetilde{u}_{h,k}$ (and $\tilde{u}_h$) depends on the meshes $\tilde{J}_N$. Because of this we need to discuss how to choose a suitable meshes $\tilde{J}_N$.

## 2.3 From geometric meshes to hybrid meshes

We first recall the geometric meshes introduced in [13] and [23].

**Definition 2.1.** *The meshes $\{\tilde{J}_N\}_{N \geqslant 2}$ is called a sequence of geometric meshes if the mesh points $\{t_n\} = \{t_n^{(N)}\}$ satisfy*

$$t_n = t_n^{(N)} = d^{N-n}T, \quad n = 1, \cdots, N, \quad (2.5)$$

*where $d$ ($0 < d < 1$; $d$ is independent of $n$) remains to be determined.*

**Remark 2.1.** *Note that the mesh diameter $h$ is given by $h_N = T(1-d)$. To guarantee $h$ satisfying that $h \to 0$ as $N \to \infty$, we require $d \to 1$ as $N \to \infty$. Therefore $d$ will depend on $N$.*

Since $h_2/h_1 = (t_2 - t_1)/t_1 - t_0 = d/1 - d \to \infty$ *as* $N \to \infty (d \to 1)$ by the definition of geometric meshes, not all Lagrange interpolation basic functions defined by geometric meshes are uniform bounded. Therefore the error estimation of the high order interpolation operator $\overline{\pi}$ is not optimal. To make the interpolation operator $\overline{\pi}$ available, we set $2p$ new points $t_{1,1}, t_{1,2}, \cdots, t_{1,2p}$ in $(t_0, t_1)$ and define $\overline{\pi} y(t)$ on $[t_0, t_1]$ as a $2p+1$ order polynomial which is determined by the values of $y(t)$ at the points $\{t_0, t_{1,1}, t_{1,2}, \cdots, t_{1,2p}, t_1\}$. That is to say, we have to define $\overline{\pi} y(t)$ on $[t_0, t_1]$ and $[t_1, t_N]$ respectively. As we will see, the high order interpolation operator $\overline{\pi}$ based on such meshes is uniform bounded. Set $t_{1,0} = t_0$ and $t_{1,2p+1} = t_1$. Since $[t_0, t_1]$ is partitioned by the new points, the set of collocation points $\widetilde{X}_1$ must be changed into $X_1 = \bigcup_{k=1}^{2p+1} X_{1,k}$ where

$$X_{1,k} := \{t_{1,k,j} := t_{1,k-1} + c_j(t_{1,k} - t_{1,k-1}),\ 0 = c_1 < \cdots < c_{m+1} = 1\}(k = 1, \cdots, 2p+1).$$

This means that the collocation method is carried out on the set $X_1 \bigcup (\bigcup_{n=2}^{N} \widetilde{X}_n)$.

Let $J_N : 0 = t_0 = t_{1,0} < t_{1,1} < t_{1,2} < \cdots < t_{1,2p} < t_{1,2p+1} = t_1 < t_2 < \cdots < t_N = T$ be the new meshes on $J$, and let $Z_N$ denote the set of the nodes except 0

$$Z_N = \{t_{1,1}, \cdots, t_{1,2p+1}, t_2, \cdots, t_N\}.$$

**Definition 2.2.** *The meshes $\{J_N\}_{N \geqslant 2}$ introduced above is called a sequence of hybrid meshes, if the nodes $\{t_n\}_{n=1}^{N}$ is defined by the geometric meshes on $J$, and the nodes $\{t_{1,k}\}_{k=1}^{2p} \subset [t_0, t_1]$ is defined by the uniform meshes on $[t_0, t_1]$.*

### 2.4 Multilevel correction based on the hybrid meshes

Set

$$e_{1,k} := [t_{1,k-1},\ t_{1,k}],\ h_{1,k} := t_{1,k} - t_{1,k-1}\ (1 \le k \le 2p+1)$$

and

$$e_n := [t_{n-1},\ t_n],\ h_n := t_n - t_{n-1}\ (1 \le n \le N).$$

Consider the finite-dimensional collocation spaces on the meshes $J_N$

$$S_m^{(0)}(J_N) := \{v \in C(J) : v\,|_{e_{1,k}} \in P_m(e_{1,k})(1 \le k \le 2p+1), v\,|_{e_n} \in P_m(e_n)(2 \le n \le N)\},$$

where $m \in \mathbf{N}$, $P_m(e_{1,k})$ and $P_m(e_n)$ denote the set of polynomials defined on $e_{1,k}$ and $e_n$ respectively, whose degrees are less than or equal to $m$. We are looking for $u_h \in S_m^{(0)}(J_N)$ satisfying

$$u_h(t) = f(t) + \int_0^t K_1(t,s)u_h(s)ds + \int_0^{qt} K_2(t,s)u_h(s)ds, \quad t \in X_n(\,1 \leqslant n \leqslant N), \quad (2.6)$$

where $X_1 := \bigcup_{k=1}^{2p+1} X_{1,k}$ with

$$X_{1,k} := \{t_{1,k,j} := t_{1,k-1} + c_j h_{1,k},\ 0 = c_1 < c_2 < \cdots < c_m < c_{m+1} = 1\}\ (1 \le k \le 2p+1),$$

and

$$X_n := \{t_{n,j} := t_{n-1} + c_j h_n,\ 0 = c_1 < c_2 < \cdots < c_m < c_{m+1} = 1\}\ (2 \le n \le N).$$

The set $X(N) := \bigcup_{n=1}^{N} X_n$ is referred to as the set of collocation points, which is completely determined by the given meshes $J_N$ and the collocation parameters $\{c_j\}_{j=1}^{m+1}$.

The collocation equation (2.6) defines an unique approximation $u_h \in S_m^{(0)}(J_N)$ whenever the mesh diameter defined below is sufficiently small. As for classical Volterra integral equations, the approximation $u_h$ will be generated recursively by successive computation of its restrictions $u_h^{1,1}, \cdots, u_h^{1,2p+1}, u_h^2, \cdots, u_h^N$ on the subintervals $e_{1,1}, \cdots, e_{1,2p+1}, e_2, \cdots, e_N$ given by the mesh $J_N$ (compare also [15]).

According to the definition of hybrid meshes, the following two assumptions are supposed to hold in the subsequent analysis.

$A_1$ : Let $\kappa$ be the maximal positive integer satisfying $q^{\frac{1}{\kappa}} \leqslant (1 - \frac{(2p+2)\ln N}{(m+2)N})$, namely,

$$\kappa := \left[ \frac{\ln q}{\ln(1 - \frac{(2p+2)\ln N}{(m+1)N})} \right].$$

For a fixed $q \in (0,1)$, we have $\kappa \geqslant 1$ as $N \to \infty$. For such $\kappa$, the parameter $d$ in the equation (2.5) is chosen as $d = q^{\frac{1}{\kappa}}$. Set $t_n = t_n^{(N)} = d^{N-n}T$ $(n = 1, \cdots, N)$.

$A_2$ : A uniform partition is further made on $[0, t_1]$, and new nodes $t_{1,k} \in [0, t_1]$ are generated by $t_{1,k} = \frac{k \cdot t_1}{2p+1}$ $(k = 0, 1, \cdots, 2p+1)$.

Set

$$h := \max_{\substack{1 \leqslant k \leqslant 2p+1 \\ 2 \leqslant n \leqslant N}} \{h_{1,k},\ h_n\}.$$

It is easy to check that $h$ satisfies $h = h_N = T(1-d) \to 0$ as $N \to \infty$.

For ease of notation, we define the operator $K:\ C(J) \to C(J)$ by setting

$$Ky(t) := \int_0^t K_1(t,s)y(s)ds + \int_0^{qt} K_2(t,s)y(s)ds,\ t \in J,\quad \forall y \in C(J).$$

For the new triangulation (hybrid meshes) $J_N$, we define the sequence of collocation operators $Q_h : C(J) \to S_m^{(0)}(J_N)$ as follow: for $\forall y \in C(J)$, $Q_h y$ is the unique solution of the discrete system

$$(I - K)Q_h y(t) = f_y(t), \quad \forall t \in X(N), \tag{2.7}$$

where $f_y = (I-K)y$ and $I$ is the identity operator. With the collocation operator $Q_h$, we have $u_h = Q_h u$ (compare a similar relation given in Subsection 2.2).

Let $N'$ be chosen as $N' = [\frac{N-1}{2p+1}] + 1$, and set $\tilde{N}' = (2p+1)(N'-2)$. Let $J$ be divided into $N'$ subinterval $\{\sigma_r\}$ such that each $\sigma_r$ $(r = 1, 2, \cdots, N'-1)$ contains $2p+2$ points in $Z_N$, and $\sigma_{N'}$ contains $N - \tilde{N}'$ points in $Z_N$. Then,

$$\sigma_1 = [t_0, t_1], \quad \sigma_r = [t_{(2p+1)(r-2)+1}, t_{(2p+1)(r-1)+1}] \ \ (2 \le r \le N') \ \text{ and } \ \sigma_{N'} = [t_{\tilde{N}'+1}, t_N].$$

It is easy to see that $N - \tilde{N}' \geqslant 2p + 2$.

Define

$$S(p, Z_N) = \{v \in C(J) : v|_{\sigma_r} \in P_{2p+1}(\sigma_r), r = 1, \cdots, N' - 1, v|_{\sigma_{N'}} \in P_{N-\tilde{N}'-1}(\sigma_{N'})\}.$$

Let $\overline{\pi} : C(J) \to S(p, Z_N)$ denote the sequence of the high order interpolation operators such that $\overline{\pi} y(t) = y(t)$ for $t \in Z_N$ and $y \in C(J)$.

In this paper the collocation parameters $\{c_j\}_{j=1}^{m+1}$ are chosen as the $m+1$ Lobatto points on $[0, 1]$. Let $k$ be a nonnegative integer. For the higher order interpolation operators $\overline{\pi}$ introduced above, define $k$ *level corrected collocation solution* of the equation (2.2)

$$u_{h,k} = (-1)^k \sum_{j=0}^{k} (-1)^j C_{k+1}^j (\overline{\pi} Q_h)^{k-j} \overline{\pi} Q_h u = (-1)^k \sum_{j=0}^{k} (-1)^j C_{k+1}^j (\overline{\pi} Q_h)^{k-j} \overline{\pi} u_h. \quad (2.8)$$

Note that, when $k = 0$, we have $u_{h,k} = \overline{\pi} Q_h u = \overline{\pi} u_h$.

The approximation $u_{h,k}$ can be regarded as a proper linear combination of the functions

$$u_{h,k}^j = (\overline{\pi} Q_h)^{k-j} \overline{\pi} Q_h u \quad (j = 0, 1, \cdots, k).$$

The $j$th approximation $u_{h,k}^j$ can be obtained by the following steps: 1) Obtaining the collocation solution $u_h = Q_h u$ defined by the system (2.6); 2) Acting $\overline{\pi}$ on $Q_h u$ to get the interpolation approximation; 3) Acting $\overline{\pi} Q_h$ on $\overline{\pi} Q_h u$ for $k - j$ times repeatedly.

**Remark 2.2.** *The reason why the collocation parameters are chosen as Lobatto points rather than Gauss points is that the high order interpolation operator $\overline{\pi}$ is defined on the nodes $Z_N$. This means that $\overline{\pi} u_h(t)$ should be determined uniquely by the values of $u_h(t)$ on $Z_N$, so the collocation solution $u_h(t)$ must be continuous on the nodes.*

As usual, let $C$ denote a generic constant independent of the meshes $J_N$, which may has different values at different places.

The following result gives a superconvergence of the multilevel correction approximation $u_{h,k}$.

**Theorem 2.1.** *Let $m \leqslant 2$, and let the meshes $J_N$ be defined by the assumptions $A_1$ and $A_2$. Assume that the functions $f$, $K_i$ $(i = 1, 2)$ have the smoothness $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t, s) :\ 0 \leqslant s \leqslant t \leqslant T\}$ and $\Omega_2 := \{(t, s) :\ 0 \leqslant s \leqslant qt,\ t \in J\}$. Let $k$ be a nonnegative integer satisfying $2m(k+1) \leqslant 2p+2$. Then the $k$ level corrected collocation solution $u_{h,k}$ possesses the superconvergence*

$$\|u_{h,k} - u\|_{0,\infty,J} \leqslant C N^{-(2(k+1)m - \varepsilon_N)}, \ N \to \infty, \quad (2.9)$$

*where $\varepsilon_N$ is an arbitrarily small positive number satisfying* $\lim_{N\to\infty} \varepsilon_N = 0$.

**Remark 2.3.** *Theorem 2.1 indicates that the multilevel correction approximation* $u_{h,k}$ *possesses very high accuracy, even if both* $m$ *and* $k$ *are small, for example,* $m = 1$ *and* $k = 2$.

# 3 Auxiliary Results

The proof of Theorem 2.1 is a bit technical. To give the proof, we first, in the section, investigate some properties of the collocation method and the high order interpolation operator based on hybrid meshes.

## 3.1 Collocation method based on hybrid meshes

In this subsection, we give some properties associated with the collocation method based on hybrid meshes. The following three Lemmas can be verified as in [13].

**Lemma 3.1.** *Assume that* $A_1$ *holds. Let* $k$ *denote any positive integer. Then, for* $N \geqslant 2$*:*

*(i)*

$$h_1 \leqslant CN^{-\frac{2p+2}{m+2}}; \tag{3.1}$$

*(ii)*

$$\sum_{n=2}^{N}(h_n)^{k+1} \leqslant CN^{-(k-\varepsilon_{N,k})}, \tag{3.2}$$

*with*

$$\varepsilon_{N,k} := log_N\left(\frac{((2p+2)(lnN)^2)^k}{(m+2)^k}\right).$$

*Here,* $\varepsilon_{N,k}$ *is an arbitrarily small positive number satisfying* $\lim_{N\to\infty}\varepsilon_{N,k} = 0$.

□

**Lemma 3.2.** *For* $\kappa + 1 \leqslant n \leqslant N$*, we have* $qt_n = t_{n-\kappa} \in Z_N$*. Here,* $\kappa$ *is defined in the assumption* $A_1$.

□

**Lemma 3.3.** *Let* $A_1$ *and* $A_2$ *hold. Assume that the functions* $f$ *and* $K_i$ $(i = 1, 2)$ *satisfy* $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, *where* $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t, s) : \ 0 \leqslant s \leqslant t \leqslant T\}$ *and* $\Omega_2 := \{(t, s) : \ 0 \leqslant s \leqslant qt, \ t \in J\}$. *Then*

$$\|u_h - u\|_{0,\infty,J} \leq Ch^m \|u\|_{m,\infty,J} \tag{3.3}$$

□

Let $\mathcal{E}_{N,p}$ denote the set of the elements $e_{1,k}$ $(1 \leq k \leq 2p + 1)$ and $e_n$ $(1 \leq n \leq N)$. In the rest of this paper, we always use $\sigma$ to denote any element in $\mathcal{E}_{N,p}$.

Let $\pi_h : C(J) \to S_m^{(0)}(J_N)$ denote the sequence of interpolation operators such that

$$\pi_h v(t_{1,k,j}) = v(t_{1,k,j}) \quad \text{and} \quad \pi_h v(t_{n,j}) = v(t_{n,j}), \quad \forall v \in C(J).$$

$$(k = 1, \cdots, 2p+1;\ n = 2, \cdots, N;\ j = 1, \cdots, m+1)$$

It is well known that the following inequalities hold for each element $\sigma$

$$\|\pi_h v\|_{0,\infty,\sigma} \leqslant C\|v\|_{0,\infty,\sigma}, \quad \forall v \in C(J), \tag{3.4}$$

and

$$\|(\pi_h - I)v\|_{j,\infty,\sigma} \leqslant Ch_\sigma^{k-j}\|v\|_{k,\infty,\sigma}, \quad 0 \leqslant j \leqslant k \leqslant m, \tag{3.5}$$

where $h_\sigma := means(\sigma)$ denotes the length of interval $\sigma$.

The following two Lemmas are the standard results in the superconvergence theory of integral equations. They can be proved by the method in [18] and [22].

**Lemma 3.4.** *For $1 \leqslant k \leqslant 2m$, assume that $\psi \in C^m(\sigma)$ and $\varphi \in C^k(\sigma)$. If the collocation parameters $\{c_j\}_{j=1}^{m+1}$ are chosen as the $m+1$ Lobatto points in $[0,1]$, then the following estimate is valid for each element $\sigma$*

$$\left| \int_\sigma \psi(t)(\pi_h - I)\varphi(t)dt \right| \leqslant Ch_\sigma^{k+1}\|\psi\|_{m,\infty,\sigma} \cdot \|\varphi\|_{k,\infty,\sigma}. \tag{3.6}$$

□

**Lemma 3.5.** *Assume that $p$ and $m$ are two non-negative integers such that $m \leqslant p$. Let $\varphi \in C^{2p+2}(\sigma)$ and $\psi \in C^{2p+2-m}(\sigma)$. Then we have for each element $\sigma$*

$$\int_\sigma (\pi_h - I)\varphi \cdot \psi dt = \sum_{j=m}^{p} h_\sigma^{2j} \sum_{i=m+1}^{2j} C_{i,j} \int_\sigma D_t^{2j-i}(D_t^i\varphi \cdot \psi)dt + O(h_\sigma^{2p+3}), \tag{3.7}$$

$$\begin{aligned}\int_\sigma D_t^\alpha(\pi_h - I)\varphi \cdot \psi dt \ &= \ \sum_{r=1}^{\alpha}\sum_{j=\alpha_1}^{\alpha_2} h_\sigma^{2j} \sum_{i=m+1}^{2j+r-1} C_{i,j,r} \int_\sigma D_t^{2j+r-i}(D_t^{i+\alpha-r}\varphi \cdot \psi)dt \\ &+ \ \sum_{j=m}^{\alpha_2} h_\sigma^{2j} \sum_{i=m+1}^{2j} C_{i,j} \int_\sigma D_t^{2j-i}(D_t^{i+\alpha}\varphi \cdot \psi)dt + O(h_\sigma^{2\alpha_2+3}). \end{aligned} \tag{3.8}$$

*Here, $1 \leqslant \alpha \leqslant m$, $\alpha_1 = [(m-r+2)/2]$, $\alpha_2 = [p - \alpha/2]$; $C_{i,j}$, $C_{i,j,r}$ are constants independent of the meshes $J_N$. $D_t$ denotes the differential operator.*

□

By Lemma 3.4, we can obtain an almost optimal superconvergence property of $Ke(t)$ at hybrid mesh points.

**Theorem 3.1.** *Let $A_1$ and $A_2$ hold. Assume that the functions $f$ and $K_i$ $(i = 1, 2)$ satisfy $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t, s) :\ 0 \leqslant s \leqslant t \leqslant T\}$ and $\Omega_2 := \{(t, s) :\ 0 \leqslant s \leqslant qt,\ t \in J\}$. If $u_h \in S_m^{(0)}(J_N)$ denotes the collocation approximation determined by the equation (2.6), then the resulting error $e := u_h - u$ satisfies*

$$\max_{t \in Z_N} |Ke(t)| \leqslant CN^{-(k-\varepsilon_{N,k})} \|u\|_{k,\infty,J},\ N \to \infty, \quad \forall 1 \leqslant k \leqslant 2m, \tag{3.9}$$

*where $\varepsilon_{N,k}$ is an arbitrarily small positive number, which satisfies $\lim_{N\to\infty} \varepsilon_{N,k} = 0$.*

*Proof.* Since $u_h \in S_m^{(0)}(J_N)$, it follows by the definition of $\pi_h$ that $\pi_h u_h = u_h$. The equalities (2.2) and (2.6) may be written in the operator form as

$$u = Ku + f, \tag{3.10}$$

and

$$u_h = \pi_h K u_h + \pi_h f. \tag{3.11}$$

Let $e := u_h - u$. Subtraction of (3.10) from (3.11) leads to

$$e = \pi_h K e + (\pi_h - I)(Ku + f).$$

Hence, by observing (3.10), we can lead to

$$e = \pi_h K e + (\pi_h - I)u. \tag{3.12}$$

In the following, we prove the inequality (3.9) for two cases: $1 \leqslant k \leqslant m + 1$, and $m + 2 \leqslant k \leqslant 2m$. For the case with $1 \leqslant k \leqslant m + 1$, it follows by (3.12) that we have for any $\psi \in C^m(\sigma)$

$$|\int_\sigma \psi(s)e(s)ds| \leqslant C\int_\sigma |\psi(s)| \cdot |Ke(s)|ds + |\int_\sigma \psi(s)(I - \pi_h)u(s)ds| .$$

This, together with (3.3) and (3.6), yields

$$\begin{aligned} |\int_\sigma \psi(s)e(s)ds| &\leqslant C(h_\sigma^2 \|\psi\|_{0,\infty,\sigma} \cdot \|e\|_{0,\infty,\sigma} + h_\sigma^{k+1} \|\psi\|_{m,\infty,\sigma} \cdot \|u\|_{k,\infty,\sigma} \\ &\leqslant Ch_\sigma(h_\sigma^{k+1} \|\psi\|_{0,\infty,\sigma} \cdot \|u\|_{k,\infty,\sigma} + h_\sigma^k \|\psi\|_{m,\infty,\sigma} \cdot \|u\|_{k,\infty,\sigma}) \\ &\leqslant Ch_\sigma^{k+1}(h_\sigma \|\psi\|_{0,\infty,\sigma} + \|\psi\|_{m,\infty,\sigma}) \cdot \|u\|_{k,\infty,\sigma} \\ &\leqslant Ch_\sigma^{k+1} \|\psi\|_{m,\infty,\sigma} \|u\|_{k,\infty,\sigma}. \end{aligned} \tag{3.13}$$

It is easy to check that

$$\frac{N}{lnN} - \frac{N}{(lnN)^2} = \frac{N}{lnN}(1 - \frac{1}{lnN}) \to \infty, \quad N \to \infty.$$

For a sufficient large $N$, we have

$$\left[\frac{lnq}{ln(1 - \frac{(2p+2)(lnN)^2}{(m+2)N})}\right] + 1 < \left[\frac{lnq}{ln(1 - \frac{(2p+2)lnN}{(m+2)N})}\right] = \kappa.$$

Thus,

$$1-d=1-q^{\frac{1}{\kappa}}<\frac{(2p+2)(lnN)^2}{(m+2)N}. \tag{3.14}$$

From the assumption $A_1$, we can obtain

$$h_n=t_n-t_{n-1}=Td^{N-n}(1-d)\leqslant Cd^{N-n}\frac{(2p+2)(lnN)^2}{(m+2)N},\quad (n=2,\cdots,N).$$

Since $0<d<1$, we have

$$h_n^k\leqslant C\frac{(2p+2)^k(lnN)^{2k}}{(m+2)^k}N^{-(k)},\quad (n=2,\cdots,N). \tag{3.15}$$

Set

$$b:=\frac{(2p+2)^k(lnN)^{2k}}{(m+2)^k}. \tag{3.16}$$

By the inequality (3.15) and the identity $b=N^{log_N b}$, we can get

$$h_n^k\leqslant CN^{-(k-log_N b)},\ N\to\infty,\quad (n=2,\cdots,N). \tag{3.17}$$

For a given constant $k$, we have from the equation (3.16)

$$\varepsilon_{N,k}=log_N b=\frac{\ln b}{\ln N}\to 0,\ N\to\infty.$$

This, together with (3.17), leads to $(1\leqslant k\leqslant m+1)$

$$h_n^k\leqslant CN^{-(k-\varepsilon_{N,k})},\ N\to\infty,\ \ (n=2,\cdots,N).$$

It is obvious that $h_{1,l}^k\leqslant h_1^k\leqslant N^{-k}$ for any $1\leqslant l\leqslant 2p+1$. Thus, the inequality

$$h_\sigma^k\leqslant CN^{-(k-\varepsilon_{N,k})},\quad N\to\infty, \tag{3.18}$$

is valid for any $\sigma\in\{e_{1,1},e_{1,2},\cdots,e_{1,2p+1},e_2,e_3,\cdots,e_N\}$. This, together with (3.13), yields

$$|\int_\sigma\psi(s)e(s)ds\ |\leqslant Ch_\sigma(N^{-(k-\varepsilon_{N,k})}\|\psi\|_{m,\infty,\sigma})\|u\|_{k,\infty,\sigma},\ N\to\infty. \tag{3.19}$$

As to the case with $m+2\leqslant k\leqslant 2m$, it can be proved (refer to [13]) that the following inequality is valid for any function $\psi\in C^m(\sigma)$

$$|\int_\sigma\psi(s)e(s)ds\ |\leqslant Ch_\sigma(N^{-(k-\varepsilon_{N,k})}\|\psi\|_{0,\infty,\sigma}+h_\sigma^k\|\psi\|_{m,\infty,\sigma})\|u\|_{k,\infty,\sigma}. \tag{3.20}$$

Now we are ready to prove Theorem 3.1. By (3.19), (3.20) and Lemma 3.1, we obtain

$$|\int_0^{t_n}\psi(s)e(s)ds\ |\leqslant CN^{-(k-\varepsilon_{N,k})}\|\psi\|_{m,\infty,[0,t_n]}\|u\|_{k,\infty,[0,t_n]},\ \forall\psi\in C^m[0,t_n],\ 1\leqslant n\leqslant N.$$

In particular, we find that

$$|\int_0^{t_n}K_1(t_n,s)e(s)ds\ |\leqslant CN^{-(k-\varepsilon_{N,k})}\|u\|_{k,\infty,[0,t_n]},\ 1\leqslant n\leqslant N, \tag{3.21}$$

and

$$| \int_0^{qt_n} K_2(t_n,s)e(s)ds | \leqslant CN^{-(k-\varepsilon_{N,k})}\|u\|_{k,\infty,[0,qt_n]}, \ \kappa+1 \leqslant n \leqslant N. \tag{3.22}$$

When $1 \leqslant n \leqslant \kappa$, we have $qt_n \leqslant t_1$. According to (3.5) and (3.1), we can lead to

$$\begin{aligned} | \int_0^{qt_n} K_2(t_n,s)e(s)ds | &\leqslant Ct_1 h_1^k \|u\|_{k,\infty,e_1} \\ &\leqslant Ch_1^{k+1}\|u\|_{k,\infty,e_1} \\ &\leqslant CN^{-k}\|u\|_{k,\infty,e_1}, \ 1 \leqslant n \leqslant \kappa. \end{aligned} \tag{3.23}$$

In a similar manner, we can prove

$$| \int_0^{t_{1,l}} K_1(t_{1,l},s)e(s)ds | \leqslant CN^{-k}\|u\|_{k,\infty,e_1}, \ 1 \leqslant l \leqslant 2p+1, \tag{3.24}$$

and

$$| \int_0^{qt_{1,l}} K_2(t_{1,l},s)e(s)ds | \leqslant CN^{-k}\|u\|_{k,\infty,e_1}, \ 1 \leqslant l \leqslant 2p+1. \tag{3.25}$$

These, together with (3.21), (3.22) and (3.23), give the desired result. □

The following result gives some properties of $\pi_h$

**Theorem 3.2.** *Let $A_1$ and $A_2$ hold. Assume that the functions $f$ and $K_i$ $(i = 1,2)$ satisfy $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t,s): \ 0 \leqslant s \leqslant t \leqslant T\}$ and $\Omega_2 := \{(t,s): \ 0 \leqslant s \leqslant qt, \ t \in J\}$. Let $\pi_h : C(J) \to S_m^{(0)}(J_N)$ denote the sequence of interpolation operators defined on $Z_N$.*

*1) When $2 \leqslant n \leqslant N$, the integral $K(\pi_h - I)u(t_n)$ has the following expansion*

$$K(\pi_h - I)u(t_n) = \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}), \tag{3.26}$$

*where $F_j^k \in C^{2p+2-2j}(J)(k = 1,\cdots,N)$. And there are functions $F_j \in C^{2p+2-2j}(J)$ $(j = m,\cdots,p)$ satisfying $F_j(t_n) = \sum_{k=1}^{n} F_j^k(t_n)$ $(n = 2,\cdots,N)$ and*

$$\|F_j\|_{\lambda,\infty,J} \leqslant C\|u\|_{\lambda+2j,\infty,J} \ \ (\lambda = 0,1,\cdots,2p+2-2j). \tag{3.27}$$

*2) When $1 \leqslant k \leqslant 2p+1$, the integral $K(\pi_h - I)u(t_{1,k})$ can be written as*

$$K(\pi_h - I)u(t_{1,k}) = O(N^{-(2p+2)}). \tag{3.28}$$

*Proof.* It is easy to see that $K(\pi_h - I)u(t)$ can be written as

$$K(\pi_h - I)u(t) = K_1(\pi_h - I)u(t) + K_2(\pi_h - I)u(t), \tag{3.29}$$

where

$$K_1(\pi_h - I)u(t) = \int_0^t K_1(t,s)(\pi_h - I)u(s)ds,$$

and

$$K_2(\pi_h - I)u(t) = \int_0^{qt} K_2(t,s)(\pi_h - I)u(s)ds.$$

Without loss of generality, we only need to verify that the second term in the right side of the equation (3.29) can be written as (3.26) or (3.28) at the mesh points.

When $n \geqslant \kappa + 2$, we have

$$\begin{aligned}\int_0^{qt_n} K_2(t_n,s)(\pi_h - I)u(s)ds &= \int_0^{t_{n-\kappa}} K_2(t_n,s)(\pi_h - I)u(s)ds \\ &= \sum_{k=2}^{n-\kappa} \int_{e_k} K_2(t_n,s)(\pi_h - I)u(s)ds \\ &\quad + \sum_{l=1}^{2p+1} \int_{e_{1,l}} K_2(t_n,s)(\pi_h - I)u(s)ds.\end{aligned} \tag{3.30}$$

By Lemma 3.5, we get the following equality

$$\begin{aligned}&\int_{e_k} K_2(t_n,s)(\pi_h - I)u(s)ds \\ = &\sum_{j=m}^{p} h_k^{2j} \sum_{i=m+1}^{2j} C_{i,j} \int_{e_k} D_s^{2j-i}(D_s^i u(s) K_2(t_n,s))ds + O(N^{-(2p+2)}) \\ = &\sum_{j=m}^{p} h_k^{2j} F_j^k(t_n) + O(N^{-(2p+2)}), \quad k = 2,3,\cdots,N,\end{aligned} \tag{3.31}$$

where

$$F_j^k(t) = \sum_{i=m+1}^{2j} C_{i,j} \int_{e_k} D_s^{2j-i}(D_s^i u(s) K_2(t,s))ds,\ k = 2,3,\cdots,N,\ j = m, m+1, \cdots, p.$$

Note that the constants $C_{i,j}$ are independent of the choice of $e_k$. Similarly, we have

$$\begin{aligned}\int_{e_{1,l}} K_2(t_n,s)(\pi_h - I)u(s)ds &= \sum_{j=m}^{p} h_{1,l}^{2j} F_j^{1,l}(t_n) + O(N^{-(2p+2)}) \\ &= \sum_{j=m}^{p} (\frac{h_1}{2p+1})^{2j} F_j^{1,l}(t_n) + O(N^{-(2p+2)}),\end{aligned} \tag{3.32}$$

where

$$F_j^{1,l}(t) = \sum_{i=m+1}^{2j} C_{i,j} \int_{e_{1,l}} D_s^{2j-i}(D_s^i u(s) K_2(t,s))ds, \quad l = 1,2,\cdots,2p+1.$$

Together with (3.30), (3.31) and (3.32), we can obtain

$$\begin{aligned}
&\int_0^{qt_n} K_2(t_n,s)(\pi_h - I)u(s)ds \\
= & \sum_{k=2}^{n-\kappa}\int_{e_k} K_2(t_n,s)(\pi_h - I)u(s)ds + \sum_{l=1}^{2p+1}\int_{e_{1,l}} K_2(t_n,s)(\pi_h - I)u(s)ds \\
= & \sum_{k=2}^{n-\kappa}\sum_{j=m}^{p} h_k^{2j}F_j^k(t_n) + \sum_{l=1}^{2p+1}\sum_{j=m}^{p}\frac{h_1}{2p+1}F_j^{1,l}(t_n) + O(N^{-(2p+2)}) \qquad (3.33)\\
= & \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}\sum_{l=1}^{2p+1}F_j^{1,l}(t_n) + \sum_{k=2}^{n-\kappa} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}) \\
= & \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^{1}(t_n) + \sum_{k=2}^{n-\kappa} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}),
\end{aligned}$$

where $F_j^1(t) = \sum_{l=1}^{2p+1} F_j^{1,l}(t)$. We can write $F_j^1(t)$ as

$$\begin{aligned}
F_j^1(t) &= \sum_{l=1}^{2p+1} F_j^{1,l}(t) \\
&= \sum_{l=1}^{2p+1}\sum_{i=m+1}^{2j} C_{i,j}\int_{e_{1,l}} D_s^{2j-i}(D_s^i u(s)K_2(t,s))ds \\
&= \sum_{i=m+1}^{2j} C_{i,j}\int_{e_1} D_s^{2j-i}(D_s^i u(s)K_2(t,s))ds.
\end{aligned}$$

Letting

$$F_j(t) = \sum_{i=m+1}^{2j} C_{i,j}\int_0^t D_s^{2j-i}(D_s^i u(s)K_2(t,s))ds,$$

we can obtain that $F_j \in C^{2p+2-2j}(J)$, $F_j^k \in C^{2p+2-2j}(J)$ $(k = 1, 2, \cdots, N; j = m, m+1, \cdots, p)$ and $F_j(t_n) = \sum_{k=1}^{n} F_j^k(t_n)$. By the definition of $F_j(t)$, it is obvious that $\|F_j\|_{\lambda,\infty,J} \leqslant C\|u\|_{\lambda+2j,\infty,J}$ $(\lambda = 0, 1, \cdots, 2p+2-2j)$.

When $n = \kappa + 1$, we have

$$\begin{aligned}
\int_0^{qt_n} K_2(t_n,s)(\pi_h - I)u(s)ds &= \int_0^{t_1} K_2(t_n,s)(\pi_h - I)u(s)ds \\
&= \sum_{l=1}^{2p+1}\int_{e_{1,l}} K_2(t_n,s)(\pi_h - I)u(s)ds.
\end{aligned}$$

It follows by (3.32) that

$$\int_0^{qt_{\kappa+1}} K_2(t_{\kappa+1},s)(\pi_h - I)u(s)ds = \sum_{j=m}^{p}(\frac{h_1}{2p+1})^{2j}F_j^1(t_{\kappa+1}) + O(N^{-(2p+2)}). \qquad (3.34)$$

When $t \in Z_N \cap \{t \mid t \leqslant t_\kappa\}$, we have $qt < t_1$. According to (3.1) and (3.5), we can get

$$\begin{aligned}
|\int_0^{qt} K_2(t,s)(\pi_h - I)u(s)ds| &\leqslant Ct_1h_1^{m+1}\|u\|_{m+1,\infty,e_1} \\
&\leqslant Ch_1^{m+2}\|u\|_{m+1,\infty,e_1} \qquad (3.35)\\
&\leqslant CN^{-(2p+2)}\|u\|_{m+1,\infty,e_1}.
\end{aligned}$$

This, together with (3.33) and (3.34), allows us to deduce that the following expansion is valid for $\kappa+1 \leqslant n \leqslant N$,

$$K_2(\pi_h - I)u(t_n) = \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n-\kappa} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}). \quad (3.36)$$

where $F_j^k \in C^{2p+2-2j}(J)(k=1,2,\cdots,N)$. Furthermore, by the definition of $F_j(t)$, we have $F_j(t_n) = \sum_{k=1}^{n} F_j^k(t_n)$ $(n=2,3,\cdots,N)$. From (3.34) and (3.35), we have

$$K_2(\pi_h - I)u(t_n) = O(N^{-(2p+2)}), \quad 1 \leqslant n \leqslant \kappa, \quad (3.37)$$

and

$$K_2(\pi_h - I)u(t_{1,k}) = O(N^{-(2p+2)}), \quad 1 \leqslant k \leqslant 2p+1. \quad (3.38)$$

By a similar manner, we are led to

$$K_1(\pi_h - I)u(t_n) = \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}), 2 \leqslant n \leqslant N,$$

$$K_1(\pi_h - I)u(t_{1,k}) = O(N^{-(2p+2)}), \quad 1 \leqslant k \leqslant 2p+1,$$

where $F_j^k \in C^{2p+2-2j}(J)(k=1,2,\cdots,N)$. Moreover, there are functions $F_j \in C^{2p+2-2j}(J)$ $(j=m,m+1,\cdots,p)$ such that $\|F_j\|_{\lambda,\infty,J} \leqslant C\|u\|_{\lambda+2j,\infty,J}$ $(\lambda=0,1,\cdots,2p+2-2j)$ and $F_j(t_n) = \sum_{k=1}^{n} F_j^k(t_n)$ $(n=2,3,\cdots,N)$.

Now, we have proved the theorem 3.2. □

**Remark 3.1.** *Theorem 3.2 is a version of Lemma 3.5. Both of them are the general form of the usual integral expansion (refer to [29]), while this theorem is helpful for us to consider the Volterra integral equation with proportional delays. The complete analysis to the multilevel correction will be based on this theorem.*

## 3.2 Error estimate of high order interpolation operator

In the subsection, we derive an error estimate of the high order interpolation operator. For the convergence, set

$$H_1 = t_1 - t_0, \quad H_{N'} = t_N - t_{\tilde{N}'+1}$$

and

$$H_r = t_{(2p+1)(r-1)+1} - t_{(2p+1)(r-2)+1} \ (2 \le r \le N'-1).$$

**Lemma 3.6.** *For a given positive integer $\mu$ satisfying $1 \leqslant \mu \leqslant 2p+2$, assume that $\forall y \in C^\mu(J)$. Let $\overline{\pi} : C(J) \to S(p, Z_N)$ be the sequence of the higher order interpolation operators. Then*

$$\|y - \overline{\pi}y\|_{k,\infty,\sigma_r} \leqslant CH_r^{\mu-k}\|y\|_{\mu,\infty,\sigma_r}. \quad (3.39)$$

*Here, $C$ is a constant independent of the meshes $J_N$, $0 \leqslant k \leqslant min\{\mu, 2p+1\}$ and $1 \leqslant r \leqslant N'$.*

*Proof.* Without loss generality, we only need to analysis $\overline{\pi}$ on a subinterval $\sigma_r$ ($r = 2, 3, \cdots, N'-1$). Let $n_r = (2p+1)(r-2)+1$. Then the restriction of $\overline{\pi}v$ on $\sigma_r$ is determined completely by the values of $v$ at the nodes $\{t_{n_r}, t_{n_r+1}, \cdots, t_{n_r+2p+1}\}$. We can regard $t_{(2p+1)(r-2)+1} = t_{n_r} < t_{n_r+1} < \cdots < t_{n_r+2p+1} = t_{(2p+1)(r-1)+1}$ as the partition of $\sigma_r$. Let $\mu$ be any given positive integer such that $1 \leqslant \mu \leqslant 2p+2$. For a function $y \in C^{\mu}(J)$, $\overline{\pi}y$ can be written as

$$\overline{\pi}y(t) = \sum_{j=0}^{2p+1} y(t_{n_r+j}) L_j^r(t), \quad \forall t \in \sigma_r,$$

where $L_j^r(t) = \prod_{l=0, l\neq j}^{2p+1} (t - t_{n_r+l})/(t_{n_r+j} - t_{n_r+l})$ is the $j$th Lagrange basic function on $\sigma_r$. It is obvious that

$$\sum_{j=0}^{2p+1} (t_{n_r+j})^k L_j^r(t) = t^k, \quad \forall\, t \in \sigma_r, \quad \forall\, 0 \leqslant k \leqslant 2p+1. \tag{3.40}$$

From the equality (3.40), we can deduce

$$\sum_{j=0}^{2p+1} (t - t_{n_r+j})^k L_j^r(t) = 0, \quad \forall\, t \in \sigma_r, \quad \forall\, 0 \leqslant k \leqslant 2p+1.$$

Noting $y \in C^{\mu}(\sigma_r)$, we can write $y(t) - y(t_{n_r+j})$ as Taylor expansion with integral remainder

$$\begin{aligned} y(t) - y(t_{n_r+j}) &= \int_{t_{n_r+j}}^{t} y'(s)ds \\ &= y'(s)(s - t_{n_r+j})|_{s=t_{n_r+j}}^{s=t} - \int_{t_{n_r+j}}^{t} y''(s)(s - t_{n_r+j})ds \\ &= y'(t)(t - t_{n_r+j}) + \cdots + \frac{(-1)^{\mu-2}}{(\mu-1)!} y^{(\mu-1)}(t)(t - t_{n_r+j})^{\mu-1} \\ &\quad + \frac{(-1)^{\mu-1}}{(\mu-1)!} \int_{t_{n_r+j}}^{t} y^{(\mu)}(s)(s - t_{n_r+j})^{\mu-1} ds. \end{aligned} \tag{3.41}$$

Both sides of the equality (3.41) are multiplied by $L_j^r(t)$ respectively, and they are summed over all $j$ ($j = 0, 1, \cdots, 2p+1$). The error can be written as

$$R(t) = y - \overline{\pi}y = \frac{(-1)^{\mu-1}}{(\mu-1)!} \sum_{j=0}^{2p+1} L_j^r(t) \int_{t_{n_r+j}}^{t} y^{(\mu)}(s)(s - t_{n_r+j})^{\mu-1} ds, \quad \forall\, t \in \sigma_r. \tag{3.42}$$

Thus, the derivative of $R(t)$ is

$$\begin{aligned} R'(t) &= \frac{(-1)^{\mu-1}}{(\mu-1)!} \{ \sum_{j=0}^{2p+1} D_t L_j^r(t) \int_{t_{n_r+j}}^{t} y^{(\mu)}(s)(s - t_{n_r+j})^{\mu-1} ds \\ &\quad + \sum_{j=0}^{2p+1} L_j^r(t) y^{(\mu)}(t)(t - t_{n_r+j})^{\mu-1} \}, \quad \forall\, t \in \sigma_r. \end{aligned} \tag{3.43}$$

Since $y^{(\mu)}(t)$ in the equation (3.43) is independent of $j$, the second term of the right side vanishes. By the same manipulation, we can deduce that the following equality is valid for any integer $k$ satisfying $0 \leqslant k \leqslant min\{\mu, 2p+1\}$,

$$D_t^k(y - \overline{\pi}y)(t) = \frac{(-1)^{\mu-1}}{(\mu-1)!} \sum_{j=0}^{2p+1} D_t^k L_j^r(t) \int_{t_{n_r+j}}^{t} y^{(\mu)}(s)(s - t_{n_r+j})^{\mu-1} ds. \tag{3.44}$$

Therefore

$$\begin{aligned} |D_t^k(y - \overline{\pi}y)(t)| &\leqslant \frac{1}{(\mu-1)!} \Big| \sum_{j=0}^{2p+1} D_t^k L_j^r(t) \int_{t_{n_r+j}}^{t} y^{(\mu)}(s)(s - t_{n_r+j})^{\mu-1} ds \Big| \\ &\leqslant \frac{1}{\mu!} \|y\|_{\mu,\infty,\sigma_r} \sum_{j=0}^{2p+1} |D_t^k L_j^r(t)(t - t_{n_r+j})^{\mu}| \\ &\leqslant \frac{1}{\mu!} H_r^{\mu} \|y\|_{\mu,\infty,\sigma_r} \sum_{j=0}^{2p+1} |D_t^k L_j^r(t)|. \end{aligned} \tag{3.45}$$

For $2 \leqslant r \leqslant N' - 1$, the $j$th Lagrange basic function on $\sigma_r$ is

$$L_j^r(t) = \prod_{\substack{l=0 \\ l \neq j}}^{2p+1} \frac{t - t_{(2p+1)(r-2)+1+l}}{t_{(2p+1)(r-2)+1+j} - t_{(2p+1)(r-2)+1+l}}, \quad j = 0, 1, \cdots, 2p+1.$$

Since the mesh on $\sigma_r$ is geometric, we have

$$\frac{t_{(2p+1)(r-1)+1} - t_{(2p+1)(r-1)}}{t_{(2p+1)(r-2)+2} - t_{(2p+1)(r-2)+1}} = \frac{d^{N-[(2p+1)(r-1)+1]}(1-d)T}{d^{N-[(2p+1)(r-2)+2]}(1-d)T} = d^{-2p}.$$

By the definition of $d$, there is a positive number $\delta$ such that $0 < \delta < d < 1$. Therefore

$$1 \leqslant \frac{t_{(2p+1)(r-1)+1} - t_{(2p+1)(r-1)}}{t_{(2p+1)(r-2)+2} - t_{(2p+1)(r-2)+1}} \leqslant \delta^{-2p}. \tag{3.46}$$

Because of the inequality (3.46), we can find a constant $C$ which is independent of $J_N$ such that

$$|L_j^r(t)| \leqslant C, \quad |D_t^k L_j^r(t)| \leqslant C/H_r^k, \ \forall t \in \sigma_r, \ 0 \leqslant k \leqslant 2p+1, \ 0 \leqslant j \leqslant 2p+1. \tag{3.47}$$

It follows by (3.45) and (3.47) that for $\forall t \in \sigma_r$,

$$\begin{aligned} |D_t^k(y - \overline{\pi}y)(t)| &\leqslant \frac{1}{\mu!} H_i^{\mu} \|y\|_{\mu,\infty,\sigma_r} \sum_{j=0}^{2p+1} |D_t^k L_j^r(t)|. \\ &\leqslant C H_r^{\mu} H_r^{-k} \|y\|_{\mu,\infty,\sigma_r}. \\ &\leqslant C H_r^{\mu-k} \|y\|_{\mu,\infty,\sigma_r}, \end{aligned} \tag{3.48}$$

where $0 \leqslant k \leqslant min\{\mu, 2p+1\}$ and $2 \leqslant r \leqslant N' - 1$.

In an analogous way with above, we can prove that the inequality (3.39) is valid for $r = 1$ and $r = N'$. □

By Lemma 3.6, we can get an error estimate of the interpolation $\overline{\pi}$ on $J$.

**Theorem 3.3.** *For a given positive integer $\mu$ satisfying $1 \leqslant \mu \leqslant 2p+2$, assume that $\forall y \in C^{\mu}(J)$. Let $\overline{\pi} : C(J) \to S(p, Z_N)$ be the sequence of the higher order interpolation operators. Then*

$$\|y - \overline{\pi}y\|_{k,\infty,J} \leqslant CN^{-(\mu-k-\varepsilon_{N,\mu-k})}\|y\|_{\mu,\infty,J}, \ N \to \infty. \tag{3.49}$$

*Here, $\varepsilon_{N,\mu-k}$ is an arbitrarily small positive number, which satisfies $\lim\limits_{N\to\infty} \varepsilon_{N,\mu-k} = 0$, and $0 \leqslant k \leqslant min\{\mu, 2p+1\}$.*

*Proof.* By Lemma 3.6, we only need to prove $H_r^{\lambda} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})}$ for any positive integer $\lambda$.

For the case of $r = 1$, we have $H_1^{\lambda} = (\sum_{l=1}^{2p+1} h_{1,l})^{\lambda} = h_1^{\lambda}$. Since $2p+2 > m+2$, it follows by (3.1) that

$$H_1^{\lambda} = h_1^{\lambda} \leqslant CN^{-\frac{2p+2}{m+2}\cdot\lambda} \leqslant CN^{-\lambda}. \tag{3.50}$$

When $2 \leqslant r \leqslant N' - 1$, we get

$$\begin{aligned} H_r^{\lambda} &= (\sum_{k=1}^{2p+1} h_{(2p+1)(r-2)+1+k})^{\lambda} \\ &\leqslant C\sum_{k=1}^{2p+1} h^{\lambda}_{(2p+1)(r-2)+1+k}. \end{aligned}$$

From the assumption $A_1$ and the inequality (3.14), we obtain

$$\begin{aligned} h_{(2p+1)(r-2)+1+k} &= t_{(2p+1)(r-2)+1+k} - t_{(2p+1)(r-2)+k} \\ &= Td^{N-((2p+1)(r-2)+1+k)}(1-d) \\ &\leqslant Cd^{N-((2p+1)(r-2)+1+k)}\frac{(2p+2)(lnN)^2}{(m+2)N}. \end{aligned}$$

Thus

$$\begin{aligned} H_r^{\lambda} &\leqslant C\sum_{k=1}^{2p+1} h^{\lambda}_{(2p+1)(r-2)+1+k} \\ &\leqslant C\frac{(1-d^{2p+1})((2p+2)(lnN)^2)^{\lambda}}{(1-d^{\lambda})(m+2)^{\lambda}}N^{-\lambda}, \quad N \to \infty. \end{aligned} \tag{3.51}$$

Because of the fact that $d \to 1$ as $N \to \infty$, we have

$$\lim_{N\to\infty}\frac{1-d^{2p+1}}{1-d^{\lambda}} = \lim_{d\to 1}\frac{1-d^{2p+1}}{1-d^{\lambda}} = \frac{2p+1}{\lambda}. \tag{3.52}$$

Since $0 < d < 1$, it follows by (3.52) that there is a constant $C$ independent of the meshes $J_N$ such that

$$\frac{1-d^{2p+1}}{1-d^{\lambda}} \leqslant C.$$

By the inequality (3.51), we can lead to

$$H_r^\lambda \leqslant C\frac{((2p+2)(lnN)^2)^\lambda}{(m+2)^\lambda}N^{-\lambda}, N \to \infty. \tag{3.53}$$

Set

$$\widetilde{b} := \frac{((2p+2)(lnN)^2)^\lambda}{(m+2)^\lambda}.$$

By the identity $\widetilde{b} = N^{log_N \widetilde{b}}$, the inequality (3.53) can be written as

$$H_r^\lambda \leqslant CN^{-(\lambda - log_N \widetilde{b})},\ N \to \infty. \tag{3.54}$$

For a given constant $\lambda$, we have

$$\varepsilon_{N,\lambda} = log_N \widetilde{b} = \frac{\ln \widetilde{b}}{\ln N} \to 0,\ N \to \infty.$$

This, together with the inequality (3.54), yields

$$H_r^\lambda \leqslant CN^{-(\lambda - \varepsilon_{N,\lambda})},\ N \to \infty,\ 2 \leqslant r \leqslant N' - 1. \tag{3.55}$$

Similarly, we can prove that

$$H_{N'}^\lambda \leqslant CN^{-(\lambda - \varepsilon_{N,\lambda})},\ N \to \infty.$$

This, together with (3.50), (3.55) and (3.39), yields (3.49). □

## 3.3 Multilevel correction for collocation solution on hybrid meshes

In the subsection, we derive an estimate of $\|[\overline{\pi}(Q_h - I)]^{k+1}u\|_{0,\infty,J}$.

It follows by (3.12) that

$$\begin{aligned} e &= \pi_h Ke + (\pi_h - I)u \\ &= Ke + (\pi_h - I)(Ke + u) \\ &= K_1 e + K_2 e + \pi_h Ke + (\pi_h - I)(Ke + u) \\ &= K_1 e + \widetilde{A}, \end{aligned}$$

where

$$\widetilde{A} = K_2 e + \pi_h Ke + (\pi_h - I)(Ke + u).$$

The standard Volterra theory implies that the resolvent kernel $R_1$ of $K_1$ inherits the smoothness of the Kernel $K_1$ and satisfies

$$e = \widetilde{A} + R\widetilde{A},$$

where

$$R\widetilde{A} = \int_0^t R_1(t,s)\widetilde{A}(s)ds.$$

Furthermore, we have

$$\begin{aligned} e &= K_2e + (\pi_h - I)(Ke + u) + R[K_2e + (\pi_h - I)(Ke + u)] \\ &= (I + R)K_2e + (I + R)(\pi_h - I)(Ke + u) \\ &= (I + R)(\pi_h - I)u + [(I + R)K_2 + (I + R)(\pi_h - I)K]e. \end{aligned} \tag{3.56}$$

Let $A = (I + R)(\pi_h - I)$ and $B = B_1 + B_2$ with

$$B_1 = (I + R)K_2, \quad B_2 = (I + R)(\pi_h - I)K. \tag{3.57}$$

It is clear that the equality (3.56) can be simplified as

$$\begin{aligned} e &= Au + Be \\ &= Au + BAu + B^2e \\ &= \sum_{i=0}^{l-1} B^iAu + B^le, \end{aligned} \tag{3.58}$$

where $l$ is any positive integer.

**Lemma 3.7.** *Let the assumptions $A_1$ and $A_2$ hold. Assume that $m \leqslant 2$ and the functions $f$, $K_i$ $(i = 1, 2)$ satisfy $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t, s) : 0 \leqslant s \leqslant t \leqslant T\}$ and $\Omega_2 := \{(t, s) :\ 0 \leqslant s \leqslant qt,\ t \in J\}$. Let $i$ denote any nonnegative integer.*

*1) When $2 \leqslant n \leqslant N$, the term $B^iAu(t_n)$ possesses the following expansion*

$$B^iAu(t_n) = \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}), \tag{3.59}$$

*where $F_j^k \in C^{2p+2-2j}(J)(k = 1, \cdots, N)$. And there are functions $F_j \in C^{2p+2-2j}(J)$ $(j = m, \cdots, p)$ satisfying $F_j(t_n) = \sum_{k=1}^{n} F_j^k(t_n)$ $(n = 2, \cdots, N)$ and*

$$\|F_j\|_{\lambda,\infty,J} \leqslant C\|u\|_{\lambda+2j,\infty,J} \ \ (\lambda = 0, 1, \cdots, 2p + 2 - 2j). \tag{3.60}$$

*2) When $1 \leqslant k \leqslant 2p + 1$, the term $B^iAu(t_{1,k})$ can be written as*

$$B^iAu(t_{1,k}) = O(N^{-(2p+2)}). \tag{3.61}$$

*Proof.* When $i = 0$, by a similar way in the proof of Theorem 3.2, we can deduce the following equalities

$$R(\pi_h - I)u(t_n) = \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}), 2 \leqslant n \leqslant N$$

and

$$R(\pi_h - I)u(t_{1,k}) = O(N^{-(2p+2)}), \quad 1 \leqslant k \leqslant 2p + 1,$$

where $F_j^k \in C^{2p+2-2j}(J)(k=1,\cdots,N)$. Moreover, there are functions $F_j \in C^{2p+2-2j}(J)$ $(j=m,m+1,\cdots,p)$ such that $\|F_j\|_{\lambda,\infty,J} \leqslant C\|u\|_{\lambda+2j,\infty,J}$ $(\lambda=0,1,\cdots,2p+2-2j)$ and $F_j(t_n)=\sum_{k=1}^{n} F_j^k(t_n)$ $(n=2,3,\cdots,N)$. Thus, by the definition of $A$ and $\pi_h$, we can obtain

$$\begin{aligned} Au(t_n) &= R(\pi_h - I)u(t) \mid_{t=t_n} + (\pi_h - I)u(t) \mid_{t=t_n} \\ &= R(\pi_h - I)u(t_n) \\ &= \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n} h_k^{2j}F_j^k(t_n)] + O(N^{-(2p+2)}),\ 2 \leqslant n \leqslant N \end{aligned}$$

and

$$\begin{aligned} Au(t_{1,k}) &= R(\pi_h - I)u(t) \mid_{t=t_{1,k}} + (\pi_h - I)u(t) \mid_{t=t_{1,k}} \\ &= R(\pi_h - I)u(t_{1,k}) \\ &= O(N^{-(2p+2)}), \quad 1 \leqslant k \leqslant 2p+1. \end{aligned}$$

When $i=1$, we first verify that $B_1Au$ can be written as (3.59) or (3.61) at the mesh points. In fact, changing the order of the integration, leads to

$$\begin{aligned} K_1R(\pi_h - I)u(t) &= \int_0^t K_1(t,x)[\int_0^x R_1(x,s)(\pi_h - I)u(s)ds]dx \\ &= \int_0^t [\int_s^t K_1(t,x)R_1(x,s)dx](\pi_h - I)u(s)ds \\ &= \int_0^t \overline{K_1}(t,s)(\pi_h - I)u(s)ds, \end{aligned} \tag{3.62}$$

where $\overline{K_1}(t,s) = \int_s^t K_1(t,x)R_1(x,s)dx$. In the same way, we have

$$K_2R(\pi_h - I)u(t) = \int_0^{qt} \overline{K_2R}(t,s)(\pi_h - I)u(s)ds, \tag{3.63}$$

$$RK_2(\pi_h - I)u(t) = \int_0^{qt} \overline{RK_2}(t,s)(\pi_h - I)u(s)ds, \tag{3.64}$$

and

$$RK_2R(\pi_h - I)u(t) = \int_0^{qt} \overline{RK_2R}(t,s)(\pi_h - I)u(s)ds, \tag{3.65}$$

where

$$\overline{K_2R}(t,s) = \int_s^{qt} K_2(t,x)R_1(x,s)dx, \quad \overline{RK_2}(t,s) = \int_{s/q}^{t} K_2(t,x)R_1(x,s)dx,$$

and

$$\overline{RK_2R}(t,s) = \int_{s/q}^{t} R_1(t,x_1)[\int_s^{qx_1} K_2(x_1,x_2)R_1(x_2,s)dx_2]dx_1.$$

By the definition of $A$ and $B_1$, we can write $B_1Au$ as

$$\begin{aligned}B_1Au(t) &= (I+R)K_2(I+R)(\pi_h-I)u \\ &= K_2(\pi_h-I)u+K_2R(\pi_h-I)u \\ &\quad +RK_2(\pi_h-I)u+RK_2R(\pi_h-I)u \\ &= \int_0^{qt}\widetilde{K_2}(t,s)(\pi_h-I)u(s)ds,\end{aligned}$$

where

$$\widetilde{K_2}(t,s)=K_2(t,s)+\overline{K_2R}(t,s)+\overline{RK_2}(t,s)+\overline{RK_2R}(t,s).$$

Furthermore, as in the proof of Theorem 3.2, we can deduce that $B_1Au$ can be written as (3.59) or (3.61) at the mesh points.

Secondly, we verify that $B_2Au$ can be written as (3.59) or (3.61) at the mesh points. By the equalities (3.7) and (3.8), we can prove that $R(\pi_h-I)K(\pi_h-I)u$ and $R(\pi_h-I)KR(\pi_h-I)u$ have the expansion (3.26) or (3.28) at the mesh points in an analogous way with the proof of Theorem 3.2. By the definition of $A$ and $B_2$, we obtain

$$\begin{aligned}B_2Au(t_n) &= (I+R)(\pi_h-I)K(I+R)(\pi_h-I)u(t)|_{t=t_n} \\ &= R(\pi_h-I)K(I+R)(\pi_h-I)u(t)|_{t=t_n}+(\pi_h-I)K(I+R)(\pi_h-I)u(t)|_{t=t_n} \\ &= R(\pi_h-I)K(\pi_h-I)u(t)|_{t=t_n}+R(\pi_h-I)KR(\pi_h-I)u(t)|_{t=t_n} \\ &= \sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n)+\sum_{k=2}^{n}h_k^{2j}F_j^k(t_n)]+O(N^{-(2p+2)}),\ 2\leqslant n\leqslant N\end{aligned}$$

and

$$\begin{aligned}B_2Au(t_{1,k}) &= (I+R)(\pi_h-I)K(I+R)(\pi_h-I)u(t)|_{t=t_{1,k}} \\ &= R(\pi_h-I)K(I+R)(\pi_h-I)u(t)|_{t=t_{1,k}} \\ &+ (\pi_h-I)K(I+R)(\pi_h-I)u(t)|_{t=t_{1,k}} \\ &= R(\pi_h-I)K(\pi_h-I)u(t)|_{t=t_{1,k}}+R(\pi_h-I)KR(\pi_h-I)u(t)|_{t=t_{1,k}} \\ &= O(N^{-(2p+2)}),\ 1\leqslant k\leqslant 2p+1,\end{aligned}$$

where $F_j^k\in C^{2p+2-2j}(J)(k=1,\cdots,N)$. Moreover, there are functions $F_j\in C^{2p+2-2j}(J)$ $(j=m,m+1,\cdots,p)$ such that $\|F_j\|_{\lambda,\infty,J}\leqslant C\|u\|_{\lambda+2j,\infty,J}$ $(\lambda=0,1,\cdots,2p+2-2j)$ and $F_j(t_n)=\sum_{k=1}^{n}F_j^k(t_n)$ $(n=2,3,\cdots,N)$. Since $B=B_1+B_2$, we deduce that $BAu$ can be written as (3.59) or (3.61) at the mesh points.

When $i\geqslant 2$, we can prove that $B^iAu$ can be written as (3.59) or (3.61) at the mesh points by the same manipulation with above. □

**Remark 3.2.** *When we prove that $R(\pi_h-I)K(\pi_h-I)u$ and $R(\pi_h-I)KR(\pi_h-I)u$ can be written as (3.26) or (3.28) at the mesh points, we need to use the condition that $m\leqslant 2$, which seems necessary. A similar situation has appeared in [25].*

Set $l = (2p+3)(2p+m+3)$ in the equality (i.e., (3.58))

$$e = \sum_{i=1}^{l-1} B^i Au + B^l e.$$

Let $B_1$ and $B_2$ be defined by (3.57). It is clear that $B^l e$ can be written as

$$B^l e = (B_1 + B_2)^l e = \sum_{i=0}^{l} B_{l-i,i} e,$$

where $B_{l-i,i}$ is the sum of $C_l^i$ terms, with each term being a product between $l-i$ operators $B_1$ and $i$ operators $B_2$, which have different order in any two terms. We need to estimate the norm $\|B_{l-i,i}e\|_{0,\infty,J}$ for each $i$.

We first consider the case with $2p+2 \leqslant i \leqslant l$. As usual, we define the norms

$$\|B_k\|_{C(J)\to C(J)} := \sup_{\substack{\forall v \in C(J) \\ \|v\|_{0,\infty,J} \neq 0}} \frac{\|B_k v\|_{0,\infty,J}}{\|v\|_{0,\infty,J}}, \quad (k = 1, 2),$$

and

$$\|B_k\|_{C^1(\sigma)\to C^1(\sigma)} := \sup_{\substack{\forall v \in C(J)\cap C^1(\sigma) \\ \|v\|_{1,\infty,\sigma} \neq 0}} \frac{\|B_k v\|_{1,\infty,\sigma}}{\|v\|_{1,\infty,\sigma}}, \quad (k = 1, 2).$$

**Lemma 3.8.** *Let $A_1$ and $A_2$ hold. Assume that the functions $f$ and $K_i$ $(i = 1, 2)$ satisfy $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t, s) : \ 0 \leqslant s \leqslant t \leqslant T\}$ and $\Omega_2 := \{(t, s) : \ 0 \leqslant s \leqslant qt, \ t \in J\}$. Let $i$ and $l$ denote two given positive integers such that $2p+2 \leqslant i \leqslant l$. Then*

$$\|B_{l-i,i}e\|_{0,\infty,J} \leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})}\|u\|_{1,\infty,J}, \tag{3.66}$$

*where $\varepsilon_{N,2p+2}$ is an arbitrarily small positive number, which satisfies $\lim_{N\to\infty} \varepsilon_{N,2p+2} = 0$.*

*Proof.* By the definition of the operator $K_2$, we can deduce that

$$\|K_2 v\|_{2,\infty,\sigma} \leqslant C\|v\|_{1,\infty,\sigma}, \quad \forall v \in C(J) \cap C^1(\sigma).$$

Therefore,

$$\|B_2 v\|_{1,\infty,\sigma} \leqslant C\|(I-\pi_h)K_2 v\|_{1,\infty,\sigma} \leqslant Ch_\sigma \|K_2 v\|_{2,\infty,\sigma} \leqslant Ch_\sigma \|v\|_{1,\infty,\sigma}.$$

Thus we obtain

$$\|B_2\|_{C^1(\sigma)\to C^1(\sigma)} \leqslant Ch_\sigma.$$

It is easy to check that

$$\|B_1\|_{C(J)\to C(J)} \leqslant C, \quad \|B_1\|_{C^1(\sigma)\to C^1(\sigma)} \leqslant C. \tag{3.67}$$

Since $i \geqslant 2p+2$, we can deduce

$$|B_{l-i,i}e(t)| \leqslant C\|B_2\|^i_{C^1(\sigma)\to C^1(\sigma)}\|e\|_{1,\infty,\sigma} \leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})}\|u\|_{1,\infty,\sigma},\ \forall t \in \sigma.$$

Furthermore, we get

$$\|B_{l-i,i}e\|_{0,\infty,J} \leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})}\|u\|_{1,\infty,J}, \quad i \geqslant 2p+2.$$

□

In the following we consider the case with $0 \leqslant i \leqslant 2p+1$. For this case, we have

$$l - i \geqslant (2p+3)(2p+2+m).$$

Set $l^* = l-(2p+2+m)$, then $l^*-i \geqslant (2p+2)(2p+2+m)$. It means that $\frac{l^*-i}{2p+2} \geqslant 2p+2+m$. Therefore $B_{l-i,i}\ e$ can be written as

$$B_{l-i,i}e = \sum_{r=0}^{l^*-i}\sum_{j=0}^{i} B_{r,j}B_1^{2p+2+m}\widetilde{B}_{l^*-i-r,i-j}e = \sum_{r=0}^{l^*-i}\sum_{j=0}^{i} B_{r,j}\widetilde{B}^{r,j},$$

where $B_{r,j}$ is the sum of $C^r_{r+j}$ terms, with each term being a product between $r$ operators $B_1$ and $j$ operators $B_2$, which have different order in any two terms, $\widetilde{B}^{r,j} = B_1^{2p+2+m}\widetilde{B}_{l^*-i-r,i-j}e \in C^{2p+2}(J)$ and $\widetilde{B}_{l^*-i-r,i-j}$ is the sum of $C^{l^*-i-r}_{l^*-r-j}$ terms, with each term being a product between $l^*-i-r$ operators $B_1$ and $i-j$ operators $B_2$, which does not contain the operator $B_1^{2p+2+m}$.

If $j \geqslant 1$, we can prove that $B_{r,j}\widetilde{B}^{r,j}$ can be written as (3.59) or (3.61) at the mesh points by the similar manipulation in the proof of the Lemma 3.7, since $\widetilde{B}^{r,j} \in C^{2p+2}(J)$. If $j = 0$, since $\|B_1\|_{C(J)\to C(J)}$is bounded, we only need to estimate $B_1^m B_{l^*-i-r,i}e$, where $r$ is a integer such that $0 \leqslant r \leqslant l^*-i$, and $B_{l^*-i-r,i}$ is the sum of $C^i_{l^*-r}$ terms, with each term being a product between $l^*-i-r$ operators $B_1$ and $i$ operators $B_2$, which have different order in any two terms.

**Lemma 3.9.** *Let $A_1$ and $A_2$ hold. Assume that $m \leqslant 2$ and the functions $f$, $K_i$ $(i=1,2)$ satisfy $f \in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega = \Omega_1 \bigcup \Omega_2$, $\Omega_1 := \{(t,s):\ 0 \leqslant s \leqslant t \leqslant T\}$ and $\Omega_2 := \{(t,s):\ 0 \leqslant s \leqslant qt,\ t \in J\}$. Let $i$, $r$ and $l^*$ denote three given positive integers such that $0 \leqslant i \leqslant 2p+1$, $i+(2p+2)(2p+2+m) \leqslant l^*$ and $0 \leqslant r \leqslant l^*-i$ . Then*

$$\|B_1^m B_{l^*-i-r,i}e\|_{0,\infty,J} \leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{2m,\infty,J},\ N \to \infty. \tag{3.68}$$

*Here $\varepsilon_{N,2m}$ is an arbitrarily small positive number such that $\lim_{N\to\infty}\varepsilon_{N,2m} = 0$.*

*Proof.* We write $B_1^m B_{l^*-i-r,i}e$ as

$$B_1^m B_{l^*-i-r,i}e = \overline{\pi} B_1^m B_{l^*-i-r,i}e + (I-\overline{\pi})B_1^m B_{l^*-i-r,i}e. \tag{3.69}$$

For $i=0$, we have $B_1^m B_{l^*-i-r,i}e = B_1^{l^*-r+m}e$, where $m \leqslant l^*-r+m$. By the definition of $B_1$, we can change the order of integration to write $B_1^{l^*-r+m}e$ as

$$B_1^{l^*-r+m}e = \sum_{i=1}^{l^*-r+m} \int_0^{q^i t} \overline{B}_1^{i,i}(t,s)e(s)ds := \overline{B}_1 e,$$

where $\overline{B}_1^{i,j}(t,s)$ is the sum of $C_{r+j}^r$ terms, with each term being a multiple integral whose integrand is a product between $i$ functions $K_2$ and $j$ functions $R_1$, which have different order in any two terms (also see [12]). By a similar method with the proof of Theorem 3.1, we can obtain

$$\begin{aligned} \|\pi B_1^{l^*-r+m}e\|_{0,\infty,J} &\leqslant \|\pi \overline{B}_1 e\|_{0,\infty,J} \\ &\leqslant C \max_{t\in Z_N} \|\overline{B}_1 e\|_{0,\infty,J} \\ &\leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{2m,\infty,J}. \end{aligned} \tag{3.70}$$

For $1 \leqslant i \leqslant 2p+1$, we have

$$\pi B_1^m B_{l^*-i-r,i}e = \pi B_1^{m+\mu} B_2 B_{l^*-i-r-\mu,i-1}e = \pi B_1^{m+\mu} B_2 v_{\mu,i-1}, \quad 0 \leqslant \mu \leqslant l^*-i-r,$$

where $\mu$ is a integer, $v_{\mu,i-1} = B_{l^*-i-r-\mu,i-1}e$, and $B_{l^*-i-r-\mu,i-1}$ is the sum of $C_{l^*-r-\mu-1}^{i-1}$ terms, with each term being a product between $l^*-i-r-\mu$ operators $B_1$ and $i-1$ operators $B_2$, which have different order in any two terms. It is easy to check that $v_{\mu,i-1} \in C(J)$. By the definition of $B_1$ and $B_2$, we can change the order of integration to write $B_1^{m+\mu}B_2 v_{\mu,i-1}$ as

$$\begin{aligned} B_1^{m+\mu}B_2 v_{\mu,i-1} &= B_1^{m+\mu}(I+R)(\pi_h - I)Kv_{\mu,i-1} \\ &= B_1^{m+\mu}(\pi_h - I)Kv_{\mu,i-1} + B_1^{m+\mu}R(\pi_h - I)Kv_{\mu,i-1} \\ &= \sum_{j=1}^{m+\mu} \int_0^{q^j t} \overline{B}_1^{j,j}(t,s)(\pi_h - I)Kv_{\mu,i-1}ds \\ &\quad + \sum_{j=1}^{m+\mu} \int_0^{q^j t} \overline{B}_1^{j,j+1}(t,s)(\pi_h - I)Kv_{\mu,i-1}ds. \end{aligned} \tag{3.71}$$

When $\kappa+j+1 \leqslant n$, we can deduce that $t_{n-j}$ is a mesh point and $t_{n-j} = q^j t_n \geqslant t_1$. Using Lemma 3.4, the inequalities (3.3), (3.18) and the smoothness $Kv_{\mu,i-1} \in C(J) \cap C^m(\sigma)$, we deduce

$$
\begin{aligned}
& | \int_0^{q^j t_n} \overline{B}_1^{j,j}(t_n,s)(\pi_h - I)Kv_{\mu,i-1}ds \,| \\
\leqslant \;& C\sum_{k=2}^{n-j} | \int_{e_k} \overline{B}_1^{j,j}(t_n,s)(\pi_h - I)Kv_{\mu,i-1}ds \,| + C\sum_{k=1}^{2p+1} | \int_{e_{1,k}} \overline{B}_1^{j,j}(t_n,s)(\pi_h - I)Kv_{\mu,i-1}ds \,| \\
\leqslant \;& C\sum_{k=2}^{n-j} N^{-(2m-\varepsilon_{N,2m})}\|Kv_{\mu,i-1}\|_{m,\infty,e_k} + C\sum_{k=1}^{2p+1} N^{-(2m-\varepsilon_{N,2m})}\|Kv_{\mu,i-1}\|_{m,\infty,e_{1,k}} \\
\leqslant \;& C\sum_{k=2}^{n-j} N^{-(2m-\varepsilon_{N,2m})}\|e\|_{m-1,\infty,e_k} + C\sum_{k=1}^{2p+1} N^{-(2m-\varepsilon_{N,2m})}\|e\|_{m-1,\infty,e_{1,k}} \qquad (3.72)\\
\leqslant \;& C\sum_{k=2}^{n-j} N^{-(2m-\varepsilon_{N,2m})}h_k\|u\|_{m,\infty,e_k} + C\sum_{k=1}^{2p+1} N^{-(2m-\varepsilon_{N,2m})}h_{1,k}\|u\|_{m,\infty,e_{1,k}} \\
\leqslant \;& CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{m,\infty,J}.
\end{aligned}
$$

When $1 \leqslant n \leqslant \kappa + j$, we have $q^j t_n \leqslant t_1$. Note that $2(n+1)m \leqslant 2p+2$, we have $m \leqslant p+1$. According to (3.5) and (3.1), we can lead to

$$
\begin{aligned}
| \int_0^{q^j t_n} \overline{B}_1^{j,j}(t_n,s)(\pi_h - I)Kv_{\mu,i-1}ds \,| \;&\leqslant\; Ct_1 h_1^m \|u\|_{m,\infty,e_1} \\
&\leqslant\; Ch_1^{m+1}\|u\|_{m,\infty,e_1} \qquad (3.73)\\
&\leqslant\; CN^{-2m}\|u\|_{m,\infty,e_1}.
\end{aligned}
$$

Similarly, we can deduce that the following inequality is valid for any integer $k$ satisfying $1 \leqslant k \leqslant 2p+1$,

$$
| \int_0^{q^j t_{1,k}} \overline{B}_1^{j,j}(t_{1,k},s)(\pi_h - I)Kv_{\mu,i-1}ds \,| \leqslant CN^{-2m}\|u\|_{m,\infty,e_1}. \qquad (3.74)
$$

By the inequalities (3.72), (3.73) and (3.74), we obtain

$$
\|\overline{\pi}[\int_0^{q^j t} \overline{B}_1^{j,j}(t,s)(\pi_h - I)Kv_{\mu,i-1}ds]\|_{0,\infty,J} \leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{m,\infty,J}. \qquad (3.75)
$$

By the same way, we can deduce

$$
\|\overline{\pi}[\int_0^{q^j t} \overline{B}_1^{j,j+1}(t,s)(\pi_h - I)Kv_{\mu,i-1}ds]\|_{0,\infty,J} \leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{m,\infty,J}. \qquad (3.76)
$$

Combining (3.71) and (3.75) with (3.76), we have

$$
\|\overline{\pi}B_1^{m+\mu}B_2 v_{\mu,i-1}\|_{0,\infty,J} \leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{m,\infty,J}, \quad 1 \leqslant i \leqslant 2p+1. \qquad (3.77)
$$

By the inequalities (3.70) and (3.77), we get

$$
\|\overline{\pi}B_1^{m}B_{l^*-i-r,i}e\|_{0,\infty,J} \leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{m,\infty,J}, \quad 0 \leqslant i \leqslant 2p+1. \qquad (3.78)
$$

By interpolation error estimate (3.49) and Lemma 3.3, we can lead to

$$\begin{aligned}\|(I-\overline{\pi})B_1^m B_{l^*-i-r,i}e\|_{0,\infty,J} &\leqslant CN^{-(m-\varepsilon_{N,m})}\|B_1^m B_{l^*-i-r,i}e\|_{m,\infty,J} \\ &\leqslant CN^{-(m-\varepsilon_{N,m})}\|B_{l^*-i-r,i}e\|_{0,\infty,J} \\ &\leqslant CN^{-(m-\varepsilon_{N,m})}\|e\|_{0,\infty,J} \\ &\leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{m,\infty,J}.\end{aligned} \tag{3.79}$$

It follows by (3.69), (3.78) and (3.79) that

$$\|B_1^m B_{l^*-i-r,i}e\|_{0,\infty,J} \leqslant CN^{-(2m-\varepsilon_{N,2m})}\|u\|_{2m,\infty,J},\ N\to\infty.$$

□

The following Theorem 3.4 plays key role in the proof of Theorem 2.1.

**Theorem 3.4.** *Let $A_1$ and $A_2$ hold. Assume that $m \leqslant 2$ and the functions $f$, $K_i$ $(i=1,2)$ satisfy $f\in C^{2p+2}(J)$, $K_i \in C^{2p+2}(\Omega)$, where $\Omega=\Omega_1\bigcup\Omega_2$, $\Omega_1 := \{(t,s):\ 0\leqslant s\leqslant t\leqslant T\}$ and $\Omega_2 := \{(t,s):\ 0\leqslant s\leqslant qt,\ t\in J\}$. Let $k$ denote a given positive integer such that $2(k+1)m\leqslant 2p+2$. Then, for any positive integer $\lambda$ satisfying $2km+1\leqslant\lambda\leqslant 2(k+1)m$, we have*

$$\|[\overline{\pi}(Q_h-I)]^{k+1}u\|_{0,\infty,J} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})}\|u\|_{\lambda,\infty,J},\ N\to\infty. \tag{3.80}$$

*Here $\varepsilon_{N,\lambda}$ is an arbitrarily small positive number such that* $\lim_{N\to\infty}\varepsilon_{N,\lambda}=0$.

*Proof.* The inequality (3.80) will be proved by the induction principle.

When $k=1$, noting the equality (3.12)

$$(Q_h-I)u = u_h-u = e = \pi_h Ke + (\pi_h - I)u,$$

we have

$$\overline{\pi}(Q_h-I)u = \overline{\pi}Ke.$$

For any positive integer $\lambda$ satisfying $1\leqslant\lambda\leqslant 2m$, it follows by Theorem 3.1 that there is a constant $C$ independent of the meshes $J_N$ such that

$$\begin{aligned}\|[\overline{\pi}(Q_h-I)]u\|_{0,\infty,J} &= \|\overline{\pi}Ke\|_{0,\infty,J} \\ &\leqslant C\max_{t\in Z_N}|(Ke)(t)|. \\ &\leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})}\|u\|_{\lambda,\infty,J},\ N\to\infty.\end{aligned}$$

When $1\leqslant k\leqslant n$, we assume that the inequality

$$\|[\overline{\pi}(Q_h-I)]^{k+1}u\|_{0,\infty,J} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})}\|u\|_{\lambda,\infty,J},\ N\to\infty, \tag{3.81}$$

is valid for any positive integer $\lambda$ satisfying $2km+1 \leqslant \lambda \leqslant 2(k+1)m$.

In the following we prove that the inequality (3.80) is valid for $k=n+1$.

Set $l=(2p+3)(2p+m+3)$ in the equality (3.58), and write $[\overline{\pi}(Q_h-I)]^{n+2}u$ as

$$\begin{aligned}
[\overline{\pi}(Q_h-I)]^{n+2}u &= [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}e \\
&= [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=1}^{l-1} B^i Au + \sum_{i=0}^{l} B_{l-i,i}e] \\
&= [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=1}^{l-1} B^i Au + \sum_{i=1}^{2p+1}\sum_{r=0}^{l^*-i}\sum_{j=1}^{i} B_{r,j}\widetilde{B}^{r,j}] \\
&\quad +[\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=0}^{2p+1}\sum_{r=0}^{l^*-i} B_{r,0}\widetilde{B}^{r,0}] \\
&\quad +[\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=2p+2}^{l} B_{l-i,i}e] \\
&= G_1+G_2+G_3,
\end{aligned} \tag{3.82}$$

where

$$G_1 = [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=1}^{l-1} B^i Au + \sum_{i=1}^{2p+1}\sum_{r=0}^{l^*-i}\sum_{j=1}^{i} B_{r,j}\widetilde{B}^{r,j}],$$

$$G_2 = [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=0}^{2p+1}\sum_{r=0}^{l^*-i} B_{r,0}\widetilde{B}^{r,0}]$$

and

$$G_3 = [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=2p+2}^{l} B_{l-i,i}e].$$

In the following we estimate $\|G_1\|_{0,\infty,J}$, $\|G_2\|_{0,\infty,J}$ and $\|G_3\|_{0,\infty,J}$ respectively.

Firstly, we estimate $\|G_1\|_{0,\infty,J}$. Note that $\widetilde{B}^{r,j} \in C^{2p+2}(J)$ and $j \geqslant 1$. Then, it is easy to verify, as in the proof of the Lemma 3.7, that $B_{r,j}\widetilde{B}^{r,j}$ can be written as (3.59) or (3.61) at the mesh points. Furthermore, we obtain by Lemma 3.7

$$\begin{aligned}
G_1 &= [\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=1}^{l-1} B^i Au + \sum_{i=1}^{2p+1}\sum_{r=0}^{l^*-i}\sum_{j=1}^{i} B_{r,j}\widetilde{B}^{r,j}] \\
&= [\overline{\pi}(Q_h-I)]^{n+1}\sum_{n=2}^{N}\sum_{j=m}^{p}[(\frac{h_1}{2p+1})^{2j}F_j^1(t_n) + \sum_{k=2}^{n} h_k^{2j}F_j^k(t_n)]L_n(t) + O(N^{-(2p+2)}),
\end{aligned}$$

where $L_n(t)$ is the Lagrange basic function of the point $t_n$. Since $F_j(t_n)=\sum_{k=1}^{n} F_j^k(t_n)$ and $F_j \in C^{2p+2-2j}(J)$, we get

$$\begin{aligned}
\|G_1\|_{0,\infty,J} &\leqslant C\sum_{j=m}^{p} h^{2j}\|[\overline{\pi}(Q_h-I)]^{n+1}\sum_{n=2}^{N}\sum_{k=1}^{n}F_j^k(t_n)L_n\|_{0,\infty,J} \\
&\leqslant C\sum_{j=m}^{p} h^{2j}\|[\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}F_j\|_{0,\infty,J} \qquad (3.83)\\
&\leqslant C\sum_{j=m}^{p} h^{2j}\|[\overline{\pi}(Q_h-I)]^{n+1}[F_j+(\overline{\pi}-I)F_j]\|_{0,\infty,J} \\
&\leqslant C\sum_{j=m}^{p} h^{2j}\|[\overline{\pi}(Q_h-I)]^{n+1}F_j\|_{0,\infty,J}+C\sum_{j=m}^{p} h^{2j}\|(\overline{\pi}-I)F_j\|_{0,\infty,J}.
\end{aligned}$$

In the following, we estimate $\|[\overline{\pi}(Q_h-I)]^{n+1}F_j\|_{0,\infty,J}$ with $j$ satisfying $m\leqslant j\leqslant p$. Because of the limitation of the smoothness of the functions $F_j$, this will be done for different cases of $j$.

By the assumption in this Lemma, we know that $2\leqslant 2(n+1)m\leqslant 2p+2$. Then $0\leqslant p+1-(n+1)m\leqslant p$. When $j$ satisfies

$$p+1-(n+1)m<j\leqslant p,$$

we have $n+1>\frac{p+1-j}{m}$. It means that $n+1\geqslant[\frac{p+1-j}{m}]+1$. It is easy to verify that both $\|Q_h\|_{C(J)\to C(J)}$ and $\|\overline{\pi}\|_{C(J)\to C(J)}$ are bounded. Therefore $\|\overline{\pi}(Q_h-I)\|_{C(J)\to C(J)}$ is also bounded. Furthermore, by the inequality (3.18) and the inductive assumption (3.81), we deduce (note that $n+1\geqslant[\frac{p+1-j}{m}]+1$)

$$\begin{aligned}
h^{2j}\|[\overline{\pi}(Q_h-I)]^{n+1}F_j\|_{0,\infty,J} &\leqslant Ch^{2j}\|[\overline{\pi}(Q_h-I)]^{[\frac{p+1-j}{m}]+1}F_j\|_{0,\infty,J} \\
&\leqslant CN^{-(2j-\varepsilon_{N,2j})}\|[\overline{\pi}(Q_h-I)]^{[\frac{p+1-j}{m}]+1}F_j\|_{0,\infty,J} \\
&\leqslant CN^{-(2j-\varepsilon_{N,2j})}N^{-(2m(\frac{p+1-j}{m})-\varepsilon_{N,2(p+1-j)})}\|F_j\|_{2p+2-2j,\infty,J} \\
&\leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})}\|F_j\|_{2p+2-2j,\infty,J},\ N\to\infty. \qquad (3.84)
\end{aligned}$$

If $p+1-(n+1)m<m$, then we have $p+1-(n+1)m<j\leqslant p$ since $m\le j\le p$. For this case, we have gotten the estimate of $\|[\overline{\pi}(Q_h-I)]^{n+1}F_j\|_{0,\infty,J}$ by the inequality (3.84). When $m\leqslant p+1-(n+1)m$, by the inequality (3.84), we only need to consider the case with $j$ satisfying $m\leqslant j\leqslant p+1-(n+1)m$. For this case, we have $2p+2-2j\geqslant 2(n+1)m$. Since $2nm+1\leqslant\lambda\leqslant 2(n+1)m$ and $F_j\in C^{2p+2-2j}(J)$, the norm $\|F_j\|_{\lambda,\infty,J}$ is well defined. Thus, using the inductive assumption (3.81), we get

$$\|[\overline{\pi}(Q_h-I)]^{n+1}F_j\|_{0,\infty,J}\leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})}\|F_j\|_{\lambda,\infty,J},\quad 2nm+1\leqslant\lambda\leqslant 2(n+1)m.$$

This, together with (3.18), (3.27) and (3.60), allows us to deduce

$$\begin{aligned} h^{2j}\|[\overline{\pi}(Q_h-I)]^{n+1}F_j\|_{0,\infty,J} &\leqslant Ch^{2j}N^{-(\lambda-\varepsilon_{N,\lambda})}\|F_j\|_{\lambda,\infty,J} \\ &\leqslant CN^{-(2j-\varepsilon_{N,2j})}N^{-(\lambda-\varepsilon_{N,\lambda})}\|F_j\|_{\lambda,\infty,J} \qquad (3.85) \\ &\leqslant CN^{-(\lambda+2j-\varepsilon_{N,\lambda+2j})}\|u\|_{\lambda+2j,\infty,J},\ N\to\infty. \end{aligned}$$

Next, we estimate $\|(\overline{\pi}-I)F_j\|_{0,\infty,J}$ $(m\leqslant j\leqslant p)$. From Theorem 3.3 and the inequality (3.18), we have

$$h^{2j}\|(\overline{\pi}-I)F_j\|_{0,\infty,J}\leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})}\|F_j\|_{2p+2-2j,\infty,J},\ N\to\infty.$$

This, together with (3.83), (3.84) and (3.85), allows us to deduce

$$\|G_1\|_{0,\infty,J}\leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})}\|u\|_{\lambda,\infty,J},N\to\infty,\ 2(n+1)m+1\leqslant\lambda\leqslant 2(n+2)m. \qquad (3.86)$$

Secondly, we estimate $\|G_2\|_{0,\infty,J}$. By the definition of $G_2$, we have

$$\begin{aligned} \|G_2\|_{0,\infty,J} &= \|[\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=0}^{2p+1}\sum_{r=0}^{l^*-i}B_{r,0}\widetilde{B}^{r,0}]\|_{0,\infty,J} \\ &\leqslant C\ \|[\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}B_1^rB_1^{2p+2+m}B_{l^*-i-r,i}e]\|_{0,\infty,J}. \end{aligned}$$

Then, we get by (3.81)

$$\begin{aligned} \|G_2\|_{0,\infty,J} &\leqslant C\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}\|[\overline{\pi}(Q_h-I)]^{n+1}\overline{\pi}[B_1^{2p+2+m+r}B_{l^*-i-r,i}e]\|_{0,\infty,J} \\ &\leqslant C\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}\|[\overline{\pi}(Q_h-I)]^{n+1}[B_1^{2p+2+m+r}B_{l^*-i-r,i}e]\|_{0,\infty,J} \\ &\quad +C\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}\|[\overline{\pi}(Q_h-I)]^{n+1}(\overline{\pi}-I)[B_1^{2p+2+m+r}B_{l^*-i-r,i}e]\|_{0,\infty,J} \\ &\leqslant C\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}N^{-(\lambda-\varepsilon_{N,\lambda})}\|B_1^{2p+2+m+r}B_{l^*-i-r,i}e\|_{\lambda,\infty,J} \qquad (3.87) \\ &\quad +C\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}\|(\overline{\pi}-I)[B_1^{2p+2+m+r}B_{l^*-i-r,i}e]\|_{0,\infty,J} \\ &= C(I_{21}+I_{22}),\ N\to\infty, \end{aligned}$$

where

$$I_{21}=\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}N^{-(\lambda-\varepsilon_{N,\lambda})}\|B_1^{2p+2+m+r}B_{l^*-i-r,i}e\|_{\lambda,\infty,J},\ 2nm+1\leqslant\lambda\leqslant 2(n+1)m,$$

and

$$I_{22}=\sum_{i=0}^{2p+2}\sum_{r=0}^{l^*-i}\|(\overline{\pi}-I)[B_1^{2p+2+m+r}B_{l^*-i-r,i}e]\|_{0,\infty,J}.$$

Noting that $2(n+1)m \leqslant 2p+2$ and the inequality (3.67), we have

$$
\begin{aligned}
I_{21} &\leqslant C \sum_{i=0}^{2p+2} \sum_{r=0}^{l^*-i} N^{-(\lambda-\varepsilon_{N,\lambda})} \|B_1^{2p+2+m+r} B_{l^*-i-r,i} e\|_{2(n+1)m,\infty,J} \\
&\leqslant C \sum_{i=0}^{2p+2} \sum_{r=0}^{l^*-i} N^{-(\lambda-\varepsilon_{N,\lambda})} \|B_1^{2p+2+m+r-2(n+1)m} B_{l^*-i-r,i} e\|_{0,\infty,J} \\
&\leqslant C \sum_{i=0}^{2p+2} \sum_{r=0}^{l^*-i} N^{-(\lambda-\varepsilon_{N,\lambda})} \|B_1^{m} B_{l^*-i-r,i} e\|_{0,\infty,J}.
\end{aligned} \tag{3.88}
$$

It follows, by the inequality (3.88) and Lemma 3.9, that the following inequality is valid for any integer $\lambda$ satisfying $2nm+1 \leqslant \lambda \leqslant 2(n+1)m$

$$
I_{21} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})} N^{-(2m-\varepsilon_{N,2m})} \|u\|_{2m,\infty,J}, \ N \to \infty.
$$

This means that

$$
I_{21} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})} \|u\|_{2m,\infty,J}, \ N \to \infty, \ 2(n+1)m+1 \leqslant \lambda \leqslant 2(n+2)m. \tag{3.89}
$$

By interpolation error estimate (3.49) and the inequalities (3.67), (3.3), we have

$$
\begin{aligned}
I_{22} &\leqslant C \sum_{i=0}^{2p+2} \sum_{r=0}^{i} \|(\overline{\pi}-I)[B_1^{2p+2+m+r} B_{l^*-i-r,i} e]\|_{0,\infty,J} \\
&\leqslant C \sum_{i=0}^{2p+2} \sum_{r=0}^{i} N^{-(2p+2-\varepsilon_{N,2p+2})} \|B_1^{2p+2+m+r} B_{l^*-i-r,i} e\|_{2p+2,\infty,J} \\
&\leqslant C \sum_{i=0}^{2p+2} \sum_{r=0}^{i} N^{-(2p+2-\varepsilon_{N,2p+2})} \|B_1^{r} B_{l^*-i-r,i} e\|_{0,\infty,J} \\
&\leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})} \|e\|_{0,\infty,J} \\
&\leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})} \|u\|_{m,\infty,J}.
\end{aligned} \tag{3.90}
$$

Combining (3.87), (3.89) and (3.90), we obtain

$$
\|G_2\|_{0,\infty,J} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})} \|u\|_{\lambda,\infty,J}, \ N \to \infty, \ 2(n+1)m+1 \leqslant \lambda \leqslant 2(n+2)m. \tag{3.91}
$$

Finally, we estimate $\|G_3\|_{0,\infty,J}$. From the inequality (3.66), it is obvious that

$$
\begin{aligned}
\|G_3\|_{0,\infty,J} &= \|[\overline{\pi}(Q_h-I)]^{n+1} \overline{\pi}[\sum_{i=2p+2}^{l} B_{l-i,i} e]\|_{0,\infty,J} \\
&\leqslant C \|\sum_{i=2p+2}^{l} B_{l-i,i} e\|_{0,\infty,J} \\
&\leqslant CN^{-(2p+2-\varepsilon_{N,2p+2})} \|u\|_{1,\infty,J}.
\end{aligned} \tag{3.92}
$$

This, together with (3.82), (3.86) and (3.91), allows us to deduce

$$
\|[\overline{\pi}(Q_h-I)]^{n+2} u\|_{0,\infty,J} \leqslant CN^{-(\lambda-\varepsilon_{N,\lambda})} \|u\|_{\lambda,\infty,J}, \ N \to \infty, \ 2(n+1)m+1 \leqslant \lambda \leqslant 2(n+2)m.
$$

Namely, the inequality (3.80) is valid for $k = n+1$.

Now we prove the inequality (3.80) by the induction principle. □

# 4 Proof of the main result

In this section, we prove Theorem 2.1 by using the auxiliary results given in the last section.

*Proof.* For convenience, we set $T_h = \overline{\pi} Q_h - I$. It is easy to check that

$$\begin{aligned} T_h^{k+1} u &= (\overline{\pi} Q_h - I)^{k+1} u \\ &= \sum_{j=0}^{k} (-1)^j C_{k+1}^j (\overline{\pi} Q_h)^{k-j} \overline{\pi} Q_h u - (-1)^k u \\ &= (-1)^k (u_{h,k} - u). \end{aligned}$$

Therefore

$$\|u_{h,k} - u\|_{0,\infty,J} = \|T_h^{k+1} u\|_{0,\infty,J}.$$

Let $\lambda$ denote a positive integer number. When $\lambda = 1$, we have

$$\begin{aligned} T_h^{\lambda} u &= T_h u \\ &= (\overline{\pi} Q_h - I) u \\ &= \overline{\pi}(Q_h - I) u + (\overline{\pi} - I) u. \end{aligned}$$

First, we prove inductively that the following equation is valid for $\lambda \geqslant 2$

$$T_h^{\lambda} u = [\overline{\pi}(Q_h - I)]^{\lambda} u + \sum_{l=0}^{\lambda-2} (-1)^{2\lambda - l} [\overline{\pi}(Q_h - I)]^{\lambda-1-l} u + (-1)^{\lambda+1} (\overline{\pi} - I) u. \tag{4.1}$$

When $\lambda = 2$, we get

$$\begin{aligned} T_h^2 u &= (\overline{\pi} Q_h - I)[\overline{\pi}(Q_h - I) u + (\overline{\pi} - I) u] \\ &= \overline{\pi}(Q_h - I)\overline{\pi}(Q_h - I) u + (\overline{\pi} Q_h - I)(\overline{\pi} - I) u \\ &= \overline{\pi}(Q_h - I)\overline{\pi}(Q_h - I) u + \overline{\pi}(Q_h - I)(\overline{\pi} - I) u - (\overline{\pi} - I) u. \end{aligned}$$

We assume that the equation

$$T_h^{\lambda} u = [\overline{\pi}(Q_h - I)]^{\lambda} u + \sum_{l=0}^{\lambda-2} (-1)^{2\lambda - l} [\overline{\pi}(Q_h - I)]^{\lambda-1-l} u + (-1)^{\lambda+1} (\overline{\pi} - I) u, \tag{4.2}$$

is valid for $\lambda$ ($\lambda \geqslant 2$). Now we only need to prove that

$$T_h^{\lambda+1} u = [\overline{\pi}(Q_h - I)]^{\lambda+1} u + \sum_{l=0}^{\lambda-1} (-1)^{2(\lambda+1) - l} [\overline{\pi}(Q_h - I)]^{\lambda - l} u + (-1)^{\lambda+2} (\overline{\pi} - I) u. \tag{4.3}$$

In fact, $T_h^{\lambda+1}u$ can be written as $T_h^{\lambda+1}u = T_h(T_h^{\lambda}u)$. It follows by the inductive assumption (4.2) that

$$\begin{aligned} T_h^{\lambda+1}u &= T_h(T_h^{\lambda}u) \\ &= (\overline{\pi}Q_h - I)\{[\overline{\pi}(Q_h - I)]^{\lambda}u \\ &\quad + \sum_{l=0}^{\lambda-2}(-1)^{2\lambda-l}[\overline{\pi}(Q_h - I)]^{\lambda-1-l}u \\ &\quad +(-1)^{\lambda+1}(\overline{\pi} - I)u\} \\ &= G_1 + G_2 + G_3, \end{aligned} \tag{4.4}$$

where $G_1 = (\overline{\pi}Q_h - I)[\overline{\pi}(Q_h - I)]^{\lambda}u$, $G_2 = (\overline{\pi}Q_h - I)\{\sum_{l=0}^{\lambda-2}(-1)^{2\lambda-l}[\overline{\pi}(Q_h - I)]^{\lambda-1-l}u\}$ and $G_3 = (\overline{\pi}Q_h - I)\{(-1)^{\lambda+1}(\overline{\pi} - I)u\}$. It is easy to obtain that

$$\begin{aligned} G_1 &= (\overline{\pi}Q_h - I)[\overline{\pi}(Q_h - I)]^{\lambda}u \\ &= [\overline{\pi}(Q_h - I)]^{\lambda+1}u + (\overline{\pi} - I)[\overline{\pi}(Q_h - I)]^{\lambda}u \\ &= [\overline{\pi}(Q_h - I)]^{\lambda+1}u, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} G_2 &= (\overline{\pi}Q_h - I)\{\sum_{l=0}^{\lambda-2}(-1)^{2\lambda-l}[\overline{\pi}(Q_h - I)]^{\lambda-1-l}u\} \\ &= \sum_{l=0}^{\lambda-2}(-1)^{2\lambda-l}([\overline{\pi}(Q_h - I)]^{\lambda-l}u + (\overline{\pi} - I)[\overline{\pi}(Q_h - I)]^{\lambda-1-l}u) \\ &= \sum_{l=0}^{\lambda-2}(-1)^{2\lambda-l}([\overline{\pi}(Q_h - I)]^{\lambda-l}u, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} G_3 &= (\overline{\pi}Q_h - I)\{(-1)^{\lambda+1}(\overline{\pi} - I)u\} \\ &= (-1)^{\lambda+1}([\overline{\pi}(Q_h - I)](\overline{\pi} - I)u - (\overline{\pi} - I)u) \\ &= (-1)^{\lambda+1}[\overline{\pi}(Q_h - I)](\overline{\pi} - I)u + (-1)^{\lambda+2}(\overline{\pi} - I)u. \end{aligned} \tag{4.7}$$

By using (4.4)-(4.7), we deduce (4.3). It then follows by the induction principle that the equation (4.1) is valid.

Now we can readily prove Theorem 2.1. Set $\lambda = k + 1$. Therefore the following inequality is a direct consequence of Theorem 3.4

$$\|[\overline{\pi}(Q - I)]^{k+1}u\|_{0,\infty,J} \leqslant CN^{-(2(k+1)m-\varepsilon_{N,2(k+1)m})}\|u\|_{2(k+1)m,\infty,J}. \tag{4.8}$$

For $0 \leqslant n \leqslant \lambda - 2 = k - 1$, we have

$$\begin{aligned} &\|(-1)^{2k+2-n}[\overline{\pi}(Q - I)]^{k-n}(\overline{\pi} - I)u\|_{0,\infty,J} \\ \leqslant\ & CN^{-(2(k-n)m-\varepsilon_{N,2(k-n)m})}\|(\overline{\pi} - I)u\|_{2(k-n)m,\infty,J} \\ \leqslant\ & CN^{-(2(p+1)-\varepsilon_{N,2(p+1)})}\|u\|_{2(p+1),\infty,J}, \quad N \to \infty, \end{aligned} \tag{4.9}$$

and

$$\|(-1)^{k+2}(\overline{\pi}-I)u\|_{0,\infty,J} \leqslant CN^{-(2(p+1)-\varepsilon_{N,2(p+1)})}\|u\|_{2(p+1),\infty,J}. \quad (4.10)$$

Combining (4.1), (4.8), (4.9) and (4.10), we lead to

$$\|T_h^{k+1}u\|_{0,\infty,J} \leqslant CN^{-(2(k+1)m-\varepsilon_{N,2(k+1)m})}\|u\|_{2(k+1)m,\infty,J}, \quad N \to \infty.$$

By the definition of $\varepsilon_{N,k}$ and the fact that $2(k+1)m \leqslant 2p+2$, we have $\varepsilon_{N,2(k+1)m} \leqslant \varepsilon_{N,2p+2}$. Letting $\varepsilon_N = \varepsilon_{N,2(p+1)}$, we can obtain

$$\|u_{h,k}-u\|_{0,\infty,J} \leqslant CN^{-(2(k+1)m-\varepsilon_N)}\|u\|_{2(k+1)m,\infty,J}, \quad N \to \infty,$$

where $\varepsilon_N$ is an arbitrarily small positive number such that $\lim\limits_{N\to\infty} \varepsilon_N = 0$. □

# 5 Numerical examples

For the numerical verification of the result stated in section 2, we consider

$$u(t) = f(t) - \int_0^t u(s)ds + \frac{1}{2}\int_0^{qt} u(s)ds, \ t \in [0,T], \quad (5.1)$$

where the function $f$ is chosen as $f(t) = \frac{1}{2}(1+e^{-qt})$ such that the exact solution is $u(t) = e^{-t}$; the delay parameter $q$ is chosen as $q = 0.9$, $q = 0.5$, or $q = 0.2$. We set $T = 10$. The equation (5.1) is solved by two collocation methods on different meshes using the space $S_1^0(J_N)$ $(m = 1)$.

The first method $M_1$ is the multilevel correction method based on the *geometric meshes* and the Lobatto collocation parameters: $c_1 = 0.0, c_2 = 1.0$; the second method $M_2$ is the multilevel correction method which uses the *hybrid meshes* introduced in this paper and the Lobatto collocation parameters $c_1$ and $c_2$. For convenience, we let $\|\cdot\|_{0,\infty}$ denote the uniform norm $\|\cdot\|_{0,\infty,J}$. The $L^\infty$ errors are reported in Table 1 ($q = 0.9$), Table 2 ($q = 0.5$) and Table 3 ($q = 0.2$).

TABLE 1 ($q = 0.9$)

| | $M_1$ | $M_1$ | $M_2$ | $M_2$ | $M_2$ |
|---|---|---|---|---|---|
| $N$ | $\|u_{h,0}-u\|_{0,\infty}$ | $\|u_{h,1}-u\|_{0,\infty}$ | $\|u_{h,0}-u\|_{0,\infty}$ | $\|u_{h,1}-u\|_{0,\infty}$ | $\|u_{h,2}-u\|_{0,\infty}$ |
| 200 | 8.08D−5 | 3.08D−8 | 3.24D−4 | 2.41D−6 | 1.87D−6 |
| 400 | 1.31D−2 | 1.79D−5 | 8.08D−5 | 3.26D−8 | 8.88D−9 |
| 800 | 3.99D−6 | 1.80D−3 | 2.02D−5 | 1.92D−9 | 2.72D−11 |
| 1600 | 1.10D−5 | 1.30D−1 | 5.05D−6 | 1.20D−10 | 9.92D−14 |

TABLE 2 ($q = 0.5$)

| | $M_1$ | $M_1$ | $M_2$ | $M_2$ | $M_2$ |
|---|---|---|---|---|---|
| $N$ | $\|u_{h,0}-u\|_{0,\infty}$ | $\|u_{h,1}-u\|_{0,\infty}$ | $\|u_{h,0}-u\|_{0,\infty}$ | $\|u_{h,1}-u\|_{0,\infty}$ | $\|u_{h,2}-u\|_{0,\infty}$ |
| 200 | 4.35D−5 | 1.30D−8 | 1.94D−4 | 2.62D−7 | 3.92D−8 |
| 400 | 1.36D−5 | 1.50D−5 | 5.43D−5 | 2.04D−8 | 1.02D−9 |
| 800 | 4.21D−6 | 6.05D−3 | 1.74D−5 | 2.09D−9 | 1.03D−11 |
| 1600 | 9.05D−6 | 3.03D−0 | 5.00D−6 | 1.71D−10 | 8.74D−14 |

TABLE 3 ($q = 0.2$)

| | $M_1$ | $M_1$ | $M_2$ | $M_2$ | $M_2$ |
|---|---|---|---|---|---|
| $N$ | $\|u_{h,0} - u\|_{0,\infty}$ | $\|u_{h,1} - u\|_{0,\infty}$ | $\|u_{h,0} - u\|_{0,\infty}$ | $\|u_{h,1} - u\|_{0,\infty}$ | $\|u_{h,2} - u\|_{0,\infty}$ |
| 200 | 4.89D−5 | 1.61D−8 | 2.15D−4 | 3.24D−7 | 5.31D−8 |
| 400 | 1.52D−5 | 1.06D−5 | 6.22D−5 | 2.61D−8 | 1.14D−9 |
| 800 | 4.63D−6 | 4.22D−3 | 1.88D−5 | 2.37D−9 | 8.78D−12 |
| 1600 | 1.45D−6 | 4.44D−0 | 5.60D−6 | 2.10D−10 | 8.48D−14 |

The above numerical results show that the correction approximations $u_{1,2}$ and $u_{h,2}$ based on the *hybrid meshes* introduced in this paper possess the superconvergence orders $4 - \varepsilon_N$ and $6 - \varepsilon_N$ respectively (with $\varepsilon_N \in [0, 0.678]$), which clearly confirm the multilevel correction estimates given in Theorem 2.1 and reveal that the method $M_2$ is effective for widely varying parameter values $q \in (0, 1)$. Note that such high superconvergence is obtained without expensive cost, since only the lowest piecewise polynomial ($m = 1$) is used in the collocation space. But the errors between the analytic solution $u$ and the correction approximation $u_{h,1}$ based on the original geometric meshes are not monotonic decreasing with mesh refinement. This is just the reason why we need to introduce the *hybrid meshes* to guarantee that the resulting $k$ level corrected approximation possesses an ideal superconvergence order (see Section 2 for the details).

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