Flexible Multi-scale Image Alignment Using B-Spline Reparametrization —-Theoretical Analysis *

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Abstract

In the first part of this companion volumes, we present a new flexible alignment model and an iterative algorithm based on an L^2-gradient flow. In this second part of the companion volumes, we focus on the theoretical analysis of our flexible alignment model presented in the first part. Three theoretical results are established. We first show that the correspondence of two images in the alignment model is an injective and surjective map under appropriate conditions. Secondly, we prove that the solution of the alignment model exists. Finally, we obtain the results on the existence and uniqueness of the solution for the ordinary differential equation derived from the finite element discretization of our flexible alignment model.

Key words: Flexible alignment, L^2-gradient flow, Bi-cubic B-spline, Existence and uniqueness, System of ordinary differential equations.

1 Introduction

Image alignment (or registration) is a fundamental task in image processing. It refers to establishing a geometric correspondence between two similar images. There are many situations where similar images are generated. One common example is that the similar images come from the same scene but taken at different sensors, or different time or different viewpoints. Another well-known example is that the similar images come from different cross sections of biological tissues [2, 24]. In the past few decades, with the rapid development of image acquisition devices and diversity of obtained images, image alignment has been used in many fields such as medical diagnoses, satellite remote sensing, weather forecast and computer vision [2, 28], and various techniques have been proposed to the alignment problem [8, 9, 10, 11, 14, 15, 25]. Once the correspondence between images is established, these images can be interpolated or compared for further study.

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Basically, the image alignment methods can be classified into two categories: rigid alignment and flexible alignment. The goal of rigid alignment is to find a few parameters such as rotation angle, scale parameter and translation components. Many rigid alignment methods have been proposed. These methods include intensity-matching based (see [13, 17, 21, 23]) and feature-matching based (see [6, 16]) techniques. Comparing with rigid alignment, flexible alignment in general is more difficulty. The aim of flexible alignment is to find a correspondence between two images with certain similarities. Several flexible alignment approaches have been proposed. Some of them are intensity-based elastic registration which was first proposed by Bajcsy and Kovacic [1, 4]. Some others include boundary mapping [5], identification of landmarks [19] and modal matching [20, 22].

The aim of this paper is to establish three theoretical results for our alignment model presented in the first part of the companion volumes [27]. In the first part, we present a new flexible alignment model and an iterative algorithm based on an L^2-gradient flow. The experiment results show that the proposed method is efficient, effective, robust and capable of capturing the variation of the initial and target images, from large to small scale. But the theoretical analysis has not been conducted. As far as we know, there are little literatures considering the existence and uniqueness about the flexible alignment problem. Many used algorithms were heuristic in nature: no proof was given of their correctness, and no attempt was made at the hypotheses under which they would work or not. We analyze in this paper our flexible alignment model from a theoretical point of view. Under appropriate conditions on the deformation $\mathbf{x}(u, v)$ we show that it is a one to one mapping and surjection. Based on the well-defined functional space, the solution of the energy model is studied. Furthermore, we prove the existence and uniqueness of the numerical solution.

The remaining of the paper is organized as follows. In section 2, we review our algorithm and list the theoretical results. In section 3, we give proof details of the regularity of the mapping $\mathbf{x}(u, v)$. Basing on the analysis results of section 3, we consider in section 4 the existence problem of the minimizer of the alignment model. An example is also presented to show that the minimizer may not be unique. In section 5, we discuss the existence and uniqueness problems of the solution for the ordinary differential equation system deduced from the finite element discretization. We conclude the paper in section 6.

2 Algorithm review and theoretical results

In this section we first review our alignment model and algorithm, and then summarize the main results of this paper.

2.1 L^2-gradient flows and theoretical results

Given two same size images $I_0(u, v)$ (target image) and $I_1(u, v)$ (initial image) defined on $[0, 1]^2 \subset \mathbb{R}^2$ with certain similarities. Suppose the size of the images is $(m + 1) \times (n + 1)$. We want to find a smooth mapping

$$\mathbf{x}(u,v) : [0,1]^2 \to [0,1]^2,$$

satisfying

(i) $\mathbf{x}$ is a C^2 mapping;
(ii) $\mathbf{x}(0,v) = [0,v]^T$, $\mathbf{x}(1,v) = [1,v]^T$, $\mathbf{x}(u,0) = [u,0]^T$ and $\mathbf{x}(u,1) = [u,1]^T$;
(iii) For a given $0 < \gamma < 1$, $\det(\mathbf{x}_u, \mathbf{x}_v) \geq \gamma$;

such that

$$\mathscr{E}(\mathbf{x}) = \int_0^1\int_0^1 (I_1(\mathbf{x}) - I_0)^2 dudv + \varepsilon \int_0^1\int_0^1 (g(\mathbf{x}) - 1)^2 dudv \tag{2.1}$$

is minimized, where $g(\mathbf{x}) = g_{11}g_{22} - g_{12}^2$ with $g_{11} = (\mathbf{x}_u)^{\mathrm{T}}\mathbf{x}_u$, $g_{12} = (\mathbf{x}_u)^{\mathrm{T}}\mathbf{x}_v$ and $g_{22} = (\mathbf{x}_v)^{\mathrm{T}}\mathbf{x}_v$. In this paper, we choose $\mathbf{x}(u,v)$ as a vector-valued bivariate bicubic B-spline function defined on $[0,1]^2$.

Assumption. Images are traditionally defined only on the integer grid. Here we assume $I_0(u,v)$ and $I_1(u,v)$ are defined as two continuous functions by assuming a bilinear interpolation is enforced in each of the pixels using the density values at the grid points.

To state the result, we need the following notation.

$$X = \Big\{\mathbf{x}(u,v) : \mathbf{x}(u,v) = [x_1(u,v), x_2(u,v)]^{\mathrm{T}} = \sum_{i=0}^{m+2}\sum_{j=0}^{n+2} \mathbf{a}_{ij} N_{i,3}(u) N_{j,3}(v) \;\; \text{satisfying} \;\; \text{(i)} - \text{(iii)}\Big\}.$$

We define the norm $\mathbf{x} \in X$ as:

$$\|\mathbf{x}\|_X = \left(\int_0^1\int_0^1 |\mathbf{x}|^2 dudv\right)^{\frac{1}{2}} = \left(\int_0^1\int_0^1 (x_1^2 + x_2^2) dudv\right)^{\frac{1}{2}}.$$

Then we have (see section 3-4)

Theorem 2.1 $\mathbf{x} : [0,1]^2 \to [0,1]^2$ *satisfying (i)-(iii) is an injection and surjection.*

Theorem 2.2 *There exists a mapping* $\mathbf{x}_0(u,v) \in X$ *such that (2.1) is minimized.*

Remark 2.1 *The uniqueness of* $\mathbf{x}_0(u,v)$ *that minimizes (2.1) is not guaranteed. To illustrate this, we present an example. Suppose* $\mathbf{x}(u,v) = \sum_{i=0}^{2}\sum_{j=0}^{2} \mathbf{x}_{ij} B_i^2(u) B_j^2(v)$ *is a bi-quadratic Bézier patch, with* $\mathbf{x}_{00} = [0,0]^{\mathrm{T}}$, $\mathbf{x}_{01} = [0,\frac{1}{2}]^{\mathrm{T}}$, $\mathbf{x}_{02} = [0,1]^{\mathrm{T}}$, $\mathbf{x}_{10} = [\frac{1}{2},0]^{\mathrm{T}}$, $\mathbf{x}_{12} = [\frac{1}{2},1]^{\mathrm{T}}$, $\mathbf{x}_{20} = [1,0]^{\mathrm{T}}$, $\mathbf{x}_{21} = [1,\frac{1}{2}]^{\mathrm{T}}$, $\mathbf{x}_{22} = [1,1]^{\mathrm{T}}$, $\mathbf{x}_{11}$ *is a free parameter. The given initial image* I_1 *is shown in Fig 2.1 (a). The initial image* I_1 *is symmetrical. Assuming* $\mathbf{x}_{11} = [u_{11}, v_{11}]^{\mathrm{T}}$ *such that* $\mathbf{x}(u,v)$ *is the*

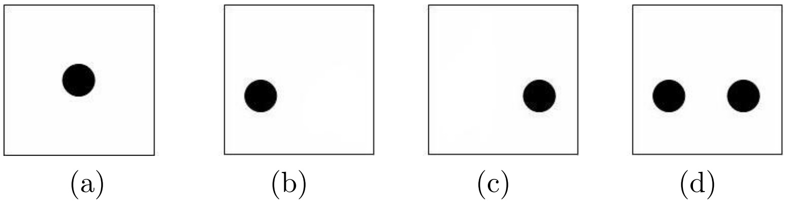

Fig 2.1: (a) is the initial image I_1. (b) is the image I_0'. (c) is the image I_0''. (d) is the target image I_0.

minimization solution that makes the disk of I_1 move toward the disk of I_0' (see Fig 2.1 (b)). Thus we conclude that $\mathbf{x}(u,v)$ is the solution minimizing (2.1), because

$$\begin{aligned} \mathscr{E}(\mathbf{x}) &= \int_0^1\int_0^1 (I_1(\mathbf{x}) - I_0)^2 dudv + \varepsilon \int_0^1\int_0^1 (g(\mathbf{x})-1)^2 dudv \\ &= \int_0^1\int_0^1 (I_1(\mathbf{x}) - I_0' + I_0' - I_0)^2 dudv + \varepsilon \int_0^1\int_0^1 (g(\mathbf{x})-1)^2 dudv \\ &= \int_0^1\int_0^1 (I_1(\mathbf{x}) - I_0')^2 dudv + \int_0^1\int_0^1 (I_0' - I_0)^2 dudv + \varepsilon \int_0^1\int_0^1 (g(\mathbf{x})-1)^2 dudv. \end{aligned} \tag{2.2}$$

Let $\tilde{\mathbf{x}}_{11} = [1-u_{11}, v_{11}]^{\mathrm{T}}$, then the mapping $\tilde{\mathbf{x}} = \sum_{\substack{i,j=0 \\ (i,j)\neq(1,1)}}^{2} \mathbf{x}_{ij} B_i^2(u) B_j^2(v) + \tilde{\mathbf{x}}_{11} B_1^2(u) B_1^2(v)$ makes the disk of I_1 move toward the disk of I_0'' (see Fig. 2.1 (c)). Similarly, we deduce that $\tilde{\mathbf{x}}(u,v)$ is the solution minimizing (2.1). It is obvious that $\mathbf{x}(u,v) \neq \tilde{\mathbf{x}}(u,v)$. Thus we conclude that $\mathbf{x}(u,v)$ and $\tilde{\mathbf{x}}(u,v)$ are two solutions minimizing (2.1).

Now we construct an L^2-gradient flow to minimize the energy functional $\mathscr{E}(\mathbf{x})$. Let

$$\underline{\mathbf{x}}(u,v,\varepsilon) = \mathbf{x} + \varepsilon\Phi(u,v), \quad \Phi \in C_0^1([0,1]^2)^2.$$

Then we have

$$\begin{aligned} \delta(\mathscr{E}(\mathbf{x}), \Phi) &= \frac{d}{d\varepsilon}\mathscr{E}(\underline{\mathbf{x}}(\cdot,\varepsilon))\Big|_{\varepsilon=0} \\ &= 2\int_0^1\int_0^1 \left((I_1(\mathbf{x}) - I_0)(\nabla_{\mathbf{x}} I_1)^T \Phi\right) dudv \\ &\quad +2\varepsilon \int_0^1\int_0^1 (\Phi_u^{\mathrm{T}}\alpha + \Phi_v^{\mathrm{T}}\beta) dudv, \end{aligned} \tag{2.3}$$

where

$$\alpha = 2(g(\mathbf{x})-1)(g_{22}\mathbf{x}_u - g_{12}\mathbf{x}_v), \qquad \beta = 2(g(\mathbf{x})-1)(g_{11}\mathbf{x}_v - g_{12}\mathbf{x}_u).$$

Let $D_1 = \frac{\partial}{\partial u}$, $D_2 = \frac{\partial}{\partial v}$. To construct an L^2-gradient flow moving $\mathbf{x}$ in the tangent $D_l\mathbf{x}$ direction $(l = 1, 2)$, we take

$$\Phi = (D_l\mathbf{x})(D_l\mathbf{x})^{\mathrm{T}}\phi, \quad \phi \in C_0^1([0,1]^2), \quad l = 1, 2. \tag{2.4}$$

Therefore, we obtain the following weak-form L^2-gradient flow moving $\mathbf{x}$ in the $D_l\mathbf{x}$ direction,

$$\begin{aligned}\int_0^1\int_0^1 \frac{\partial \mathbf{x}}{\partial t}\phi dudv &= -2\int_0^1\int_0^1 \left((I_1(\mathbf{x})-I_0)(D_l\mathbf{x})^{\mathrm{T}}(\nabla_{\mathbf{x}}I_1)(D_l\mathbf{x})\phi\right) dudv \\ &\quad -2\varepsilon\int_0^1\int_0^1 (\Phi_u^{\mathrm{T}}\alpha+\Phi_v^{\mathrm{T}}\beta)dudv, \quad l=1,2,\end{aligned} \tag{2.5}$$

where t is a time parameter introduced in $\mathbf{x}$.

2.2 Numerical solutions and theoretical results

To minimize $\mathscr{E}(\mathbf{x})$, we solve (2.5) interchangeably using finite element method in the spatial discretization and explicit Euler scheme in the temporal discretization. Let

$$\mathbf{x}(u,v)=\sum_{j=0}^{n_0}\mathbf{x}_j\phi_j(u,v)+\sum_{j=n_0+1}^{n_1}\mathbf{x}_j\phi_j(u,v) \tag{2.6}$$

be the cubic B-spline representation of $\mathbf{x}(u,v)$, where ϕ_j is the tensor product form B-spline basis function, $\mathbf{x}_0,\cdots,\mathbf{x}_{n_0}$ are unknown control points, n_1 is the number of all the control points.

According to the specific feature of the given images, we can set three types of boundary conditions: the first one is that four boundaries are fixed; the second case is that two vertical boundaries are fixed; the third is that two horizontal boundaries are fixed.
Substituting $\mathbf{x}(u,v)$ into (2.5), and taking the test function ϕ as ϕ_i, for $i=0,\cdots,n_0$, we can discretize (2.5) as a set of nonlinear systems of ordinary differential equations (ODE) with the internal control points $\mathbf{x}_j, j=0,\cdots,n_0$, as unknowns.

$$\sum_{j=0}^{n_0} m_{ij}\frac{d\mathbf{x}_j(t)}{dt}=-q_i^{(l)}, \quad i=0,\cdots,n_0, \tag{2.7}$$

for $l=1,2$, where

$$m_{ij}=\int_0^1\int_0^1 \phi_i\phi_j I_2 dudv,$$

$$\begin{aligned}q_i^{(l)} &= 2\varepsilon\int_0^1\int_0^1 (\Phi_u^{\mathrm{T}}\alpha+\Phi_v^{\mathrm{T}}\beta)dudv \\ &\quad +2\int_0^1\int_0^1 \left((I_1(\mathbf{x})-I_0)(D_l\mathbf{x})^{\mathrm{T}}(\nabla_{\mathbf{x}}I_1)(D_l\mathbf{x})\phi_i\right) dudv.\end{aligned}$$

For equation (2.7), the following result can be proved (see section 5).

Theorem 2.3 *For $\delta>0$ given by (5.6), there exists a unique solution of the problem (2.7) for a given initial map I on $0\le t\le\delta$.*

For the temporal direction discretization of these ODE systems, we use a forward Euler scheme

$$\frac{d\mathbf{x}_j(t)}{dt} \approx \frac{\mathbf{x}_j^{(s)} - \mathbf{x}_j^{(s-1)}}{\tau}, \tag{2.8}$$

where τ is a temporal step-size. Solving the linear system using the inverse of the matrix $[m_{ij}]$ for $l = 1, 2$, we obtain $\frac{d\mathbf{x}_j(t)}{dt}$ and then the new inner control points of $\mathbf{x}$ from (2.8).

3 Regularity analysis of mapping x

In this section, we first introduce the used definitions, terminologies and theorems. Then we provide the details of proof for Theorem 2.1.

3.1 The used definitions and theorems

First, we introduce the used definitions and theorems in this section.

Definition 3.1 *[18] Let X and Y be topological spaces; let f:$X \to Y$ be a bijection. If both the function f and the inverse function f^{-1}:$Y \to X$ are continuous, then f is called a homeomorphism. f is a local homeomorphism, if for every point x in X, there exists an open set U containing x, such that $f(U)$ is open in Y and is a homeomorphism.*

Definition 3.2 *[7] Let $\tilde{B}$ and B be subsets of $\mathbb{R}^2$. We say that $\pi : \tilde{B} \to B$ is a covering map if*
1. π is continuous and $\pi(\tilde{B}) = B$.
2. Each point $p \in B$ has a neighborhood U in B (to be called a distinguished neighborhood of p) such that

$$\pi^{-1}(U) = \bigcup_{\alpha} V_\alpha,$$

where the V_α's are pairwise disjoint open sets such that the restriction of π to V_α is a homeomorphism of V_α onto U.

Definition 3.3 *[26] Assuming that (X, ρ) is a metric space, A is a subspace of X. If every sequence in A has a convergence subsequence and the limit point of the convergence subsequence lies in A, we named that A is a self-sequentially compact set.*

Definition 3.4 *[7] $A \subset \mathbb{R}^n$ is arcwise connected if, given two points $p, q \in A$, there exists an arc in A joining p to q.*

Definition 3.5 *[7] $A \subset \mathbb{R}^n$ is connected when it is not possible to write $A = U_1 \cup U_2$, where U_1 and U_2 are nonempty open sets in A and $U_1 \cap U_2 = \emptyset$.*

Theorem 3.1 *[3] Let T be of class C^1 in a set D with $J(p) \neq 0$ for each $p \in D$, and let T map D one-to-one onto a set $T(D)$. Then, the inverse T^{-1} of T is of class C^1 on $T(D)$ and the differential of T^{-1} is $(dT)^{-1}$, the inverse of the differential of T. $J(p)$ denotes the Jacobian determinant of T in the point p.*

Theorem 3.2 *[7] Let $\pi : \widetilde{B} \to B$ be a local homeomorphism, $\widetilde{B}$ compact and B connected. Then π is a covering map.*

Theorem 3.3 *[7] Let $\pi : \widetilde{B} \to B$ be a covering map, $\widetilde{B}$ arcwise connected, and B simply connected. Then π is a homeomorphism.*

3.2 Theoretical analysis of mapping $\mathbf{x}$

Now, we prove the correspondence $\mathbf{x}(u, v)$ is an injection and surjection.

Remark 3.1 $[0,1]^2$ *is regarded as a topological space, i.e.* $[0,1]^2$ *is a clopen set.*

Lemma 3.1 $\mathbf{x} : [0,1]^2 \to [0,1]^2$ *satisfying (i)-(iii) is a locally one to one mapping.*

Proof. The idea of the proof is borrowed from [3] . Let a point $p \in [0,1]^2$, we determine a neighborhood B of p in which $\mathbf{x}$ is one to one. Let p' and p'' be two points near p such that the line segment jointing p' and p'' lies in $[0,1]^2$. According to the mean value theorem, we may choose two points p_1^*, p_2^* on this line segment such that

$$\mathbf{x}(p'') - \mathbf{x}(p') = L(p'' - p'), \tag{3.1}$$

where L is the linear transformation represented by $L = (\mathbf{x}_u(p_1^*), \mathbf{x}_v(p_2^*))$.
Let

$$F(p_1, p_2) = \det(\mathbf{x}_u(p_1), \mathbf{x}_v(p_2)),$$

then $F(p_1^*, p_2^*) = \det(L)$. Moreover, since $\mathbf{x}$ is C^2, then F is C^1 continuous, and $F(p, p) \geq \gamma$, there exists a circular neighborhood B of p lying in $[0,1]^2$, such that $F(p_1, p_2) \geq \gamma$ for all choices of the points p_1, p_2 in B. We shall prove that $\mathbf{x}$ is a one to one mapping in B. Assuming that p' and p'' lies in B and $\mathbf{x}(p') = \mathbf{x}(p'')$, we will prove $p' = p''$. Since p' and p'' lies in B and B is convex, the entire line segment joining p' to p'' also lies in B, hence both p_1^* and p_2^* are points of B. Using the property of B, we have $F(p_1^*, p_2^*) = \det(L) \neq 0$. The linear transformation L is therefore nonsingular. According to (3.1) and using the assumption that $\mathbf{x}(p') = \mathbf{x}(p'')$, we have $L(p'' - p') = 0$. Since L is nonsingular, therefore we deduce that $p' = p''$, i.e. $\mathbf{x}$ is a one to one mapping in B.

Lemma 3.2 $\mathbf{x} : [0,1]^2 \to [0,1]^2$ *satisfying (i)-(iii) is a locally homeomorphism.*

Proof. According to Lemma 3.1, $\mathbf{x}$ is a locally one to one mapping in $[0,1]^2$. Given a point $p \in [0,1]^2$, assuming that $\mathbf{x}$ is one to one in a neighborhood B of p, then it is obvious that $\mathbf{x} : B \to \mathbf{x}(B)$ is surjective, where $\mathbf{x}(B)$ denotes the range of $\mathbf{x}$ in B. And from Theorem 3.1, we know that $\mathbf{x}^{-1}$ is continuous in $\mathbf{x}(B)$. Thus, $\mathbf{x}$ is homeomorphic in B, i.e. $\mathbf{x}$ is a locally homeomorphism.
Proof of Theorem 2.1 It is obvious that $[0,1]^2$ is connected and compact. From Lemma 3.2 and Theorem 3.2, we deduce that $\mathbf{x}$ is a covering map. And since $[0,1]^2$ is arcwise connected and simply connected, according to Theorem 3.3, $\mathbf{x}$ is a homeomorphism. Thus, we obtain that $\mathbf{x}$ is an injection and surjection. The result is deduced.

4 Existence and uniqueness of x

This section devotes to the proof of Theorem 2.2.

Lemma 4.1 *Let* $\rho(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_X$*, then* X *is a metric space.*

Proof. It is obvious that X is a nonempty set, and $\rho(\mathbf{x},\mathbf{y})$ satisfies,
(i) $\rho(\mathbf{x},\mathbf{y}) \geq 0$, and $\rho(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$;
(ii)$\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{y},\mathbf{x})$;

$$\begin{aligned}
\text{(iii)}\rho(\mathbf{x},\mathbf{z}) \quad &= \left(\int_0^1\int_0^1 |\mathbf{x}-\mathbf{z}|^2 dudv\right)^{\frac{1}{2}} \\
&= \left(\int_0^1\int_0^1 |\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{z}|^2 dudv\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1\int_0^1 |\mathbf{x}-\mathbf{y}|^2 dudv\right)^{\frac{1}{2}} + \left(\int_0^1\int_0^1 |\mathbf{y}-\mathbf{z}|^2 dudv\right)^{\frac{1}{2}} \\
&= \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z}).
\end{aligned}$$

Hence X is a metric space. Under the distance ρ, we denote this metric space as (X,ρ).

Lemma 4.2 (X,ρ) *is a closed set.*

Proof. Suppose that $\{\mathbf{x}_k\}$ is a fundamental sequence in the space (X,ρ). This sequence can be written as $\mathbf{x}_k(u,v) = \sum_{i=0}^{m+2}\sum_{j=0}^{n+2} (\mathbf{a}_{ij})_k N_{i,3}(u)N_{j,3}(v)$. It is easy to deduce that $(\mathbf{a}_{ij})_k$ are bounded, hence there exists a subsequence $(\mathbf{a}_{ij})_{k_l}$ converging to $(\mathbf{a}_{ij})_0$. Because $\{\mathbf{x}_k\}$ is a fundamental sequence, we obtain that,

$$\begin{aligned}\lim_{k\to\infty}\mathbf{x}_k(u,v) &= \lim_{l\to\infty}\mathbf{x}_{k_l}(u,v)\\ &= \lim_{l\to\infty}\sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_{k_l}N_{i,3}(u)N_{j,3}(v)\\ &= \sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_0N_{i,3}(u)N_{j,3}(v).\end{aligned}$$

Let $\mathbf{x}_0(u,v)=\sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_0N_{i,3}(u)N_{j,3}(v)$. Because the range of $\mathbf{x}_k$ is $[0,1]^2$ which is a closed set, we obtain that $\mathbf{x}_0(u,v)\in[0,1]^2$.

On the other hand, it is obvious that $\mathbf{x}_0(0,v)=[0,v]^T$, $\mathbf{x}_0(1,v)=[1,v]^T$, $\mathbf{x}_0(u,0)=[u,0]^T$ and $\mathbf{x}_0(u,1)=[u,1]^T$. Moreover, because $\det((\mathrm{x_k})_\mathrm{u},(\mathrm{x_k})_\mathrm{v})\geq\gamma$, hence $\det((\mathbf{x}_0)_u,(\mathbf{x}_0)_v)=\lim_{k\to\infty}\det((\mathbf{x}_k)_u,(\mathbf{x}_k)_v)\geq\gamma$, i.e. $\mathbf{x}_0(u,v)\in X$. Therefore, X is a closed set.

Lemma 4.3 *X is a self-sequentially compact set.*

Proof. Because the number of the bicubic B-spline bases is finite, X is a finite dimensional space. On the other hand, $\mathbf{x}(u,v):[0,1]^2\to[0,1]^2$, hence $\|\mathbf{x}(u,v)\|\leq 2$. According to Lemma 4.2, X is a closed set. Then we conclude that X is a self-sequentially compact set.

Proof of Theorem 2.2 Let $\mathbf{x}_n$ be a minimizing sequence for the model (2.1), i.e.

$$\lim_{n\to\infty}\mathscr{E}(\mathbf{x}_n)=\inf_{\mathbf{x}\in X}\mathscr{E}(\mathbf{x}).$$

According to Lemma 4.3, X is a self-sequentially compact set. Thus, there exists a subsequence $\mathbf{x}_{n_k}$ and $\mathbf{x}_0$ in X such that $\mathbf{x}_{n_k}\to\mathbf{x}_0$.

Finally, because $I_1(\mathbf{x})$ is defined as a continuous function, X is a bounded closed set, hence $I_1(\mathbf{x})$ is a uniformly continuous function with respect to $\mathbf{x}$. For every $\varepsilon>0$, there exists a $\delta>0$ such that $|I_1(\mathbf{x})-I_1(\mathbf{y})|<\varepsilon$ when $|\mathbf{x}-\mathbf{y}|<\delta$. Therefore,

$$\begin{aligned}&\left|\int_0^1\!\!\int_0^1|I_1(\mathbf{x})-I_0|^2dudv-\int_0^1\!\!\int_0^1|I_1(\mathbf{y})-I_0|^2dudv\right|\\ &\leq\int_0^1\!\!\int_0^1|I_1(\mathbf{x})-I_1(\mathbf{y})||I_1(\mathbf{x})+I_1(\mathbf{y})-2I_0|dudv\leq M\varepsilon\end{aligned}$$

where M is a constant. Similarly, we can get that $\int_0^1\!\int_0^1(g(\mathbf{x}(u,v))-1)^2dudv$ is continuous about $\mathbf{x}$. Hence the energy functional

$$\mathscr{E}(\mathbf{x})=\int_0^1\!\!\int_0^1|I_1(\mathbf{x})-I_0|^2dudv+\varepsilon\int_0^1\!\!\int_0^1(g(\mathbf{x}(u,v))-1)^2dudv$$

is continuous with respect to $\mathbf{x}$. Thus,

$$\mathscr{E}(\mathbf{x_0})=\lim_{k\to\infty}\mathscr{E}(\mathbf{x}_{n_k})=\inf_{\mathbf{x}\in X}\mathscr{E}(\mathbf{x}),$$

i.e., $\mathbf{x}_0$ is a minimum point of $\varepsilon(\mathbf{x})$.

5 Existence and uniqueness of ODE's solution

In this section, we show that equation (2.7) has a unique solution. First, we introduce Gronwall's inequality.

Gronwall's inequality [12] If (i) $g(t)$ is continuous on $t_0 \leq t \leq t_1$, (ii) $g(t)$ satisfies the inequality

$$0 \leq g(t) \leq K + L\int_{t_0}^{t} g(s)ds \quad \text{on } t_0 \leq t \leq t_1,$$

then

$$0 \leq g(t) \leq Ke^{L(t-t_0)} \quad \text{on} \quad t_0 \leq t \leq t_1.$$

Hence, if $K = 0$, then $g(t) = 0$.

Now we prove the existence and uniqueness for the equation (2.7). The problem (2.7) can be written as

$$\begin{cases} \dfrac{d\vec{x}(t)}{dt} = M^{-1}\vec{Q}(\vec{x}(t)), \\ \vec{x}(0) = \vec{c}_0, \end{cases} \tag{5.1}$$

where

$$\vec{x}(t) = \left[\mathbf{x}_0^{\mathrm{T}}(t), \mathbf{x}_1^{\mathrm{T}}(t), \cdots, \mathbf{x}_{n_0-1}^{\mathrm{T}}(t), \mathbf{x}_{n_0}^{\mathrm{T}}(t)\right]^{\mathrm{T}},$$

$$M^{-1} = [m_{ij}]^{-1} \otimes I_2, \qquad i, j = 1, 2, \cdots, n_0,$$

$$\vec{Q}(\vec{x}(t)) = -\left[\mathbf{q}_0^{\mathrm{T}}, \mathbf{q}_1^{\mathrm{T}}, \cdots, \mathbf{q}_{n_0-1}^{\mathrm{T}}, \mathbf{q}_{n_0}^{\mathrm{T}}\right]^{\mathrm{T}},$$

and

$$\vec{c}_0 = \left[\mathbf{c}_0^{\mathrm{T}}, \mathbf{c}_1^{\mathrm{T}}, \cdots, \mathbf{c}_{n_0-1}^{\mathrm{T}}, \mathbf{c}_{n_0}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{5.2}$$

with $\vec{c}_0$ satisfing

$$\sum_{j=0}^{n_0} \mathbf{c}_j \phi_j(u,v) + \sum_{j=n_0+1}^{n_1} \mathbf{x}_j \phi_j(u,v) = I(u,v), \tag{5.3}$$

where $I(u,v) = [u,v]^{\mathrm{T}}$.

Remark 5.1 *Because* $\mathbf{x}_{n_0+1}$, $\mathbf{x}_{n_0+2}$, $\cdots$, $\mathbf{x}_{n_1}$ *are fixed on the whole iterative process, hence* $\mathbf{x} = [\mathbf{x}_{n_0+1}^{\mathrm{T}}, \mathbf{x}_{n_0+2}^{\mathrm{T}}, \cdots, \mathbf{x}_{n_1-1}^{\mathrm{T}}, \mathbf{x}_{n_1}^{\mathrm{T}}]^{\mathrm{T}}$ *is a constant vector.*

For the simplicity, we denote $\vec{f}(\vec{x}(t)) = M^{-1}\vec{Q}(\vec{x}(t))$ where $\vec{f}(\vec{x}(t)) = [\mathbf{f}_0^{\mathrm{T}}(\vec{x}(t)), \mathbf{f}_1^{\mathrm{T}}(\vec{x}(t)) \cdots, \mathbf{f}_{n_0-1}^{\mathrm{T}}(\vec{x}(t)), \mathbf{f}_{n_0}^{\mathrm{T}}(\vec{x}(t))]^{\mathrm{T}}$. Now we prove the existence and uniqueness of the solution for the equation (5.1). First, a positive number δ needs to be determined using the mapping $\mathbf{x}(u, v)$. Integrating both sides of (5.1), we have

$$\vec{x}(t) = \vec{c}_0 + \int_0^t \vec{f}(\vec{x}(t))dt.$$

Taking inner product of both sides with $[\phi_0, \cdots, \phi_{n_0}]^{\mathrm{T}}$ and using (5.3), we have

$$\mathbf{x}(u, v) = I(u, v) + \sum_{j=0}^{n_0} \int_0^t \mathbf{f}_j(\vec{x}(s))\phi_j(u, v)ds.$$

Let $\mathbf{F}(\vec{x}(t), u, v) = \sum_{j=0}^{n_0} \mathbf{f}_j(\vec{x}(t))\phi_j(u, v)$. Then it is obvious that $\mathbf{F}(\vec{x}(t), u, v)$ is continuous with respect to t. According mean value theorem,

$$\int_0^t \sum_{j=0}^{n_0} f_j^{(l)}(\vec{x}(s))\phi_j(u, v)ds = tF^{(l)}(\vec{x}(\xi_l), u, v), \quad where \;\; 0 < \xi_l < t, \qquad l = 1, 2.$$

Then let

$$\det(\mathbf{x}_u, \mathbf{x}_v) = \begin{vmatrix} 1 + tF_u^{(1)}(\vec{x}(\xi_1)) & tF_v^{(1)}(\vec{x}(\xi_1)) \\ tF_u^{(2)}(\vec{x}(\xi_2)) & 1 + tF_v^{(2)}(\vec{x}(\xi_2)) \end{vmatrix} = \gamma,$$

By computing the determinant, we get

$$\begin{array}{l}(F_u^{(1)}(\vec{x}(\xi_1))F_v^{(2)}(\vec{x}(\xi_2)) - F_u^{(2)}(\vec{x}(\xi_2))F_v^{(1)}(\vec{x}(\xi_1)))t^2 \\ +(F_u^{(1)}(\vec{x}(\xi_1)) + F_v^{(2)}(\vec{x}(\xi_2)))t + 1 - \gamma = 0.\end{array} \tag{5.4}$$

For convenience, let

$$\begin{aligned} a(\mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) &= F_u^{(1)}(\vec{x}(\xi_1))F_v^{(2)}(\vec{x}(\xi_2)) - F_u^{(2)}(\vec{x}(\xi_2))F_v^{(1)}(\vec{x}(\xi_1)), \\ b(\mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) &= F_u^{(1)}(\vec{x}(\xi_1)) + F_v^{(2)}(\vec{x}(\xi_2)), \end{aligned}$$

where $\mathbf{x}_{\xi_1}$ and $\mathbf{x}_{\xi_2}$ are the mappings in X defined by the coefficients $\vec{x}(\xi_1)$ and $\vec{x}(\xi_2)$, respectively. Then we can get a minimal positive root as following,

$$t(\mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) = \frac{2(1-\gamma)}{-b(\mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) + \sqrt{b^2(\mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) - 4a(\mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2})(1-\gamma)}}. \tag{5.5}$$

Let

$$\begin{aligned} \Omega(\mathbf{x}, \mathbf{y}) = \Big\{ & [u, v]^T \in [0, 1]^2 : b^2(\mathbf{x}(u, v), \mathbf{y}(u, v)) - 4a(\mathbf{x}(u, v), \mathbf{y}(u, v))(1-\gamma) \geq 0, \\ & \sqrt{b^2(\mathbf{x}(u, v), \mathbf{y}(u, v)) - 4a(\mathbf{x}(u, v), \mathbf{y}(u, v))(1-\gamma)} \geq b(\mathbf{x}(u, v), \mathbf{y}(u, v)) \Big\}. \end{aligned}$$

Then $\Omega(\mathbf{x}, \mathbf{y})$ is closed set.

Lemma 5.4 *Let*

$$\delta = \min_{\mathbf{x},\mathbf{y}\in X} \min_{[u,v]^T\in\Omega(\mathbf{x},\mathbf{y})} t(\mathbf{x},\mathbf{y}). \tag{5.6}$$

then $\delta > 0$. *Furthermore, if* $0 \le t \le \delta$, *then* $det(\mathbf{x}_u, \mathbf{x}_v) \ge \gamma$.

Proof. Let

$$y(t) = det(\mathbf{x}_u, \mathbf{x}_v) - \gamma = at^2 + bt + 1 - \gamma, \quad \mathbf{F}(\vec{x}(t), u, v) = \sum_{j=0}^{n_0} \mathbf{f}_j(\vec{x}(t))\phi_j(u,v).$$

We can easily prove that $\mathbf{F}(\vec{x}(t), u, v)$ are C^2 continuous functions in the space X and $[0,1]^2$. Because X and $[0,1]^2$ are closed sets, hence $F_u^{(l)}(\vec{x}(\xi_l), u, v)$ and $F_v^{(l)}(\vec{x}(\xi_l), u, v)$ $(l = 1, 2)$ are bounded functions. $\delta > 0$ is proved. On the other hand, because $y(0) = 1 - \gamma > 0$, therefore $y(t) > 0$ is true when $0 \le t \le \delta$.

Defining a region $\Re$ as following,

$$\Re = \left\{ \vec{x}(t) : det(\mathbf{x}_u, \mathbf{x}_v) \ge \gamma, \text{where } \ \mathbf{x} = \sum_{j=0}^{n_0} \mathbf{x}_j\phi_j(u,v) + \sum_{j=n_0+1}^{n_1} \mathbf{x}_j\phi_j(u,v) \right\}. \tag{5.7}$$

Lemma 5.5 $M^{-1}\vec{Q}(\vec{x}(t))$ *is Lipschitz continuous with respect to* $\vec{x}$ *on* $\Re$ *when* $0 \le t \le \delta$.

Proof. Because

$$D_1\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u} = \sum_{j=0}^{n_1} \mathbf{x}_j \frac{\partial \phi_j(u,v)}{\partial u}, \quad D_2\mathbf{x} = \frac{\partial \mathbf{x}}{\partial v} = \sum_{j=0}^{n_1} \mathbf{x}_j \frac{\partial \phi_j(u,v)}{\partial v}.$$

It is obvious that $D_l\mathbf{x}$ are the polynomials of $\mathbf{x}_j(t)$ $(j = 0, 1, \cdots, n_0)$, $l = 1, 2$.

Compute

$$(D_1\mathbf{x})_u = \frac{\partial^2 \mathbf{x}}{\partial u^2} = \sum_{j=0}^{n_1} \mathbf{x}_j \frac{\partial^2 \phi_j(u,v)}{\partial u^2}, \quad (D_1\mathbf{x})_v = \frac{\partial^2 \mathbf{x}}{\partial u \partial v} = \sum_{j=0}^{n_1} \mathbf{x}_j \frac{\partial^2 \phi_j(u,v)}{\partial u \partial v},$$

we deduce that $(D_1\mathbf{x})_u$ and $(D_1\mathbf{x})_v$ are the polynomials of $\mathbf{x}_j(t)$ $(j = 0, 1, \cdots, n_0)$. Similarly, $(D_2\mathbf{x})_u$ and $(D_2\mathbf{x})_v$ are the polynomials of $\mathbf{x}_j(t)$ $(j = 0, 1, \cdots, n_0)$. From

$$\Phi_u = (D_l\mathbf{x})_u(D_l\mathbf{x})^{\mathrm{T}}\phi_i + (D_l\mathbf{x})(D_l\mathbf{x})_u^{\mathrm{T}}\phi_i + (D_l\mathbf{x})(D_l\mathbf{x})^{\mathrm{T}}(\phi_i)_u,$$

it is easy to deduce that Φ_u is the polynomial of $\mathbf{x}_j(t)$. Similarly Φ_v and $g(\mathbf{x})$ are the polynomials of $\mathbf{x}_j(t)$. Because on the interval $[0, \delta]$, $\Re$ is a closed set, hence we conclude that $\int_0^1\int_0^1 (\Phi_u^{\mathrm{T}}\alpha + \Phi_v^{\mathrm{T}}\beta) dudv$ is Lipschitz continuous on $\Re$. Furthermore, since $I_1(\mathbf{x})$ and I_0 are C^2, $\int_0^1\int_0^1 \big((I_1(\mathbf{x}) - I_0)(D_l\mathbf{x})^{\mathrm{T}}(\nabla_{\mathbf{x}} I_1)(D_l\mathbf{x})\phi_i\big)\, dudv$ is Lipschitz continuous on $\Re$. Thus, $M^{-1}\vec{Q}(\vec{x}(t))$ is Lipschitz continuous on $\Re$.

Now we construct successive approximations which are defined as follows:

$$\begin{cases} \vec{x}_{k+1}(t) = \vec{c}_0 + \int_0^t \vec{f}(\vec{x}_k(s))ds, & k = 0, 1, 2, \cdots \\ \vec{x}_0(t) = \vec{c}_0, \end{cases} \tag{5.8}$$

Theorem 5.1 *Each function $\vec{x}_k$ defined by (5.8) lies in $\Re$ for $0 \le t \le \delta$, $k = 0, 1, 2, \cdots$.*

Proof. Because $\vec{x}_0(t) = \vec{c}_0$ and the equality (5.3), we deduce that $\vec{x}_0$ lies in $\Re$. Moreover, according to

$$\vec{x}_{k+1}(t) = \vec{c}_0 + \int_0^t \vec{f}(\vec{x}_k(s))ds, \tag{5.9}$$

we obtain that

$$\begin{aligned} \mathbf{x}_{k+1} &= \sum_{j=0}^{n_0} (\mathbf{x}_{k+1})_j(t)\phi_j(u,v) + \sum_{j=n_0+1}^{n_1} \mathbf{x}_j\phi_j(u,v) \\ &= I(u,v) + \sum_{j=0}^{n_0} \int_0^t \mathbf{f}_j(\vec{x}_k(s))\phi_j(u,v)ds. \end{aligned}$$

Let $\mathbf{F}_k(t,u,v) = \sum_{j=0}^{n_0} \mathbf{f}_j(\vec{x}_k(t))\phi_j(u,v)$. It is obvious that $\mathbf{F}_k(t,u,v)$ is continuous with respect to t. According the mean value theorem,

$$\int_0^t \sum_{j=0}^{n_0} f_j^{(l)}(\vec{x}_k(s))\phi_j(u,v)ds = tF_k^{(l)}(\xi_l, u, v), \quad l = 1, 2,$$

where $0 \le \xi_l \le t \le \delta$. From Lemma 5.4, we have

$$\det((\mathbf{x}_{k+1})_u, (\mathbf{x}_{k+1})_v) = \begin{vmatrix} 1 + t(F_k)_u^{(1)}(\xi_1) & t(F_k)_v^{(1)}(\xi_1) \\ t(F_k)_u^{(2)}(\xi_2) & 1 + t(F_k)_v^{(2)}(\xi_2) \end{vmatrix} \ge \gamma \ \ when \ \ 0 \le t \le \delta.$$

Thus, $\vec{x}_{k+1} \in \Re$ for $0 \le t \le \delta$.

The proofs of Lemma 5.6, Lemma 5.7 in the following are similar as that of Theorem I-1-4 in [12]. For completeness, we give the details.

Lemma 5.6 *The successive approximations given by (5.8) satisfy the estimates*

$$|\vec{x}_{k+1}(t) - \vec{x}_k(t)| \le \frac{ML^k}{(k+1)!}t^{k+1} \quad for \quad 0 \le t \le \delta, \quad k = 0, 1, 2, \cdots . \tag{5.10}$$

Proof. We use mathematical induction. When $k = 0$, we have $|\vec{x}_1(t) - \vec{x}_0(t)| = |\int_0^t \vec{f}(\vec{c}_0)ds| \le Mt$. Assume that the result is true for k, then

$$\begin{aligned} |\vec{x}_{k+1}(t) - \vec{x}_k(t)| &= \left|\int_0^t \left(\vec{f}(\vec{x}_k(s)) - \vec{f}(\vec{x}_{k-1}(s))\right) ds\right| \\ &\le L\left|\int_0^t |\vec{x}_k(s) - \vec{x}_{k-1}(s)|ds\right| \\ &\le \frac{ML^k}{k!}\left|\int_0^t s^k ds\right| = \frac{ML^k}{(k+1)!}t^{k+1}. \end{aligned} \tag{5.11}$$

Lemma 5.7 *The sequence* $\vec{x}_k(t)$, $k = 0, 1, 2, \cdots$, *converges to*

$$\vec{x}(t) = \vec{c}_0(t) + \sum_{k=1}^{\infty}(\vec{x}_k(t) - \vec{x}_{k-1}(t)) \tag{5.12}$$

uniformly on $0 \le t \le \delta$ *as* $k \to \infty$.

Proof. According to Lemma 5.6, for a given $\varepsilon > 0$, there exists a positive integer N such that

$$\sum_{k=N}^{\infty} |\vec{x}_k(t) - \vec{x}_{k-1}(t)| \le \frac{M}{L}\sum_{k=N}^{\infty}\frac{(Lt)^k}{k!} \le \frac{M}{L}\sum_{k=N}^{\infty}\frac{(L\delta)^k}{k!} < \varepsilon. \tag{5.13}$$

Thus the series $\sum_{k=1}^{\infty}(\vec{x}_k(t) - \vec{x}_{k-1}(t))$ is uniformly convergent on $0 \le t \le \delta$. On the other hand, since $\vec{x}_N(t) = \vec{c}_0(t) + \sum_{k=1}^{N}(\vec{x}_k(t) - \vec{x}_{k-1}(t))$, the result is proved.

Proof of Theorem 2.3. Because

$$\vec{x}_{k+1}(t) = \vec{c}_0 + \int_0^t \vec{f}(\vec{x}_k(s))ds, \tag{5.14}$$

the continuity of $\vec{x}$ and Lemma 5.7, we conclude that

$$\vec{x}(t) = \vec{c}_0 + \int_0^t \vec{f}(\vec{x}(s))ds. \tag{5.15}$$

Then

$$\frac{d\vec{x}(t)}{dt} = \vec{f}(\vec{x}(t)), \tag{5.16}$$

hence $\vec{x}(t)$ is a solution of (5.1).

To prove the uniqueness, suppose that $\vec{y}(t)$ is another solution of problem (5.1) on the interval $[0, \delta]$. Note that

$$\vec{y}(t) = \vec{c}_0 + \int_0^t \vec{f}(\vec{y}(s))ds \tag{5.17}$$

on $0 \le t \le \delta$. Hence using the Lipschitz continuity of $\vec{f}$, we have

$$|\vec{x}(t) - \vec{y}(t)| = \left|\int_0^t (\vec{f}(\vec{x}(s)) - \vec{f}(\vec{y}(s))ds\right| \le L\int_0^t |\vec{x}(s) - \vec{y}(s)|ds \tag{5.18}$$

on $0 \le t \le \delta$. Applying Gronwall's inequality, we conclude that $|\vec{x}(t) - \vec{y}(t)| = 0$ on $0 \le t \le \delta$. Hence we complete the proof of the Theorem.

6 Conclusion

We have presented an algorithm for solving the flexible alignment problem in [27]. This paper analyzes the validity of the model from the theoretical point of view. We have proved the regularity of mapping $\mathbf{x}(u, v)$ under certain conditions. We also proved that there exists a mapping $\mathbf{x}_0(u, v) \in X$ satisfying (i)-(iii) such that the energy functional (2.1) is minimized. For the systems of the ordinary differential equations derived from the finite element discretization in the spatial direction, the existence and uniqueness of the solution have been proved.

References

[1] R. Bajcsy and S. Kovacic. Multiresolution elastic matching. *Computer Vision, Graphics and Image Processing*, 46:1–21, 1989.

[2] L. G. Brown. A survey of image registration techniques. *ACM Computing Surveys*, 24:326–376, 1992.

[3] R. C. Buck. *Advanced Calculus.* the McGraw-Hill Book Company, 1956.

[4] U. Clarenz, M. Droske, and M. Rumpf. Towards fast non-rigid registration. In *Inverse Problems, Image Analysis And Medical Imaging, AMS Special Session Interaction of Inverse Problems and Image Analysis*, pages 67–84, 2002.

[5] C. Davatzikos, J. L. Prince, and R. N. Bryan. Spatial transformation and registration of brain images using elastically deformable models. *Computer Vision and Image Understanding* , 66:207–222, 1997.

[6] P. A. Van den Elsen, J. B. A. Maintz, E. J. D. Pol, and M. A. Viergever. Automatic registration of CT and MR brain images using correlation of geometrical features. *IEEE Transactions on Medical Imaging*, 2:384–396, 1995.

[7] M. P. do Carmo. *Differential Geometry of Curves and Surfaces.* China Machine Press, 2004.

[8] L. M. G. Fonseca and B. S. Manjunath. Registration techniques for multisensor remotely sensed imagery. *Photogrammetric Engineering and Remote Sensing*, 62:1049–1056, 1996.

[9] E. Gülch. Results of test on image matching of isprs wg. *ISPRS Journal of Photogrammetry and Remote Sensing*, 46:1–18, 1991.

[10] A. A. Goshtasby. *2-D and 3-D image registration for medical, remote sensing, and industrial applications.* Wiley Press, 2005.

[11] D. L. G. Hill, P. G. Batchelor, M. Holden, and D. J. Hawkes. Medical image registration. *Physics in Medicine and Biology*, 46:R1–R45, 2001.

[12] P. F. Hsieh and Y. Sibuya. *Basic Theory of Ordinary Differential Equations.* Springer, 1999.

[13] Y. Keller, Y. Shkolnisky, and A. Averbuch. The angular difference function and its application to image registration. *IEEE Transactions on Pattern Analysis and Machine Intelligence* , 27:969–976, 2005.

[14] H. Lester and S. R. Arridge. A survey of hierarchical non-linear medical image registration. *Pattern Recognition*, 32:129–149, 1999.

[15] J. B. A. Maintz and M. A. Viergever. A survey of medical image registration. *Medical Image Analysis*, 2:1–36, 1998.

[16] P. F. Mclauchlan and A. Jaenicke. Image mosaicing using sequential bundle adjustment. *Image and Vision Computing*, 20:751–759, 2002.

[17] P. Milanfar. Two-dimensional matched filtering for motion estimation. *IEEE Transactions on Image Processing*, 8:438–444, 1999.

[18] J. R. Munkres. *Topology*. China Machine Press, 2004.

[19] C. Pelizzari, G. Chen, D. Spelbring, R. Weichselbaum, and C. Chen. Accurate three-dimensional registration of CT, PET and/or MR images of the brain. *Journal Computer assisted tomography*, 13:20–26, 1989.

[20] A. Pentland and S. Sclaroff. Closed-form solutions for physically based shape modeling and recognition. *IEEE Transactions Pattern Analysis and Machine Intelligence* , 13:715–729, 1991.

[21] B. Porat. *A Course in Digital Signal Processing*. John Wiley Publication, 1996.

[22] S. Sclaroff and A. Pentland. Modal matching for correspondence and recognition. *IEEE Transactions Pattern Analysis and Machine Intelligence* , 17:545–561, 1995.

[23] J. Segman. Fourier cross correlation and invariance transformations for an optimal recognition of functions deformed by affine groups. *Journal of the Optical Society of America. A.*, 9:895–902, 1992.

[24] R. Szeliski. Image alignment and stitching: a tutorial. *Foundations and Trends in Computer Graphics and Computer Vision*, 2:1–104, 2006.

[25] P. A. van den Elsen, E. J. D. Pol, and M. A. Viergever. Medical image matching–a review with classification. *IEEE Engineering in Medicine and Biology*, 12:26–39, 1993.

[26] G. Q. Zhang. *Functional Analysis Lecture*. Peking University Press, 1997.

[27] Y. Zheng and G. Xu. Flexible multi-scale alignment of images using B-spline reparameterization-computational methods. *Technical Report ICM-10-06* .

[28] B. Zitová and J. Flusser. Image registration methods: a survey. *Image and Vision Computing*, 21:977–1000, 2003.