

层合厚板混合状态 Hamiltonian 元的半解析解

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A SEMI-ANALYTICAL SOLUTION FOR LAMINATED COMPOSITE PLATES IN HAMILTON SYSTEM

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摘要 给出任意角铺设或斜交铺设层合厚板的混合状态 Hamilton 正则方程及其有效的半解析法。该方法在板厚方向未作任何有关应力或位移的人为假设, 采用状态空间法给出解析解答, 且通过状态转移矩阵的建立, 保证了层间位移和应力的连续, 减少了计算中未知量数目。

关键词 层压板, 哈密尔顿函数, 半解析法

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Abstract The mixed state Hamilton canonical equation and a semi-analytical solution are presented for analyzing the laminated composite plates. The method accounts for the separation of variables, the discrete element is employed in the plane of lamina, and the exact solution in thick direction is derived by state space approach. Furthermore, for applying the transfer matrix method, the continuity of interlaminar stresses and displacements is satisfied, the relational expression at the top and bottom plate surfaces is established, and then the spot method is employed. The present results are compared with those obtained using the classical theory and higher-order theory.

Key words laminates, Hamiltonian functions, semi-analytical solution

Hamilton 体系在弹性力学中有着重要应用^[1~4], 如何对其求解现在还没有很好开拓。用解析法求解^[4,5]是比较繁琐的, 且会受到结构几何形状和边界条件的很大限制。文献[1]从 Hellinger-Reissner 变分原理以及修正后导出的 Hamilton 正则方程出发, 给出一种求解条形域平面问题的半离散半解析方法, 并提出了在正交异性板中的可能应用前景。本文在文献[1,2]工作基础上, 进一步给出任意角铺设或斜交铺设层合板的 Hamilton 正则方程及其半解析法。该方法在板平面内采用等参单元插值, 在板厚方向采用状态空间法给出解析解答, 且采用传递矩阵法, 建立了层合板上下表面力学量的关系, 利用打靶法进行求解。数值结果表明, 本文方法与解析解吻合得很好。

1 混合状态 Hamiltonian 元

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对于角铺设层合板, 坐标系取 x - y 坐标沿板平面内方向, z 取板厚方向。选择坐标 z 为展开方向以建立 Hamilton 体系, 写出修正后的 Hamilton 型 Hellinger-Reissner 变分原理为

$$\pi^* = \iiint_{\Omega} \{P^T \dot{q} - H\} dx dy dz - \iint_{S_\sigma} (\bar{T}_x u + \bar{T}_y v + \bar{T}_z w) ds \quad (1)$$

其中 Ω 为体积; S_σ 为应力规定边界的面积; $\bar{T}_i (i = x, y, z)$ 为应力边界上给定面力。

$$P = (\tau_{xz}, \tau_{yz}, \sigma_z)^T, \quad q = (u, v, w)^T, \quad \dot{q} = \partial q / \partial z \quad (2)$$

H 为 Hamilton 函数的二次式

$$\begin{aligned} -H = & a_2 \left(\frac{\partial u}{\partial x}\right)^2 + a_{13} \left(\frac{\partial u}{\partial y}\right)^2 + a_4 \left(\frac{\partial v}{\partial y}\right)^2 + a_{13} \left(\frac{\partial v}{\partial x}\right)^2 + 2a_{14} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2a_{13} \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \\ & + 2a_3 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + 2a_{14} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + 2a_{15} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + 2a_{15} \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} - a_1 \sigma_z \frac{\partial u}{\partial x} - a_5 \sigma_z \frac{\partial v}{\partial x} \\ & - a_{16} \sigma_z \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{yz} \frac{\partial w}{\partial y} + \tau_{xz} \frac{\partial w}{\partial x} - f_x u - f_y v - f_z w - \Gamma^* \end{aligned} \quad (3)$$

式中 $f_i (i = x, y, z)$ 为体积力; a'_s 为与斜交铺设偏轴刚度系数有关的常系数; 斜交铺设时偏轴刚度系数 $c_{ij} (i, j = 1, 2, \dots, 6)$ 的表示形式见文献 [6]。

$$\begin{aligned} a_1 = & -c_{13} / c_{33}, \quad a_2 = c_{11} - c_{13}^2 / c_{33}, \quad a_3 = c_{12} - c_{13} c_{23} / c_{33}, \\ a_4 = & c_{22} - c_{23}^2 / c_{33}, \quad a_5 = -c_{23} / c_{33}, \quad a_6 = c_{66}, \\ a_7 = & 1 / c_{33}, \quad a_8 = 1 / c_{55}, \quad a_9 = 1 / c_{44}, \quad a_{10} = c_{44} / (c_{44} c_{55} - c_{45}^2), \\ a_{11} = & c_{45} / (c_{44} c_{55} - c_{45}^2), \quad a_{12} = c_{55} / (c_{44} c_{55} - c_{45}^2), \quad a_{13} = c_{66} - c_{36}^2 / c_{33}, \\ a_{14} = & c_{16} - c_{13} c_{36} / c_{33}, \quad a_{15} = c_{26} - c_{23} c_{36} / c_{33}, \quad a_{16} = -c_{36} / c_{33} \end{aligned} \quad (4)$$

$$\Gamma^* = \{\sigma^*\}^T [c]^{-1} \{\sigma^*\} / 2 \quad (5)$$

$$\begin{aligned} \{\sigma^*\} = & [(a_2 \frac{\partial u}{\partial x} + a_{14} \frac{\partial u}{\partial y} + a_3 \frac{\partial v}{\partial y} + a_{14} \frac{\partial v}{\partial x} - a_1 \sigma_z), (a_3 \frac{\partial u}{\partial x} + a_{15} \frac{\partial u}{\partial y} + a_4 \frac{\partial v}{\partial y} + a_{15} \frac{\partial v}{\partial x} \\ & - a_5 \sigma_z), \sigma_z, \tau_{yz}, \tau_{xz}, (a_{14} \frac{\partial u}{\partial x} + a_{13} \frac{\partial u}{\partial y} + a_{15} \frac{\partial v}{\partial y} + a_{13} \frac{\partial v}{\partial x} - a_{16} \sigma_z)]^T \end{aligned} \quad (6)$$

由 $\delta \pi^* = 0$ 就得到

$$\partial q / \partial z = \partial H / \partial P, \quad \partial P / \partial z = -\partial H / \partial q \quad (7)$$

就是典型的 Hamilton 正则方程。这说明混合状态结构是一种很基本的 Hamiltonian 结构, 它的力学意义是它们构成混合辛空间, 当沿 z 方向介质模量发生变化时或两种介质接触时, 这 6 个相变量 $(u, v, w, \tau_{xz}, \tau_{yz}, \sigma_z)$ 必须是连续的, 而 $(\sigma_x, \sigma_y, \tau_{xy})$ 则可以是间断的。

由于式 (7) 是 Hamilton 正则方程, 所以可以采用分离变量法^[2,3]。在 x - y 平面内采用四边形等参单元离散, z 方向保留原形式, 其场函数假设为

$$\left. \begin{aligned} u = & [N(x, y)]\{u(z)\}, \quad v = [N(x, y)]\{v(z)\}, \quad \sigma_z = [N(x, y)]\{\sigma(z)\} \\ \tau_{xz} = & [N(x, y)]\{\tau_{xz}(z)\}, \quad \tau_{yz} = [N(x, y)]\{\tau_{yz}(z)\}, \quad w = [N(x, y)]\{w(z)\} \end{aligned} \right\} \quad (8)$$

$$\text{其中} \quad N_i(\xi, \eta) = (1 + \xi_i \xi)(1 + \eta_i \eta) / 4 \quad (9)$$

将式(8)代入式(1)中, 变分并分部积分得

$$c^e \frac{\partial F^e(z)}{\partial z} = K^e F^e(z) + r^e(z) \quad (10)$$

$$\text{其中} \quad F^e(z) = [u^e(z) \quad v^e(z) \quad \sigma^e(z) \quad \tau_{xz}^e(z) \quad \tau_{yz}^e(z) \quad w^e(z)]^T \quad (11)$$

$$c^e = (\iint N^T N |J| d\xi d\eta) [I], \quad [I]_{6 \times 6} \text{ 是一个单位阵} \quad (12)$$

$$K^e = \iiint \begin{bmatrix} 0 & A^e \\ B^e & 0 \end{bmatrix} |J| d\xi d\eta \quad (13)$$

$$A^e = \begin{bmatrix} a_{10} N^T N & -a_{11} N^T N & -N^T \frac{\partial N}{\partial x} \\ -a_{11} N^T N & a_{12} N^T N & -N^T \frac{\partial N}{\partial y} \\ \frac{\partial N^T}{\partial x} N & \frac{\partial N^T}{\partial y} N & 0 \end{bmatrix} \quad (14)$$

$$B^e = \begin{bmatrix} \alpha & \beta & -(a_1 \frac{\partial N^T}{\partial x} N + a_{16} \frac{\partial N^T}{\partial y} N) \\ \gamma & \delta & -(a_5 \frac{\partial N^T}{\partial y} N + a_{16} \frac{\partial N^T}{\partial x} N) \\ \varepsilon & \zeta & a_7 N^T N \end{bmatrix} \quad (15)$$

$$\text{式中} \quad \alpha = a_2 \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} + a_{13} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial y} + a_{14} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial y} + a_{14} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial x};$$

$$\beta = a_{14} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} + a_{15} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial y} + a_{13} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial x} + a_3 \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial y};$$

$$\gamma = a_{14} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} + a_{15} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial y} + a_{13} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial y} + a_3 \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial x};$$

$$\delta = a_3 \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} + a_4 \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial y} + a_{15} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial y} + a_{15} \frac{\partial N^T}{\partial y} \frac{\partial N}{\partial x};$$

$$\varepsilon = a_1 N^T \frac{\partial N}{\partial x} + a_{16} N^T \frac{\partial N}{\partial y}; \quad \zeta = a_5 N^T \frac{\partial N}{\partial y} + a_{16} N^T \frac{\partial N}{\partial x};$$

$$r^e = r_1^e + r_2^e \quad (16)$$

$$r_1^e = -[0 \quad 0 \quad \iint N^T f_z |J| d\xi d\eta \quad \iint N^T f_x |J| d\xi d\eta \quad \iint N^T f_y |J| d\xi d\eta \quad 0]^T \quad (17)$$

$$r_2^e = -[0 \quad 0 \quad \int_{S_x} N^T \bar{T}_z dS \quad \int_{S_x} N^T \bar{T}_x dS \quad \int_{S_x} N^T \bar{T}_y dS \quad 0]^T \quad (18)$$

2 Hamiltonian 元控制方程的解析解

经过上面沿 x - y 平面的离散, 得到层合板中第 m 层 Hamiltonian 元的控制方程为

$$c_m \partial F_m / \partial z = K_m F_m + r_m \quad (19)$$

其中
$$c_m = \sum_c c_m^e, K_m = \sum_c K_m^e, r_m = \sum_c r_m^e \quad (20)$$

两端同乘以逆阵 c_m^{-1} , 有

$$\partial F_m / \partial z = G_m F_m + R_m \quad (21)$$

式中
$$G_m = c_m^{-1} K_m, R_m = c_m^{-1} r_m \quad (22)$$

它的解析解形式为

$$F_m(z) = T_m(z) F_m(0) + \tilde{R}_m(z) \quad (23)$$

其中
$$T_m(z) = e^{G_m z} \quad (24)$$

$$\tilde{R}_m(z) = \int_0^z e^{G_m(z-\tau)} R_m(\tau) d\tau \quad (25)$$

$e^{G_m z}$ 的展开解为

$$e^{G_m z} = \sum_{k=0}^l \frac{G_m^k z^k}{k!} = \left[I + G_m z \left(I + \frac{G_m z}{2} \left\{ I + \dots + \frac{G_m z}{l-1} \left(I + \frac{G_m z}{l} \right) \dots \right\} \right) \right] \quad (26)$$

l 为截断项数。

若层合板由 n 层铺层组成, 则采用迁移矩阵法并考虑层间位移和应力连续条件

$$F_j(h_j) = F_{j+1}(0) \quad (j = 1, 2, \dots, n-1) \quad (27)$$

就得到层合板上下表面位移和应力的关系式为

$$F_n(h_n) = \left(\prod_{i=1}^n T_i \right) F_1(0) + \left(\prod_{i=2}^n T_i \right) \tilde{R}_1(h_1) + \left(\prod_{i=3}^n T_i \right) \tilde{R}_2(h_2) + \dots + \tilde{R}_n(h_n) \quad (28)$$

其中
$$\left(\prod_{i=1}^n T_i \right) = T_n T_{n-1} \dots T_i T_{i-1} \dots T_2 T_1 \quad (29)$$

一般情况下, 层合板上下表面的应力 (τ_{xz} , τ_{yz} , σ_z) 是已知的, 这样式 (28) 就相当于两点边值问题, 利用打靶法将其分解成两个方程

$$\left. \begin{aligned} \{q_n(h_n)\} &= T_{11}^* \{q_1(0)\} + T_{12}^* \{P_1(0)\} + \{\tilde{R}\}_q \\ \{P_n(h_n)\} &= T_{21}^* \{q_1(0)\} + T_{22}^* \{P_1(0)\} + \{\tilde{R}\}_p \end{aligned} \right\} \quad (30)$$

其中 $q_1(0)$ 、 $P_1(0)$ 为上表面 ($z=0$) 处的位移和应力向量; $q_n(h_n)$ 、 $P_n(h_n)$ 为下表面 ($z=h$) 处的位移和应力向量; $\{\tilde{R}\}_q$ 、 $\{\tilde{R}\}_p$ 为与外载及边界相关的等效荷载。

代入层合板上下表面的已知应力边界条件, 就得到关于层合板上表面位移初值问题的求解方程为

$$[D]\{q_1(0)\} = \{Q\} \quad (31)$$

其中

$$\left. \begin{aligned} [D] &= T_{21}^* \\ \{Q\} &= \{P_n(h_n)\} - T_{22}^* \{P_1(0)\} - \{\tilde{R}\}_p \end{aligned} \right\} \quad (32)$$

这样不管层合板的层数有多少,都只剩下上表面的初始位移作为 Hamiltonian 元的节点未知量。与通常的层板有限元法相比,大大减少了计算中的未知量数目。

3 数值算例

为了考证本文的精度,考虑受双向正弦荷载作用下的层合板,其铺设方式为:①(0/90/0);②(0/90/0/90/0/90/0/90/0)。材料的弹性常数为: $E_1/E_2 = 25$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\mu_{12} = \mu_{13} = \mu_{23} = 0.25$ 。网格划分如图 1 所示。解析解取之 Pagano^[6] 三维弹性理论解,高阶理论结果取之文献 [7]。无量纲应力为 $\bar{\sigma}_x = \sigma_x(l/2, l/2, \pm h/2)/q_0 S^2$, $\bar{\sigma}_y = \sigma_y(l/2, l/2, \pm h/4)/q_0 S^2$, $\bar{\tau}_{xz} = \tau_{xz}(0, l/2, 0)/q_0 S$, $\bar{\tau}_{yz} = \tau_{yz}(l/2, 0, 0)/q_0 S$ 。

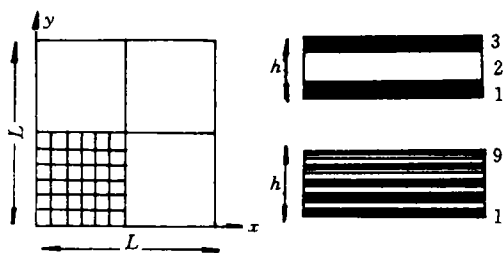


图1 网格划分

表 1 给出 3 层板的应力计算值,表 2 给出 9 层板的应力计算值,并与其它方法进行了比较。从表中可以看出本文结果与解析解吻合得很好,精度高于高阶理论。

表 1 3 层(0/90/0)方板的应力值

l/h	方法	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	本文 6×6	± 0.694	± 0.621	0.224	0.317
	高阶理论 ^[7]	± 0.518	± 0.372	0.202	0.422
	解析解 ^[6]	± 0.702	± 0.663	0.219	0.292
10	本文 6×6	± 0.526	± 0.384	0.307	0.221
	高阶理论 ^[7]	± 0.494	± 0.253	0.213	0.409
	解析解 ^[6]	± 0.559	± 0.401	0.301	0.196
50	本文 6×6	± 0.530	± 0.348	0.345	0.166
	高阶理论 ^[7]	± 0.494	± 0.331	0.160	0.436
	解析解 ^[6]	± 0.539	± 0.276	0.337	0.141

表 2 9层(0/90/0/90/0/90/0/90/0)方板的应力值

l/h	方法	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	本文 6×6	± 0.626	± 0.598	0.228	0.239
	高阶理论 ^[7]	± 0.547	± 0.445	0.225	0.231
	解析解 ^[6]	± 0.666	± 0.670	0.223	0.223
10	本文 6×6	± 0.546	± 0.443	0.254	0.259
	高阶理论 ^[7]	± 0.541	± 0.425	0.219	0.257
	解析解 ^[6]	± 0.551	± 0.477	0.247	0.226
50	本文 6×6	± 0.526	± 0.451	0.271	0.252
	高阶理论 ^[7]	± 0.522	± 0.447	0.190	0.263
	解析解 ^[6]	± 0.539	± 0.433	0.258	0.219

4 结 论

(1) 本文沿层合板厚度方向未作任何有关应力或位移的人为假设, 采用状态空间法给出真解, 且通过状态转移矩阵的建立, 保证了层间位移和应力的连续, 减少了计算中的未知量数目。

(2) 本文采用 c^0 连续函数构造 c^1 类连续问题, 避免了 c^1 类连续问题单元形函数选取上的困难。

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