# Criteria for Generalized $\alpha\eta$-Monotonicities Without Condition C*

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**Abstract** If $\alpha$ and $\eta$ are affine in the first argument and skew instead of Condition C, the following results are obtained: (i) if the gradient of a function is (strictly) $\alpha\eta$-pseudomonotone, the function is (strictly) pseudo $\alpha\eta$-invex; (ii) if the gradient of a function is quasi $\alpha\eta$-monotone, the function is quasi $\alpha\eta$-invex.

**Keywords** mathematical programming, generalized $\alpha\eta$-monotonicity, generalized $\alpha\eta$-invex functions, affine

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# 无条件 C 的广义 $\alpha\eta$- 单调性的判别标准

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**摘要** 用 $\alpha$ 和 $\eta$ 关于第一分量是仿射的且是斜对称的条件代替条件 C, 得到如下结论：(1) 如果一个函数的梯度是 (严格)$\alpha\eta$- 伪单调的，则该函数是 (严格) 伪 $\alpha\eta$- 不变凸的；(2) 如果一个函数的梯度是拟 $\alpha\eta$- 单调的，则该函数是拟 $\alpha\eta$- 不变凸的.

**关键词** 数学规划，广义 $\alpha\eta$- 单调性，广义 $\alpha\eta$- 不变凸函数，仿射

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## 0 Introduction

In recent years, the concept of convexity has been generalized and extended in several directions using novel and innovative techniques. An important and significant generalization

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of convex functions is the introduction of invex function, which was introduced by Hanson[1]. Convexity and generalized convexity play a central role in mathematical economics, engineering and optimization theory. Therefore, the research on convexity and generalized convexity is one of most important aspects in mathematical programming. Invex functions and invex monotonicities are interesting topics in the study of generalized convexity. Generalized $\alpha\eta$-invexity and $\alpha\eta$-monotonicities have been investigated in [2-4]. Noor and Noor[12] introduced a new class of generalized convex functions, which is called the strongly $\alpha$-preinvex functions, and established the equivalence among the strongly $\alpha$-preinvex functions, strongly $\alpha$-invex functions and strongly $\alpha\eta$-monotonicity of their differential under some suitable conditions. Fan and Guo[13] have studied the relationships among (pseudo, quasi) $\alpha$-preinvexity, (strict, strong, pseudo, quasi) $\alpha$-invexity and (strict, strong, pseudo, quasi) $\alpha\eta$-monotonicity in a systematic way. In this paper, under the conditions that $\alpha$ and $\eta$ is affine in the first argument and skew rather than Condition C, the $\alpha\eta$-invex functions are further discussed, and we establish some new relations between generalized $\alpha\eta$-monotonicity and generalized $\alpha\eta$-invexity.

## 1 Preliminaries

Let $K$ be a nonempty subset of $R^n$, We denote by $\langle\ .,.\rangle$ and $\|.\|$ the inner product and norm respectively. Let $F : K \longrightarrow R^n$ and $\eta(.\ ,.) : K \times K \longrightarrow R^n$ be continuous functions. Let $f : K \longrightarrow R$ be a differentiable function and $f'(y)$ be the differential of a function $f$ at $y \in K$. Let $\alpha : K \times K \longrightarrow R\backslash\{0\}$ be a bifunction[2−3].

**Definition 1.1**[2] Let $y \in K$. Then the set $K$ is said to be $\alpha$-invex at $y$ with respect to $\eta(.,.)$ and $\alpha(.,.)$, if, for all $x \in K, t \in [0,1]$,

$$y + t\alpha(x,y)\eta(x,y) \in K.$$

$K$ is said to be an $\alpha$-invex set with respect to $\eta$ and $\alpha$, if $K$ is $\alpha$-invex at each $y \in K$. The $\alpha$-invex set $K$ is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(x,y) = 1$ and $\eta(x,y) = x - y$ is an invex set, but the converse is not true.

**Definition 1.2**[2] A differentiable function $f$ on an open $\alpha$-invex subset $K$ of $R$ is a pseudo $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $K$ if for any $x, y \in K$,

$$\langle \alpha(x,y)f'(y), \eta(x,y)\rangle \geqslant 0,$$

implies

$$f(x) \geqslant f(y).$$

**Definition 1.3**[2,5−6] A differentiable function $f$ on an open $\alpha$-invex subset $K$ of $R^n$ is a strictly pseudo $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $K$ if for any $x, y \in K, x \neq y$,

$$\langle \alpha(x,y) f'(y), \eta(x,y) \rangle \geqslant 0,$$

implies

$$f(x) > f(y).$$

**Definition 1.4**[2,5−6] A differentiable function $f$ on an open $\alpha$-invex subset $K$ of $R^n$ is a quasi $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $K$ if for any $x, y \in K$,

$$f(x) \geqslant f(y),$$

implies

$$\langle \alpha(y,x) f'(x), \eta(y,x) \rangle \leqslant 0.$$

**Definition 1.5**[2] Let $K$ of $R^n$ be an $\alpha$-invex set with respect to $\alpha$ and $\eta$. Then, function $F : K \longrightarrow R$ is said to be $\alpha\eta$-pseudomonotone with respect to $\alpha$ and $\eta$ on $K$ if for every pair of points $x, y \in K$,

$$\langle \alpha(x,y) F(y), \eta(x,y) \rangle \geqslant 0,$$

implies

$$\langle \alpha(y,x) F(x), \eta(y,x) \rangle \leqslant 0.$$

**Definition 1.6**[2] Let $K$ of $R^n$ be an $\alpha$-invex set with respect to $\alpha$ and $\eta$. Then, function $F : K \longrightarrow R^n$ is said to be strictly $\alpha\eta$-pseudomonotone with respect to $\alpha$ and $\eta$ on $K$ if for every pair of points $x, y \in K, x \neq y$,

$$\langle \alpha(x,y) F(y), \eta(x,y) \rangle \geqslant 0,$$

implies

$$\langle \alpha(y,x) F(x), \eta(y,x) \rangle < 0.$$

**Definition 1.7**[2,5−6] Let $K$ of $R^n$ be an $\alpha$-invex set with respect to $\alpha$ and $\eta$. Then, function $F : K \longrightarrow R^n$ is said to be quasi $\alpha\eta$-monotone with respect to $\alpha$ and $\eta$ on $K$ if for every pair of points $x, y \in K$,

$$\langle \alpha(x,y) F(y), \eta(x,y) \rangle > 0,$$

implies

$$\langle \alpha(y,x) F(x), \eta(y,x) \rangle \leqslant 0.$$

**Definition 1.8**[7] Let us say that the function $\eta : K \times K \longrightarrow R^n$ is a skew function if

$$\eta(x,y) + \eta(y,x) = 0, \quad \forall x, y \in K \subseteq R^n.$$

**Condition C**[8] The vector-valued map $\eta : X \times X \longrightarrow R^n$ is said to satisfy the Condition C, if for any $x, y \in R^n$ and for $\lambda \in [\,0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$

$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

**Example 1.1**[9] Let $\eta : \Gamma \times \Gamma \longrightarrow R^n$ be defined as

$$\eta(x, y) = \begin{cases} x - y & if\ x \geqslant 0,\ y \geqslant 0, \\ x - y & if\ x \leqslant 0,\ y \leqslant 0, \\ -2 - y & if\ x > 0,\ y \leqslant 0, \\ 2 - y & if\ x \leqslant 0,\ y > 0. \end{cases}$$

From Example 2.1 in [5], we know that $\eta$ satisfies Condition C. And it is easy to verify that $\eta$ is not affine[10−11] in the first argument and skew.

**Example 1.2**[9] Let $\eta : \Gamma \times \Gamma \longrightarrow R^n$ be defined as $\eta(x, y) = 3(x - y),\ \forall x, y \in \Gamma$. Then, it is easy to verify that $\eta$ is affine in the first argument and skew. However, $\eta$ does not satisfy Condition C.

## 2 Main results

**Proposition 2.1** Let $K$ of $R^n$ be an $\alpha$-invex set with respect to $\alpha$ and $\eta$. Then, function $F : K \longrightarrow R$ is said to be $\alpha\eta$-pseudomonotone with respect to $\alpha$ and $\eta$ on $K$ if and only if, for every pair of points $x, y \in K$,

$$\langle \alpha(x, y)F(y), \eta(x, y)\rangle > 0,$$

implies

$$\langle \alpha(y, x)F(x), \eta(y, x)\rangle < 0.$$

**Proof** From Definition 1.5, $\alpha\eta$-pseudomonotonocity is equivalent to

$$\langle \alpha(y, x)F(x), \eta(y, x)\rangle > 0,$$

implies

$$\langle \alpha(x, y)F(y), \eta(x, y)\rangle < 0, \quad \forall x, y \in K.$$

Thus, changing the role of $x$ and $y$, we have

$$\langle \alpha(x, y)F(y), \eta(x, y)\rangle > 0,$$

implies

$$\langle \alpha(y, x)F(x), \eta(y, x)\rangle < 0, \quad \forall x, y \in K.$$

In the following theorems improves the theorems of Peng[9].

**Theorem 2.1** Let $\Gamma$ be an open $\alpha$-invex subset of $R^n$. Suppose that:

$(i)$ $f'$ is strictly $\alpha\eta$-pseudomonotone with respect to $\alpha$ and $\eta$;

$(ii)$ $\alpha$ and $\eta$ are affine in the first argument and skew respectively;

$(iii)$ for each $x, y \in \Gamma, x \neq y$, $f(x) \geqslant f(y)$, implies

$$\langle \alpha(x,\overline{x}) f'(\overline{x}), \eta(x,\overline{x}) \rangle \geqslant 0$$

for some $\overline{x}$ which lies on the line segment joining $x$ and $y$, then $f$ is strictly pseudo $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $\Gamma$.

**Proof** Let $x, y \in \Gamma, x \neq y$ be such that

$$\langle \alpha(y,x) f'(x),\ \eta(y,x) \rangle \geqslant 0. \tag{2.1}$$

We need to show that

$$f(y) > f(x). \tag{2.2}$$

On the contrary, we assume that

$$f(y) \leqslant f(x). \tag{2.3}$$

By hypothesis $(iii)$,

$$\langle \alpha(x,\overline{x}) f'(\overline{x}),\ \eta(x,\overline{x}) \rangle \geqslant 0. \tag{2.4}$$

where $\overline{x} = \overline{\lambda}x + (1-\overline{\lambda})y$ for some $0 < \overline{\lambda} < 1$. By (2.4) and the strictly $\alpha\eta$-pseudomonotonicity of $f'$ it follows that

$$\langle \alpha(\overline{x},x) f'(x),\ \eta(\overline{x},x) \rangle < 0. \tag{2.5}$$

Now, from the hypothesis $(ii)$, we know

$$\begin{aligned} \alpha(\overline{x},x) &= \alpha(\overline{\lambda}x + (1-\overline{\lambda})y, x) \\ &= \overline{\lambda}\alpha(x,x) + (1-\overline{\lambda})\alpha(y,x) \\ &= (1-\overline{\lambda})\alpha(y,x), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \eta(\overline{x},x) &= \eta(\overline{\lambda}x + (1-\overline{\lambda})y, x) \\ &= \overline{\lambda}\eta(x,x) + (1-\overline{\lambda})\eta(y,x) \\ &= (1-\overline{\lambda})\eta(y,x). \end{aligned} \tag{2.7}$$

Therefor, by (2.5)-(2.7), we have

$$\langle \alpha(y,x) f'(x),\ \eta(y,x) \rangle < 0,$$

which contradicts (2.1). Hence, $f$ is strictly pseudo $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $\Gamma$.

**Theorem 2.2** Let $\Gamma$ be an open $\alpha$-invex subset of $R^n$. Suppose that:

$(i)$ $f'$ is $\alpha\eta$-pseudomonotone with respect to $\alpha$ and $\eta$

$(ii)$ $\alpha$ and $\eta$ are affine in the first argument and skew respectively,

$(iii)$ for each $x, y \in \Gamma$, $f(x) > f(y)$, implies

$$\langle \alpha(x,\overline{x}) f'(\overline{x}), \eta(x,\overline{x})\rangle > 0$$

for some $\overline{x}$ which lies on the line segment joining $x$ and $y$, then $f$ is pseudo $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $\Gamma$.

**Proof** Let $x, y \in \Gamma$ be such that

$$\langle \alpha(y,x) f'(x),\ \eta(y,x)\rangle \geqslant 0. \tag{2.8}$$

We need to show that

$$f(y) \geqslant f(x). \tag{2.9}$$

On the contrary, we assume that

$$f(y) < f(x). \tag{2.10}$$

By hypothesis $(iii)$,

$$\langle \alpha(x,\overline{x}) f'(\overline{x}),\ \eta(x,\overline{x})\rangle > 0. \tag{2.11}$$

where $\overline{x} = \overline{\lambda} x + (1-\overline{\lambda}) y$ for some $0 < \overline{\lambda} < 1$. By (2.11) and the $\alpha\eta$-pseudomonotonicity of $f'$ and Proposition 2.1 it follows that

$$\langle \alpha(\overline{x},x) f'(x),\ \eta(\overline{x},x)\rangle < 0. \tag{2.12}$$

Now, from the hypothesis $(ii)$, we know

$$\begin{aligned} \alpha(\overline{x},x) &= \alpha(\overline{\lambda} x + (1-\overline{\lambda}) y, x) \\ &= \overline{\lambda}\alpha(x,x) + (1-\overline{\lambda})\alpha(y,x) \\ &= (1-\overline{\lambda})\alpha(y,x), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \eta(\overline{x},x) &= \eta(\overline{\lambda} x + (1-\overline{\lambda}) y, x) \\ &= \overline{\lambda}\eta(x,x) + (1-\overline{\lambda})\eta(y,x) \\ &= (1-\overline{\lambda})\eta(y,x). \end{aligned} \tag{2.14}$$

Therefor, by (2.12)-(2.14), we have

$$\langle \alpha(y,x) f'(x),\ \eta(y,x)\rangle < 0,$$

which contradicts (2.8). Hence, $f$ is pseudo $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $\Gamma$.

**Theorem 2.3** Let $\Gamma$ be an open $\alpha$-invex subset of $R^n$. Suppose that:

$(i)$ $f'$ is quasi $\alpha\eta$-monotone with respect to $\alpha$ and $\eta$;

(*ii*) $\alpha$ and $\eta$ are affine in the first argument and skew respectively;

(*iii*) for each $x, y \in \Gamma, x \neq y,\ f(y) \leqslant f(x)$, implies

$$\langle \alpha(x,\overline{x})f'(\overline{x}), \eta(x,\overline{x})\rangle > 0$$

for some $\overline{x}$ which lies on the line segment joining $x$ and $y$, then $f$ is quasi $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $K$.

**Proof** Assume that $f$ is not quasi $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $K$. Then there exists $x, y \in \Gamma$ such that

$$f(y) \leqslant f(x). \tag{2.15}$$

But

$$\langle \alpha(y,x)f'(x),\ \eta(y,x)\rangle > 0. \tag{2.16}$$

By (2.15) and hypothesis (*iii*),

$$\langle \alpha(x,\overline{x})f'(\overline{x}),\ \eta(x,\overline{x})\rangle > 0. \tag{2.17}$$

where $\overline{x} = \overline{\lambda}x + (1-\overline{\lambda})y$ for some $0 < \overline{\lambda} < 1$. By (2.17) and the quasi $\alpha\eta$-monotonicity of $f'$ it follows that

$$\langle \alpha(\overline{x},x)f'(x),\ \eta(\overline{x},x)\rangle \leqslant 0. \tag{2.18}$$

Now, from the hypothesis (*ii*), we know

$$\begin{aligned} \alpha(\overline{x},x) &= \alpha(\overline{\lambda}x + (1-\overline{\lambda})y, x) \\ &= \overline{\lambda}\alpha(x,x) + (1-\overline{\lambda})\alpha(y,x) \\ &= (1-\overline{\lambda})\alpha(y,x), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \eta(\overline{x},x) &= \eta(\overline{\lambda}x + (1-\overline{\lambda})y, x) \\ &= \overline{\lambda}\eta(x,x) + (1-\overline{\lambda})\eta(y,x) \\ &= (1-\overline{\lambda})\eta(y,x). \end{aligned} \tag{2.20}$$

Therefor, by (2.18)-(2.20), we have

$$\langle \alpha(y,x)f'(x),\ \eta(y,x)\rangle \leqslant 0,$$

which contradicts (2.17). Hence, $f$ is quasi $\alpha\eta$-invex function with respect to $\alpha$ and $\eta$ on $K$.

## 3 Conclusions

In this paper, under the conditions that $\alpha$ and $\eta$ is affine in the first argument and skew rather than Condition C, some new relations between generalized $\alpha\eta$-monotonicity and generalized $\alpha\eta$-invexity are established, i.e. (i) if the gradient of a function is (strictly) $\alpha\eta$-pseudomonotone, then the function is (strictly) pseudo $\alpha\eta$-invex; (ii) if the gradient of a function is quasi $\alpha\eta$-monotone, then the function is quasi $\alpha\eta$-invex.

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