Duality for Minmax Fractional Problems Involving Generalized Arcwise Connected Type I*

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Abstract This paper deals with a minmax fractional problems in terms of the right derivative of the function with respect to an arc. Under arcwise connected type I and generalized arcwise connected type I assumptions, a dual model is proposed for the minmax fractional problems. Furthermore, weak duality theorem, strong duality theorem and strict converse duality theorem are established.

Keywords arcwise connected type I, generalized arcwise connected type I, duality
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广义 I 型弧连通极小极大分式问题的对偶

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摘要 在 I 型弧连通和广义 I 型弧连通假设下，建立了极大极小分式优化问题的对偶模型，并提出了弱对偶定理、强对偶定理和严格逆对偶定理.

关键词 I 型弧连通，广义 I 型弧连通，对偶
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0 Introduction

Liu and Wu[1] investigated a minimax fractional programming involving η-invex, η-pseudoinvex and η-quasi-invex functions, and presented sufficient Kuhn-Tucker conditions and three dual models. Avriel and Zang[2] defined the right derivative of a real-valued function with respect to a continuous vector-valued function called an arc. Mehra and Bhatia[3] discussed a static minmax programming problem in term of the right derivative of functions involved with respect to the same arc, and obtained sufficient optimality conditions. Mond-Weir type dual was proposed and duality results were established.

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This paper is going to develop a minmax fractional programming problem by using the notion of generalized arewise connected type I. Some notations and preliminary results are given in section 2. A dual problem are formulated and duality results are developed in section 3.

1 Notations and Preliminaries

Definition 1.1[2] A set $X \subseteq R^n$ is said to be arcwise connected (AC) if for every pair of points $x_1, x_2 \in X$, there exists a continuous vector-valued function $H_{x_1,x_2} : [0,1] \to X$, called an arc, such that $H_{x_1,x_2}(0) = x_1$ and $H_{x_1,x_2}(1) = x_2$.

Definition 1.2[2] Let $\varphi : X \to R$, where $X \subseteq R^n$ is an AC set. Let $x_1, x_2 \in X$ and H_{x_1,x_2} is an arc connecting x_1 and x_2 in X. The function φ is said to possess a right derivative, denoted by $\varphi^+(H_{x_1,x_2}(0))$, with respect to the arc H_{x_1,x_2} at $\theta = 0$ if

$$\varphi^+(H_{x_1,x_2}(0)) := \lim_{\theta \to 0^+} \frac{\varphi(H_{x_1,x_2}(\theta)) - \varphi(x_1)}{\theta}$$

exists.

Clearly, if $\varphi : X \to R$ has a right derivative with respect to the arc H_{x_1,x_2} at $\theta = 0$, then

$$\varphi(H_{x_1,x_2}(\theta)) = \varphi(x_1) + \theta\varphi^+(H_{x_1,x_2}(0)) + \theta\alpha(\theta),$$

where $\theta \in [0,1]$ and $\alpha : [0,1] \to R$ satisfies $\lim\limits_{\theta \to 0^+} \alpha(\theta) = 0$.

Let $X \subseteq R^n$ be AC. For any x_1 and x_2 in X, let H_{x_1,x_2} be an arc connecting them. Let $\varphi : X \to R$ and $\psi : X \to R$ possess right derivatives with respect to the arc H_{x_1,x_2} at $\theta = 0$.

Now we introduce the following concepts.

Definition 1.3 $(\varphi(\cdot), \psi(\cdot))$ is called arcwise connected type I (CN-type I), if for every $x_1, x_2 \in X$,

$$\varphi(x_2) - \varphi(x_1) \geqslant \varphi^+(H_{x_1,x_2}(0)),$$
$$-\psi(x_1) \geqslant \psi^+(H_{x_1,x_2}(0)).$$

Definition 1.4 $(\varphi(\cdot), \psi(\cdot))$ is called P-Q-arcwise connected type I (PQCN-type I), if for every $x_1, x_2 \in X$,

$$\varphi^+(H_{x_1,x_2}(0)) \geqslant 0 \Rightarrow \varphi(x_2) \geqslant \varphi(x_1),$$
$$-\psi(x_1) \leqslant 0 \Rightarrow \psi^+(H_{x_1,x_2}(0)) \leqslant 0.$$

Definition 1.5 $(\varphi(\cdot), \psi(\cdot))$ is called Q-P-arcwise connected type I (QPCN-type I), if for every $x_1, x_2 \in X$,

$$\varphi(x_2) \leqslant \varphi(x_1) \Rightarrow \varphi^+(H_{x_1,x_2}(0)) \leqslant 0,$$
$$\psi^+(H_{x_1,x_2}(0)) \geqslant 0 \Rightarrow -\psi(x_1) \geqslant 0. \tag{1.1}$$

If in the above definition, inequality (1.1) is satisfied as

$$\psi^+(H_{x_1,x_2}(0)) \geqslant 0 \Rightarrow -\psi(x_1) > 0 \quad \text{for } x_1 \neq x_2,$$

then we say that $(\varphi(\cdot), \psi(\cdot))$ is Q-strictly-P-arcwise connected type I (QSTPCN-type I).

Let A be a topological vector space and B a nonempty set in A. Let $\bar{B}$ denote the closure of B and $B^* = \{b^* \in A^* | \langle b, b^* \rangle \geqslant 0, \forall b \in B\}$, where A^* is the dual space of A.

For some nonempty set Y, let $R^Y = \pi_Y R$ denote the product space under a product topology. Then the topological dual space of R^Y is the generalized finite sequence space consisting of all the functions $u : Y \to R$ with finite support. The set $R_+^Y = \pi_Y R_+$ denotes the convex cone of all nonnegative functions on Y. Then $(R_+^Y)^* = \Lambda = \{\lambda = (\lambda_y)_{y \in Y} : \exists$ a finite set $Y_0 \subseteq Y$ such that $\lambda_y = 0, \forall y \in Y \backslash Y_0$ and $\lambda_y \geqslant 0, \forall y \in Y_0\}$.

Definition 1.6[3] Let $\varphi : X \to R$ and $G : X \times Y \to R$, where X and Y are arbitrary sets. The pair (φ, G) is called convexlike on X, if for every $x_1, x_2 \in X$ there exist $x_3 \in X$ and $\theta \in (0, 1)$ such that

$$\varphi(x_3) \leqslant (1-\theta)\varphi(x_1) + \theta\varphi(x_2)$$

and

$$G(x_3, y) \leqslant (1-\theta)G(x_1, y) + \theta G(x_2, y), \ \forall y \in Y.$$

We now consider the following minmax fractional programming problem:

$$\begin{aligned} \text{(P)} \qquad \min \qquad & F(x) = \sup_{y \in Y} \frac{f(x,y)}{h(x,y)}, \\ \text{s.t.} \qquad & g(x) \leqslant 0, x \in X, \end{aligned}$$

where

(a) X is an open AC subset of R^n, and Y is a compact subset of R^n;

(b) $f : X \times Y \to R$ is nonnegative, and $f(x, \cdot)$ is continuous on Y for any $x \in X$;

(c) $h : X \times Y \to R$ is positive, and $h(x, \cdot)$ is continuous on Y for any $x \in X$;

(d) $g : X \to R^p$;

(e) the right derivatives of the functions $f(\cdot, y), h(\cdot, y)$ and $g(\cdot)$ with respect to an arc H_{x_1,x_2} at $\theta = 0$ exist, $\forall x_1, x_2 \in X, \forall y \in Y$;

(f) $f^+(H_{x_1,x_2}(0), \cdot)$ and $h^+(H_{x_1,x_2}(0), \cdot)$ are continuous on $Y, \forall x_1, x_2 \in X$.

We let $J = \{1, 2, \cdots, p\}, J(x) = \{j \in J | g_j(x) = 0\}$,

$$Y(x) = \left\{ y \in Y \,\middle|\, \frac{f(x,y)}{h(x,y)} = \sup_{z \in Y} \frac{f(x,z)}{h(x,z)} \right\},$$

The feasible set of problem (P) is defined as $K = \{x \in X | g(x) \leqslant 0\}$.

Definition 1.7 $x^* \in K$ is called a (P)-optimal solution, if

$$\sup_{y\in Y} \frac{f(x^*,y)}{h(x^*,y)} \leqslant \sup_{y\in Y} \frac{f(x,y)}{h(x,y)},\ \forall x \in K.$$

Consider the following problem (P_v) associated with (P):

$$\begin{aligned} (\mathrm{P}_v) \quad \min \quad & \sup_{y\in Y}[f(x,y) - vh(x,y)], \\ \text{s.t.} \quad & g(x) \leqslant 0. \end{aligned}$$

Lemma 1.1 Let x^* be a (P)-optimal solution and $v = \sup_{y\in Y} \frac{f(x^*,y)}{h(x^*,y)}$. Then x^* is a (P_v)-optimal solution.

Proof If x^* is not a (P_v)-optimal solution, then there exists a feasible x of (P_v) such that

$$\sup_{y\in Y}[f(x,y) - vh(x,y)] < \sup_{y\in Y}[f(x^*,y) - vh(x^*,y)]. \tag{1.2}$$

Because

$$\sup_{y\in Y}\left[\frac{f(x^*,y)}{h(x^*,y)} - v\right] = 0 \Rightarrow \sup_{y\in Y}\left[\frac{f(x^*,y) - vh(x^*,y)}{h(x^*,y)}\right] = 0$$

and $h(x^*,y) > 0$, we have

$$\sup_{y\in Y}[f(x^*,y) - vh(x^*,y)] \leqslant 0.$$

By (1.2), we get

$$\sup_{y\in Y}[f(x,y) - vh(x,y)] < 0,$$

and hence

$$\sup_{y\in Y}\left[\frac{f(x,y) - vh(x,y)}{h(x,y)}\right] < 0.$$

Therefore

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} < v = \sup_{y\in Y}\frac{f(x^*,y)}{h(x^*,y)},$$

which contradicts the (P)-optimality of x^*. Hence the result follows.

For any $t > 0$ and $x \in X$, define

$$\Phi_j(x) = g_j^+(H_{x^*,x}(0)) + t, \quad j \in J(x^*),$$

$$G(x,y) = f^+(H_{x^*,x}(0),y) - vh^+(H_{x^*,x}(0),y) + t, \quad y \in Y(x^*),$$

and

$$\begin{aligned} \Omega(x^*,t) = \{(u,r)| \quad & r = (r_j)_{j\in J(x^*)} \text{ and there exists } x \in X \text{ such that} \\ & \Phi_j(x) \leqslant r_j, j \in J(x^*), G(x,y) \leqslant u(y), \forall y \in Y(x^*)\}. \end{aligned}$$

Lemma 1.2(Necessary Optimality Condition) Suppose that

(i) x^* is a (P)-optimal solution;

(ii) $(\Phi_j, G)_{j\in J(x^*)}$ are convexlike on X;

(iii) There exist a neighborhood U of the zero element "0" in $R^{Y(x^*)}$ and constants $\delta = (\delta_j)_{j\in J(x^*)}$ such that the set $\Omega(x^*, t) \cap U \times \pi_{j\in J(x^*)}(-\infty, \delta_j]$ is a nonempty closed set, $\forall t > 0$.

Then, there exist an integer $\alpha > 0$, scalars $\lambda_i \geqslant 0, (1 \leqslant i \leqslant \alpha), \mu_j \geqslant 0, (1 \leqslant j \leqslant p)$, vectors $y_i \in Y(x^*), (1 \leqslant i \leqslant \alpha)$, and $v \in R_+$ such that

$$\sum_{i=1}^{\alpha} \lambda_i[f^+(H_{x^*,x}(0), y_i) - vh^+(H_{x^*,x}(0), y_i)] + \sum_{j=1}^{p} \mu_j g_j^+(H_{x^*,x}(0)) \geqslant 0,\ \forall x \in X;$$

$$f(x^*, y_i) - vh(x^*, y_i) = 0,\quad \forall 1 \leqslant i \leqslant \alpha;$$

$$\mu_j g_j(x^*) = 0,\quad \forall 1 \leqslant j \leqslant p;$$

$$\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{p} \mu_j \neq 0.$$

Proof Since x^* is a (P)-optimal solution, by Lemma 1.1, there exists a v such that x^* is a (P$_v$)-optimal solution. The theorem now follows by applying Theorem 3.1 of [3] to (P$_v$) at (x^*, v).

2 Duality

In this section, we introduce a kind of dual model to the minmax problem (P). Let

$$G = \Big\{(\alpha, \lambda, \bar{y})| \quad \alpha \text{ is a positive integer}, \lambda \in R_+^{\alpha}, \sum_{i=1}^{\alpha} \lambda_i = 1, \bar{y} = (y_1, y_2, \cdots, y_\alpha) \text{ with } y_i \in Y(x), 1 \leqslant i \leqslant \alpha \text{ for some } x \in X\Big\}.$$

For $\bar{y} = (y_1, y_2, \cdots, y_\alpha) \subset Y(x)$, we define

$$\Omega(\alpha, \lambda, \bar{y}) = \Big\{(z, \mu, v) \in X \times R_+^p \times R_+ \ |f(z, y_i) - vh(z, y_i) = 0, 1 \leqslant i \leqslant \alpha;\ \sum_{j=1}^{p} \mu_j g_j(z) \geqslant 0;$$
$$\sum_{i=1}^{\alpha} \lambda_i[f^+(H_{z,w}(0), y_i) - vh^+(H_{z,w}(0), y_i)] + \sum_{j=1}^{p} \mu_j g_j^+(H_{z,w}(0)) \geqslant 0, \forall w \in X\Big\}.$$

Thus, we can define the following dual problem

$$\text{(D)} \qquad \max_{(\alpha,\lambda,\bar{y})\in G} \sup_{(z,\mu,v)\in\Omega(\alpha,\lambda,\bar{y})} F(z) = \sup_{y\in Y} \frac{f(z, y)}{h(z, y)}$$

The feasible set of problem (D) is defined as

$$S=\{(z,\mu,v,\alpha,\lambda,\bar{y})|(\alpha,\lambda,\bar{y})\in G,(z,\mu,v)\in\Omega(\alpha,\lambda,\bar{y})\}$$

Theorem 2.1(Weak Duality) Suppose that

(i) $x\in K$;

(ii) $(z,\mu,v,\alpha,\lambda,\bar{y})\in S$;

(iii) one of the following conditions hold:

(a) $\Big[\sum\limits_{i=1}^{\alpha}\lambda_i(f(\cdot,y_i)-vh(\cdot,y_i)),\sum\limits_{j=1}^{p}\mu_jg_j(\cdot)\Big]$ is PQCN-type I on K;

(b) $[f(\cdot,y_i)-vh(\cdot,y_i),g(\cdot)],(1\leqslant i\leqslant\alpha)$ are CN-type I on K ;

(c) $\Big[\sum\limits_{i=1}^{\alpha}\lambda_i(f(\cdot,y_i)-vh(\cdot,y_i)),\sum\limits_{j=1}^{p}\mu_jg_j(\cdot)\Big]$ is QSTPCN-type I on K.

Then,

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geqslant F(z).$$

Proof (a) Because $-\sum\limits_{j=1}^{p}\mu_jg_j(z)\leqslant 0$ and the PQCN-ness of $\Big[\sum\limits_{i=1}^{\alpha}\lambda_i(f(\cdot,y_i)-vh(\cdot,y_i)),$ $\sum\limits_{j=1}^{p}\mu_jg_j(\cdot)\Big]$, we have

$$\sum_{j=1}^{p}\mu_jg_j^+(H_{z,x}(0))\leqslant 0. \tag{2.1}$$

Consequently, $(z,\mu,v,\alpha,\lambda,\bar{y})\in S$ and (2.1) yield

$$\sum_{i=1}^{\alpha}\lambda_i[f^+(H_{z,x}(0),y_i)-vh^+(H_{z,x}(0),y_i)]\geqslant 0. \tag{2.2}$$

By the PQCN-ness of $\Big[\sum\limits_{i=1}^{\alpha}\lambda_i(f(\cdot,y_i)-vh(\cdot,y_i)),\sum\limits_{j=1}^{p}\mu_jg_j(\cdot)\Big]$ and (2.2), we have

$$\sum_{i=1}^{\alpha}\lambda_i[f(x,y_i)-vh(x,y_i)]\geqslant\sum_{i=1}^{\alpha}\lambda_i[f(z,y_i)-vh(z,y_i)]=0. \tag{2.3}$$

Therefore there exists i_0 such that $f(x,y_{i_0})-vh(x,y_{i_0})\geqslant 0$. It follows that

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geqslant\frac{f(x,y_{i_0})}{h(x,y_{i_0})}\geqslant v=\frac{f(z,y_{i_0})}{h(z,y_{i_0})}.$$

Since $y_{i_0}\in Y(z)$, we have

$$\frac{f(z,y_{i_0})}{h(z,y_{i_0})}=\sup_{y\in Y}\frac{f(z,y)}{h(z,y)}=F(z).$$

Thus the conclusion of the theorem is true under condition (a).

(b) By the given hypothesis, we have

$$[f(x,y_i)-vh(x,y_i)]-[f(z,y_i)-vh(z,y_i)]\geqslant f^+(H_{z,x}(0),y_i)-vh^+(H_{z,x}(0),y_i),$$
$$-g_j(z)\geqslant g_j^+(H_{z,x}(0)),\quad 1\leqslant j\leqslant p.$$

Therefore

$$\sum_{i=1}^{\alpha}\lambda_i[(f(x,y_i)-vh(x,y_i))-(f(z,y_i)-vh(z,y_i))]\geqslant \sum_{i=1}^{\alpha}\lambda_i[f^+(H_{z,x}(0),y_i)-vh^+(H_{z,x}(0),y_i)], \tag{2.4}$$

$$-\sum_{j=1}^{p}\mu_j g_j(z)\geqslant\sum_{j=1}^{p}\mu_j g_j^+(H_{z,x}(0)). \tag{2.5}$$

By $-\sum_{j=1}^{p}\mu_j g_j(z)\leqslant 0$ and (2.5), we get

$$\sum_{j=1}^{p}\mu_j g_j^+(H_{z,x}(0))\leqslant 0. \tag{2.6}$$

Consequently, $(z,\mu,v,\alpha,\lambda,\bar{y})\in S$, (2.6) and (2.4) yield

$$\sum_{i=1}^{\alpha}\lambda_i[f(x,y_i)-vh(x,y_i)]\geqslant\sum_{i=1}^{\alpha}\lambda_i[f(z,y_i)-vh(z,y_i)]=0,$$

which is the same as (2.3). The remaining part of the proof is the same as that under hypothesis (a).

(c) By the QSTPCN-ness of $\left[\sum_{i=1}^{\alpha}\lambda_i(f(\cdot,y_i)-vh(\cdot,y_i)),\sum_{j=1}^{p}\mu_j g_j(\cdot)\right]$ and the condition (ii), we in turn have the followings:

$$\begin{aligned}&-\sum_{j=1}^{p}\mu_j g_j(z)\leqslant 0\Rightarrow\sum_{j=1}^{p}\mu_j g_j^+(H_{z,x}(0))<0\\ \Rightarrow\quad&\sum_{i=1}^{\alpha}\lambda_i[f^+(H_{z,x}(0),y_i)-vh^+(H_{z,x}(0),y_i)]>0\\ \Rightarrow\quad&\sum_{i=1}^{\alpha}\lambda_i[f(x,y_i)-vh(x,y_i)]>\sum_{i=1}^{\alpha}\lambda_i[f(z,y_i)-vh(z,y_i)]=0.\end{aligned}$$

The remaining part of the proof is the same as that under hypothesis (a).

Theorem 2.2 (Strong Duality) Suppose that

(i) x^* is a (P)-optimal solution;

(ii) there exists $w^*\in X$ such that $g_j^+(H_{x^*,w^*}(0))<0,\ 1\leqslant j\leqslant p$;

(iii) the conditions of Lemma 2.2 are satisfied;

Then, there exists $(\alpha^*, \lambda^*, y^*) \in G$ and $\mu^* \in R_+^p,\ v \in R_+$ such that $(x^*, \mu^*, v^*) \in \Omega(\alpha^*, \lambda^*, y^*)$. Further, if one of the following conditions hold:

(a) $\Big[\sum\limits_{i=1}^{\alpha} \lambda_i(f(\cdot, y_i) - vh(\cdot, y_i)), \sum\limits_{j=1}^{p} \mu_j g_j(\cdot)\Big]$ is PQCN-type I on K;

(b) $[f(\cdot, y_i) - vh(\cdot, y_i), g(\cdot)], (1 \leqslant i \leqslant \alpha)$ are CN-type I on K ;

(c) $\Big[\sum\limits_{i=1}^{\alpha} \lambda_i(f(\cdot, y_i) - vh(\cdot, y_i)), \sum\limits_{j=1}^{p} \mu_j g_j(\cdot)\Big]$ is QSTPCN-type I on K.

Then $(x^*, \mu^*, v^*, \alpha^*, \lambda^*, y^*)$ is a (D)-optimal solution, and the two problems (P) and (D) have the same extremal value.

Proof By Lemma 1.2, there exist an integer $\alpha^* > 0$, scalars $\bar{\lambda} \in R_+^{\alpha^*},\ \bar{\mu} \in R_+^p,\ v^* \in R_+$, and vectors $y_i^* \in Y(x^*), 1 \leqslant i \leqslant \alpha^*$ such that

$$\sum_{i=1}^{\alpha^*} \bar{\lambda}_i[f^+(H_{x^*,w}(0), y_i^*) - v^* h^+(H_{x^*,w}(0), y_i^*)] + \sum_{j=1}^{p} \bar{\mu_j} g_j^+(H_{x^*,w}(0)) \geqslant 0,\ \forall w \in X \tag{2.7}$$

$$f(x^*, y_i^*) - v^* h(x^*, y_i^*) = 0, \quad 1 \leqslant i \leqslant \alpha^* \tag{2.8}$$

$$\sum_{j=1}^{p} \bar{\mu_j} g_j(x^*) \geqslant 0 \tag{2.9}$$

$$\sum_{i=1}^{\alpha^*} \bar{\lambda}_i + \sum_{j=1}^{p} \bar{\mu_j} \neq 0 \tag{2.10}$$

If $\bar{\lambda} = 0$, then (2.10) gives $\sum\limits_{j=1}^{p} \bar{\mu_j} \neq 0$ and (2.7) reduces to

$$\sum_{j=1}^{p} \bar{\mu_j} g_j^+(H_{x^*,w}(0)) \geqslant 0, \forall w \in X. \tag{2.11}$$

On the other hand, by the hypothesis (ii), we can get

$$\sum_{j=1}^{p} \bar{\mu_j} g_j^+(H_{x^*,w^*}(0)) < 0,$$

which contradicts (2.11) for $w = w^*$. Therefore, $\bar{\lambda} \neq 0$, i.e., $\sum\limits_{i=1}^{\alpha^*} \bar{\lambda}_i \neq 0$. Set $\tau = \sum\limits_{i=1}^{\alpha^*} \bar{\lambda}_i,\ \lambda^* = \tau^{-1}\bar{\lambda},\ \mu^* = \tau^{-1}\bar{\mu}$, and $y^* = (y_1^*, y_2^*, \cdots, y_{\alpha^*}^*)$. Then $(\alpha^*, \lambda^*, y^*) \in G$ and $(x^*, \mu^*, v^*, \alpha^*, \lambda^*, y^*)$

is (D)-feasible. Now, if $(x^*,\mu^*,v^*,\alpha^*,\lambda^*,y^*)$ is not (D)-optimal, then there exists a (D)-feasible point $(x,\mu,v,\alpha,\lambda,\bar{y})$ such that

$$F(x)=\sup_{z\in Y}\frac{f(x,z)}{h(x,z)}>F(x^*)=\sup_{z\in Y}\frac{f(x^*,z)}{h(x^*,z)},$$

which contradicts the weak duality between (P)-feasible point x^* and (D)-feasible point $(x,\mu,v,\alpha,\lambda,\bar{y})$ by Theorem 2.1. Thus, $(x^*,\mu^*,v^*,\alpha^*,\lambda^*,y^*)$ is (D)-optimal solution. Since

$$F(x^*)=\sup_{z\in Y}\frac{f(x^*,z)}{h(x^*,z)},$$

the two problems (P) and (D) have the same extremal value.

Theorem 2.3 (Strict Converse Duality) Suppose that

(i) x^* is a (P)-optimal solution and $(z^*,\mu^*,v^*,\alpha^*,\lambda^*,y^*)$ is a (D)-optimal solution;

(ii) there exists $w^*\in X$ such that $g_j^+(H_{x^*,w^*}(0))<0,\ 1\leqslant j\leqslant p$;

(iii) the conditions of Lemma 1.2 are satisfied;

(iv) $\Big[\sum\limits_{i=1}^{\alpha}\lambda_i(f(\cdot,y_i)-vh(\cdot,y_i)),\sum\limits_{j=1}^{p}\mu_jg_j(\cdot)\Big]$ is QSTPCN-type I on K.

Then, $x^*=z^*$; that is, z^* is a (P)-optimal solution.

Proof We shall assume that $x^*\neq z^*$ and reach a contradiction. From Theorem 2.2, we know that the two problem (P) and (D) have the same extremal value, i.e.,

$$\sup_{y\in Y}\frac{f(z^*,y)}{h(z^*,y)}=F(z^*)=\sup_{y\in Y}\frac{f(x^*,y)}{h(x^*,y)} \tag{2.12}$$

Using the fact that $-\sum\limits_{j=1}^{p}\mu_j^*g_j(z^*)\leqslant 0$ and the assumption (iv) , we have

$$\sum_{j=1}^{p}\mu_j^*g_j^+(H_{z^*,x^*}(0))<0. \tag{2.13}$$

(2.13) together with $(z^*,\mu^*,v^*,\alpha^*,\lambda^*,y^*)\in S$ yields

$$\sum_{i=1}^{\alpha^*}\lambda_i^*[f^+(H_{z^*,x^*}(0),y_i^*)-v^*h^+(H_{z^*,x^*}(0),y_i^*)]>0. \tag{2.14}$$

By (2.14) and the suppose (iv), we have

$$\sum_{i=1}^{\alpha^*}\lambda_i^*[f(x^*,y_i^*)-v^*h(x^*,y_i^*)]>\sum_{i=1}^{\alpha^*}\lambda_i^*[f(z^*,y_i^*)-v^*h(z^*,y_i^*)]=0.$$

Therefore, there exists i_0 such that $f(x^*, y^*_{i_0}) - v^* h(x^*, y^*_{i_0}) > 0$. It follows that

$$\sup_{y\in Y} \frac{f(x^*, y)}{h(x^*, y)} \geqslant \frac{f(x^*, y^*_{i_0})}{h(x^*, y^*_{i_0})} > v^* = \frac{f(z^*, y^*_{i_0})}{h(z^*, y^*_{i_0})}. \tag{2.15}$$

Using the fact that $y^*_{i_0} \in Y(z^*)$, we get

$$\frac{f(z^*, y^*_{i_0})}{h(z^*, y^*_{i_0})} = \sup_{y\in Y} \frac{f(z^*, y)}{h(z^*, y)} = F(z^*). \tag{2.16}$$

By (2.15) and (2.16), we have

$$\sup_{y\in Y} \frac{f(x^*, y)}{h(x^*, y)} > \sup_{y\in Y} \frac{f(z^*, y)}{h(z^*, y)} = F(z^*),$$

which contradicts (2.12). Hence, $x^* = z^*$.

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