The Wiener Index of Trees with Prescribed Diameter*

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Abstract The Wiener index $W(G)$ of a graph G is defined as the sum of $d_G(u,v)$ over all pairs of vertices, where $d_G(u,v)$ is the distance between vertices u and v in G. In this paper, we characterize the tree with third-minimum Wiener index and introduce the method of obtaining the order of the Wiener indices among all the trees with given order and diameter, respectively.

Keywords Wiener index, diameter, tree, distance
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固定直径的树的 Wiener 指数

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摘要 图 G 的 Wiener 指数定义为图中所有点对 u, v 的距离之和 $\sum d_G(u,v)$. 在给定顶点和直径的所有树中具有第三小 Wiener 指数的树的特征，得到这类树的 Wiener 指数排序的方法.

关键词 Wiener 指数，直径，树，距离
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0 Introduction

The Wiener index of a graph G, defined in [1], is

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v),$$

where $d_G(u,v)$ is the distance between u and v in G and the sum goes over all the pairs of vertices. Since the Wiener index was introduced by Wiener in 1947[1], numerous of its chemical applications were reported and its mathematical properties were understood (see [1-5]). Recently, finding the graphs with minimum or maximum topological indices including Wiener index attracted the attention of many researchers and lots of results are achieved(see

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[6-12]). For example, Liu and Pan[6] obtained the graphs with the minimum and second-minimum Wiener indices among all graphs with given order and diameter. Deng and Xiao[8] gave the maximum Wiener polarity index of trees with n vertices and k pendants. Chen and Zhang[9] characterized all extremal graphs which attain the lower bound of Wiener index of unicyclic graphs of order n with girth and the matching number $\beta \geqslant 3g/2$. Lin[11] characterized the graphs with the minimum and maximum Wiener indices among all graphs with given order and clique number. More results of the Wiener index, we refer the readers to [2].

All graphs considered in this paper are finite, simple and undirected. For a vertex x of a graph G, we denote the degree of x by $d(x)$. The diameter of a graph G, diam(G), is the maximum distance between any two vertices of G. The distance of a vertex x, $D_G(x)$, is the sum of distances between x and all other vertices of G. The order of a graph G is denoted by $n(G)$. Let P_n and S_n denote the path and star of order n, $T^1_{n-3,1}$ is a tree of order n obtained from the star S_{n-1} by attaching a pendant vertex to one pendant vertex of S_{n-1}. In order to formulate our results, we define some trees as follows.

We use $G = P_d^{v_1v_2\cdots v_k}(T_1, T_2, \cdots, T_k)$ to denote the tree of order n obtained from a path $P_d = v_0v_1\cdots v_d$ by identifying a tree T_i to vertex v_i of $P_d(1 \leqslant i \leqslant k,\ 1 \leqslant k \leqslant d-1)$, where v_i is the root of T_i, let $l_k = n(T_k) - 1$ and $l = l_1 + l_2 + \cdots + l_k$, then $l = n - d - 1$. If $T_i \cong S_i$, let

$$G_1 = P_d^{v_1v_2\cdots v_k}(S_{l_1+1}, S_{l_2+1}, \cdots, S_{l_k+1}),$$

the root v_i is the center of the star S_{l_i+1}. Write

$H_1 = \{P_d^{v_i}(T_i) : 1 \leqslant i \leqslant d-1\}$ and $T_{n,d,i} = P_d^{v_i}(S_{n-d})$, $X_{n,d,i} = P_d^{v_i}(T^1_{n-d-3,1})$;

$H_2 = \{P_d^{v_iv_j}(T_i, T_j) : 1 \leqslant i < j \leqslant d-1\}$ and $Z_{n,d,i,j} = P_d^{v_iv_j}(S_{n-d-1}, S_2)$,
$A_{n,d,i,j} = P_d^{v_iv_j}(S_{n-d-2}, S_3)$;

$H_3 = \{P_d^{v_iv_jv_m}(T_i, T_j, T_m) : 1 \leqslant i < j < m \leqslant d-1\}$ and $B_{n,d,i,j,m} = P_d^{v_iv_jv_m}(S_2, S_{n-d-2}, S_2)$.

Some of these graphs are depicted in Fig.1. We follow [3] for other graph-theoretical terminologies and notations not defined here.

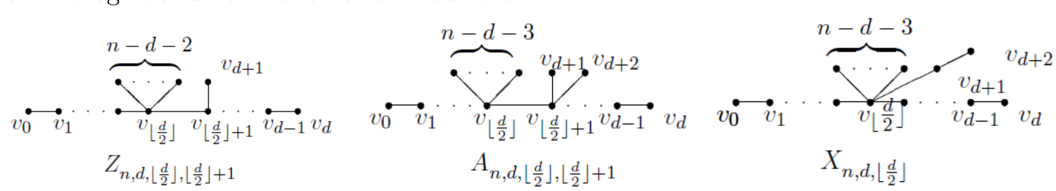

Fig. 1

Let T be a tree of order n with diameter$d$. If $d = n-1$, then $T \cong P_n$; if $d = n-2$,then T is a tree obtained from the path $P_d = v_0v_1\cdots v_d$ by attaching a pendant vertex to the vertex $v_i$of $P_d, (d(v_i) = 2)$; if $n-3 \leqslant d \leqslant n-5$, then the research on Wiener index of T is easy; and if $d = 2$, then $T \cong S_n$. Therefore, in the following, we assume that $3 \leqslant d \leqslant n-6$. Let $T_{n,d} = \{G : G = P_d^{v_1v_2\cdots v_k}(T_1, T_2, \cdots, T_k)$ is a tree with order n and diameter $d, 3 \leqslant d \leqslant n-6\}$. In this paper, the third-minimum Wiener index of tree in the set $T_{n,d}(3 \leqslant d \leqslant n-6)$ is characterized.

1 Lemmas

Lemma 1.1[2] Let T be a tree of order n. Then $W(T) \geqslant W(S_n)$ and the equality holds if and only if $T \cong S_n$.

By Lemma 1.1, we have the result. Let H be a connected graph and T_l be a tree of order l with $V(H) \cap V(T_l) = \{v\}$. Then $W(HvT_l) \geqslant W(HvS_l)$, and equality holds if and only if $HvT_l \cong HvS_l$, where v is identified with the center of the star S_l in HvS_l.

Lemma 1.2[11] Let $H,\ X,\ Y$ be three connected, pairwise disjoint graphs. Suppose that $u,\ v$ are two vertices of $H,\ v'$ is a vertex of $X,\ u'$ is a vertex of Y. Let G be the graph obtained from $H,\ X,\ Y$ by identifying v with v' and u with u', respectively. Let G_1^* be the graph obtained from $H,\ X,\ Y$ by identifying vertices $v,\ v',\ u'$, and let G_2^* be the graph obtained from $H,\ X,\ Y$ by identifying vertices $u,\ v',\ u'$. Then

$$W(G_1^*) < W(G) \qquad or \qquad W(G_2^*) < W(G).$$

Lemma 1.3[6] Let $P = v_0v_1\cdots v_d$ be a path of order $d+1$. Then

$$D_p(v_j) = \frac{2j^2 - 2dj + d^2 + d}{2},$$

for $1 \leqslant j \leqslant d-1$. Moreover, if $1 \leqslant i < j \leqslant d/2$, then

$$D_p(v_i) > D_p(v_j),$$

and if $d/2 \leqslant i < j \leqslant d-1$, then

$$D_p(v_i) < D_p(v_j).$$

Lemma 1.4 Let $G = P_d^{v_1v_2\cdots v_k}(T_1, T_2, \cdots, T_k)$ be a tree of order n with diameter d, then

$$\begin{aligned} W(G) =& \frac{(d+1)^3 - (d+1)}{6} + \sum_{i=1}^{k} l_i D_{P_d}(v_i) + \sum_{i=1}^{k} W(T_i) \\ &+ d\sum_{i=1}^{k} w_i + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} (l_iw_j + l_jw_i + l_il_jd_{P_d}(v_i, v_j)), \end{aligned}$$

where $l_i = n(T_i) - 1, w_i = D_{T_i}(v_i)$ is the sum of distances between v_i and all other vertices of T_i.

Proof In order to verify the result, we divide the vertices of G into four cases:

(1) $x, y \in V(P_d)$, the sum of distances between x and y, denoted by W_1, then

$$W_1 = \frac{(d+1)^3 - (d+1)}{6}.$$

(2) $x, y \in V(T_i - v_i)$, $1 \leqslant i \leqslant k$, the sum of distances between x and y, denoted by W_2, then

$$W_2 = \sum_{i=1}^{k} (W(T_i) - w_i),$$

where $w_i = D_{T_i}(v_i)$ is the sum of distances between v_i and all other vertices of T_i.

(3) $x \in V(T_i - v_i), y \in V(T_j - v_j), 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant k$ and $i \neq j$, the sum of distances between x and y, denoted by W_3,then

$$d_G(x, y) = d_{T_i}(x, v_i) + d_{P_d}(v_i, v_j) + d_{T_j}(v_j, y),$$

$$\begin{aligned}\sum_{x\in V(T_i-v_i)} \sum_{y\in V(T_j-v_j)} d_G(x, y) &= \sum_{x\in V(T_i-v_i)} \sum_{y\in V(T_j-v_j)} (d_{T_i}(x, v_i) + d_{P_d}(v_i, v_j) + d_{T_j}(v_j, y)) \\ &= \sum_{x\in V(T_i-v_i)} (l_j d_{T_i}(x, v_i) + w_j + l_j d_{P_d}(v_i, v_j)) \\ &= l_j w_i + l_i w_j + l_i l_j d_{P_d}(v_i, v_j),\end{aligned}$$

hence

$$W_3 = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (l_i w_j + l_j w_i + l_i l_j d_{P_d}(v_i, v_j)).$$

Where $l_i = n(T_i) - 1$.

(4) $x \in V(T_i - v_i),\ y \in V(P_d), 1 \leqslant i \leqslant k$, the sum of distances between x and y, denoted by W_4, then

$$\begin{aligned}W_4 &= \sum_{i=1}^{k} \sum_{y\in V(P_d)} \sum_{x\in V(T_i-v_i)} d_G(x, y) \\ &= \sum_{i=1}^{k} \sum_{y\in V(P_d)} \sum_{x\in V(T_i-v_i)} (d_G(x, v_i) + d_G(v_i, y)) \\ &= \sum_{i=1}^{k} \sum_{y\in V(P_d)} (l_i d_G(v_i, y) + w_i) \\ &= \sum_{i=1}^{k} (l_i D_{P_d}(v_i) + (d+1) w_i),\end{aligned}$$

Therefore,

$$\begin{aligned}W(G) &= W_1 + W_2 + W_3 + W_4 \\ &= \frac{(d+1)^3 - (d+1)}{6} + \sum_{i=1}^{k} l_i D_{P_d}(v_i) + \sum_{i=1}^{k} W(T_i) \\ &\quad + d\sum_{i=1}^{k} w_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (l_i w_j + l_j w_i + l_i l_j d_{P_d}(v_i, v_j)),\end{aligned}$$

where $l_i = n(T_i) - 1, w_i = D_{T_i}(v_i), i = 1, 2, \cdots, k$.

Lemma 1.5[6] Let T be a non-trivial connected graph and let $v \in V(T)$. Suppose that two paths $P = vv_1v_2\cdots v_k, Q = vu_1u_2\cdots u_m$ of lengths $k,\ m(k \geqslant m \geqslant 1)$ are attached to T by their ends vertices at v, respectively, to form $T^*_{k,m}$. Then

$$W(T^*_{k,m}) < W(T^*_{k+1,m-1}).$$

2 Main results

In this section, we will give the tree with the third-minimum Wiener index in the set $T_{n,d}(3 \leqslant d \leqslant n-6)$. Liu and Pan[6] obtained the tree with the minimum Wiener index is $T_{n,d,\lfloor\frac{d}{2}\rfloor}$ and the tree with the second-minimum Wiener index is $Z_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}$ in the set $T_{n,d}(3 \leqslant d \leqslant n-3)$. In fact, by Lemma 1.1, 1.3–1.5, we have in the set H_1, the Wiener index is minimized by the tree $T_{n,d,\lfloor\frac{d}{2}\rfloor}$ and in the set H_2, the Wiener index is minimized by the tree $Z_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}$.

Lemma 2.1 In the set H_2 with $3 \leqslant d \leqslant n-6$,

(1) for d is even, the second-minimum Wiener index of tree is

$$Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor} \quad \text{or} \quad A_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1};$$

(2) for d is odd, the second-minimum Wiener index of tree is

$$Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}.$$

Proof By lemmas in section 1, we get the tree with the second-minimum Wiener index in the set H_2 maybe

$$Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor} \ or\ A_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1} or\ Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2} \quad or\ P_d^{v_{\lfloor\frac{d}{2}\rfloor}v_{\lfloor\frac{d}{2}\rfloor+1}}(T^1_{n-d-4,1}, S_2).$$

By Lemma 1.3 and 1.4, it is easy to obtain that

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor}) - W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}) = (n-d-2)(2d-4\left\lfloor\frac{d}{2}\right\rfloor) + (2d-4-4\left\lfloor\frac{d}{2}\right\rfloor),$$

if d is even, the result is $-4 < 0$; if d is odd, the result is $2(n-d-3) > 0$.

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor}) - W(A_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}) = (n-d)(d-2\left\lfloor\frac{d}{2}\right\rfloor),$$

if d is even, the result is 0; if $d$is odd, the result is $n-d > 0$.

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor}) - W(P_d^{v_{\lfloor\frac{d}{2}\rfloor}v_{\lfloor\frac{d}{2}\rfloor+1}}(T^1_{n-d-4,1}, S_2)) = (n-d-1)(d-2\left\lfloor\frac{d}{2}\right\rfloor) - d,$$

if d is even, the result is $-d < 0$.

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}) - W(A_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}) = -(n-d-2)(d-2\left\lfloor\frac{d}{2}\right\rfloor) + 4,$$

if d is odd, the result is$-(n-d-6) \leq 0$.

$$W(Z_{n,d,\lfloor \frac{d}{2}\rfloor+1,\lfloor \frac{d}{2}\rfloor+2}) - W(P_d^{v_{\lfloor \frac{d}{2}\rfloor} v_{\lfloor \frac{d}{2}\rfloor+1}}(T^1_{n-d-4,1}, S_2)) = (n-d-1)(2\left\lfloor \frac{d}{2}\right\rfloor - d) - d + 4,$$

if d is odd, the result is $5-n<0$.

Thus the result holds.

Lemma 2.2 Let $G \in T_{n,d} \backslash (H_1 \cup H_2), 3 \leqslant d \leqslant n-6$. Then

$$W(G) \geqslant W(B_{n,d,\lfloor \frac{d}{2}\rfloor-1,\lfloor \frac{d}{2}\rfloor,\lfloor \frac{d}{2}\rfloor+1}),$$

with equality if and only if $G \cong B_{n,d,\lfloor \frac{d}{2}\rfloor-1,\lfloor \frac{d}{2}\rfloor,\lfloor \frac{d}{2}\rfloor+1}$.

Proof Let $P_d = v_0 v_1 \cdots v_{d-1} v_d$ is a path of length d in G with $d(v_0) = d(v_d) = 1$. Let $V_d = \{v_i : d(v_i) \geqslant 3, 1 \leqslant i \leqslant d-1\}$. Since $n \geqslant d+3, V_d \neq \phi$. We discuss in two cases.

Case 1 $|V_d| \geqslant 3$. In this case, we first obtain a tree $T_1 \cong G_1$, by Lemma 1.1, $W(G) \geqslant W(T_1)$ and equality holds if and only if $G \cong T_1$. Since$G \notin (H_1 \cup H_2)$, by Lemma 1.2, we can obtain a tree $T_2 \cong B_{n,d,i,j,m}$ such that

$$W(T_1) > W(T_2) = W(B_{n,d,i,j,m}).$$

Hence

$$W(G) \geqslant W(T_1) > W(T_2) = W(B_{n,d,i,j,m}).$$

Case 2 $|V_d| = 3$ and $G \cong B_{n,d,i,j,m}$. In this case, by Lemma 1.4, we have

$$\begin{aligned} W(B_{n,d,i,j,m}) =& \frac{(d+1)^3-(d+1)}{6} + D_{P_d}(v_i) + D_{P_d}(v_m) + (n-d-3)D_{P_d}(v_j) \\ &+ (n-d-3)(n-d+1) + d(n-d-1) + 4 + d_{P_d}(v_i, v_m) \\ &+ (n-d-3)d_{P_d}(v_i, v_j) + (n-d-3)d_{P_d}(v_m, v_j), \end{aligned}$$

by Lemma 1.3, it is easy to proof that when $v_i = v_{\lfloor \frac{d}{2}\rfloor-1}$,$v_j = v_{\lfloor \frac{d}{2}\rfloor}, v_m = v_{\lfloor \frac{d}{2}\rfloor+1}$, the Wiener index of $B_{n,d,i,j,m}$ achieve the minimum.

By case 1 and case 2, the proof of the Lemma is complete.

It is easy to proof that the second-minimum Wiener index among all the trees of order n is $T^1_{n-3,1}$ and $W(T^1_{n-3,1}) = (n-2)(n+1)$. Therefore, let $H \in P_d^{v_{\lfloor \frac{d}{2}\rfloor}}(T_{\lfloor \frac{d}{2}\rfloor})$, then

$$W(H) \geqslant W(X_{n,d,\lfloor \frac{d}{2}\rfloor}) > W(T_{n,d,\lfloor \frac{d}{2}\rfloor}).$$

By Lemma 1.3 and 1.4, we have the trees with the second and third-minimum Wiener indices in the set H_1 maybe $T_{n,d,\lfloor \frac{d}{2}\rfloor-1}$, $T_{n,d,\lfloor \frac{d}{2}\rfloor-2}$ *or* $X_{n,d,\lfloor \frac{d}{2}\rfloor}$.

Lemma 2.3 In the set $H_1 \cup H_2 \cup \{B_{n,d,\lfloor \frac{d}{2}\rfloor-1,\lfloor \frac{d}{2}\rfloor,\lfloor \frac{d}{2}\rfloor+1}\}(3 \leq d \leq n-6)$,

(1) for d is even, the third-minimum Wiener index is

$$Z_{n,d,\lfloor \frac{d}{2}\rfloor-1,\lfloor \frac{d}{2}\rfloor} \quad or \quad A_{n,d,\lfloor \frac{d}{2}\rfloor,\lfloor \frac{d}{2}\rfloor+1} \, (n < 2d+1);$$

$$X_{n,d,\lfloor \frac{d}{2}\rfloor} \, (n > 2d+1);$$

(2) for d is odd, the third-minimum Wiener index is $Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}$.

Proof Let

$$T\in H_1\cup H_2\cup\{B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}\}$$

is the tree with the third-minimum Wiener index. By Lemma 2.1 and 2.2, T maybe $T_{n,d,\lfloor\frac{d}{2}\rfloor-1}$ or $X_{n,d,\lfloor\frac{d}{2}\rfloor}$ or $T_{n,d,\lfloor\frac{d}{2}\rfloor-2}$ or $Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}$ or $Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor}$ or $B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}$.

By Lemma 1.3 and 1.4, for d is even,

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor})-W(X_{n,d,\lfloor\frac{d}{2}\rfloor})=(n-d-2)(1+d-2\left\lfloor\frac{d}{2}\right\rfloor)+1-d=n-2d-1,$$

if $n>2d+1$, the result is>0; if $n<2d+1$, the result is<0.

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor})-W(T_{n,d,\lfloor\frac{d}{2}\rfloor-2})=-2(n-d)<0,$$

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor})-W(B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1})=-2<0,$$

for d is odd,

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2})-W(B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1})=-(n-d-2)<0,$$

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2})-W(X_{n,d,\lfloor\frac{d}{2}\rfloor})=3-d<0,$$

$$W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2})-W(T_{n,d,\lfloor\frac{d}{2}\rfloor-1})=-(n-d-2)<0.$$

Thus the lemma is proved.

Theorem 2.4 Let

$$G\in T_{n,d}\backslash\{T_{n,d,\lfloor\frac{d}{2}\rfloor},T_{n,d,\lfloor\frac{d}{2}\rfloor-1}Z_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}\},$$

$(3\leqslant d\leqslant n-6)$. Then

(1) for d is even,

$$W(G)\geqslant W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor})$$

or

$$W(G)\geqslant W(A_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1})\ (n<2d+1),$$
$$W(G)\geqslant W(X_{n,d,\lfloor\frac{d}{2}\rfloor})\ (n>2d+1);$$

(2) for d is odd,

$$W(G)\geqslant W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}).$$

Proof Let $P_d=v_0v_1\cdots v_{d-1}v_d$ is a path of length d in G with $d(v_0)=d(v_d)=1$. Let $V_d=\{v_i:d(v_i)\geqslant 3,1\leqslant i\leqslant d-1\}$. Since $n\geqslant d+3,V_d\neq\phi$. We consider three cases.

Case 1 $|V_d|\geqslant 3$.

In this case, by Lemma 2.2 and 2.3, we have

$$W(G) \geqslant W(B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}),$$

and for d is even,

$$W(B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}) > W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor});$$

for d is odd,

$$W(B_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1}) > W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}).$$

So, for d is even,

$$W(G) > W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor})$$

and for d is odd,

$$W(G) > W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}).$$

Case 2 $|V_d| = 2$**.**

In this case, by Lemma 2.1, we have for d is even,

$$W(G) \geqslant W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor}) \quad or \quad W(G) \geqslant W(A_{n,d,\lfloor\frac{d}{2}\rfloor,\lfloor\frac{d}{2}\rfloor+1});$$

for d is odd,

$$W(G) \geqslant W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}).$$

Case 3 $|V_d| = 1$**.**

In this case, by Lemma 2.3, we have for d is even, when $n > 2d+1$,

$$W(G) \geqslant W(X_{n,d,\lfloor\frac{d}{2}\rfloor}),$$

and when $n < 2d+1$,

$$W(G) \geqslant W(Z_{n,d,\lfloor\frac{d}{2}\rfloor-1,\lfloor\frac{d}{2}\rfloor}).$$

For d is odd,

$$W(G) \geqslant W(Z_{n,d,\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2}).$$

According to the above method, if we want to obtain the order of trees in the set $T_{n,d}(3 \leqslant d \leqslant n-6)$ on Wiener index, we should divide $T_{n,d}(3 \leqslant d \leqslant n-6)$ into some smaller sets $H_k(1 \leqslant k \leqslant d-1)$. And then, by Lemma 1.1, 1.3 and 1.4, we characterize the order of trees in the set $H_k(1 \leqslant k \leqslant d-1)$ respectively, at last we will obtain the order of trees in the set $T_{n,d}(3 \leqslant d \leqslant n-6)$ on wiener index.

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