The Number of Arcs of Strongly Connected Oriented Graphs with Two Noncritical Vertices*

LIN Shangwei[1†] LI Chunfang[1] WANG Shiying[1]

Abstract It is proved that a strongly connected oriented graph D with $n \geqslant 4$ vertices and at least $\binom{n-1}{2}+3$ arcs has two distinct vertices u^*, v^* such that both $D-u^*$ and $D-v^*$ are strongly connected. The examples show that the above lower bound on the number of arcs is sharp.

Keywords digraph, strongly connected subdigraph, critical vertex
Chinese Library Classification O157.5
2010 Mathematics Subject Classification 05C40

含有两个非临界点的强连通定向图的弧数

林上为 [1†] 李春芳 [1] 王世英 [1]

摘要 证明顶点数为 $n \geqslant 4$, 弧数为 $m \geqslant \binom{n-1}{2}+3$ 的强连通定向图 D 中存在两点 u^* 、v^*, 使得 $D-u^*$ 和 $D-v^*$ 都是强连通的，并用例子说明这里所给的关于弧数的下界是紧的.

关键词 有向图，强连通子图，临界点
中图分类号 O157.5
数学分类号 05C40

1 Terminology and introduction

For graph-theoretical terminology and notation not defined here we follow [1]. We only consider finite (di)graphs without loops and multiple edges (arcs). A digraph H is a subdigraph of a digraph D (written $H \subseteq D$) if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. When $H \subseteq D$ but $H \neq D$, we call H a proper subdigraph of D. A spanning subdigraph of D is a subdigraph H with $V(H) = V(D)$. Suppose that U is a nonempty subset of $V = V(D)$. The subdigraph of D whose vertex set is U and whose arc set is the set of those arcs of D that have both end-vertices in U is called the subdigraph of D induced by U and is denoted by $D[U]$; we say that $D[U]$ is an induced subdigraph of D. The induced subdigraph $G[V \setminus U]$ is denoted by $D-U$; it is the

收稿日期： 2011 年 8 月 13 日.
* Supported by the National Natural Science Foundation of China (No. 11026163, 61070229) and the Natural Science Foundation for Young Scientists of Shanxi Province (No. 2011021004).
1. School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China; 山西大学数学科学学院，太原 030006
† 通讯作者 Corresponding author

subdigraph obtained from D by deleting the vertices in U together with their incident arcs. If $U=\{v\}$ we write $D-v$ for $D-\{v\}$.

A digraph D is strongly connected (or, just, strong) if every vertex of D is reachable from every other vertex of D. We define a digraph with one vertex to be strong. A vertex v in a strong digraph D is said to be noncritical if the digraph $D-v$ is also strong. Similar notions can be introduced for an undirected graph G. It is well known that a connected undirected graph G with at least two vertices contains a spanning tree T , whence at least two noncritical vertices (in particular, any leaf in T is noncritical both for T and G). So we have the following statement.

Observation 1.1 Every connected undirected graph on at least 2 vertices has at least two noncritical vertices.

An oriented graph is a digraph with no cycle of length two. An out-arborescence rooted at s is an oriented tree T such that $s\in V(T)$ and the in-degree of every vertex $x\in V(T)\backslash\{s\}$ is 1. A tournament is an oriented graph where every pair of distinct vertices are adjacent. We have a similar result to Observation 1.1 for tournaments.

Theorem 1.1[1] Every strong tournament on at least 4 vertices contains at least two noncritical vertices.

Consider the directed cycle C_n with $n\geqslant 3$ vertices. It is easy to see that C_n is strong, but contains no noncritical vertex. So Observation 1.1 can not be extended to general digraphs. In 1999, Schwarz[2] conjectured that every strong digraph with $n\geqslant 3$ vertices and $m\geqslant\binom{n}{2}+1$ arcs contains at least one noncritical vertex. In the same year, London[3] proved this conjecture. Later, Aharoni and Berger[4] showed that the lower bound on the number of arcs can be sharpened from $\binom{n}{2}+1$ to $\binom{n-1}{2}+4$.

Theorem 1.2[4] If a strong digraph D with $n\geqslant 4$ vertices has at least $\binom{n-1}{2}+4$ arcs, it has at least one noncritical vertex.

For oriented graphs, the above bound on the number of arcs can be improved to $\binom{n-2}{2}+5$.

Theorem 1.3[5] If a strong oriented graph D with $n\geqslant 4$ vertices has at least $\binom{n-2}{2}+5$ arcs, it has at least one noncritical vertex.

In this paper, we will show that $\binom{n-1}{2}+3$ arcs in a strong oriented graph with $n\geqslant 4$ vertices can guarantee the existence of two noncritical vertices. Clearly, Theorem 1.1 is an immediate consequence of this result.

2 Main result

We start with the notion of maximal strong proper subdigraphs and a characterization of such subdigraphs. A strong proper subdigraph H of a strong digraph D is maximal if any strong subdigraph in D containing H coincides either with H or with D.

Lemma 2.1[6] Let H be a strong proper subdigraph of a strong digraph D, and let $H'=D-V(H)$. Then H is a maximal strong proper subdigraph in D iff the following three conditions are satisfied:

(1) there is a vertex w_{in} in H' such that any arc going from $V(H)$ to $V(H')$ enters w_{in};

(2) there is a vertex w_{out} in H' such that any arc going from $V(H')$ to $V(H)$ leaves from w_{out};

(3) the distance between w_{in} and w_{out} is one less than the order of H'.

The main result of this paper is the following.

Theorem 2.1 If a strong oriented graph D with $n \geqslant 4$ vertices has at least $\binom{n-1}{2}+3$ arcs, it has at least two noncritical vertices.

Proof By induction on n. The statement is clearly true for $n=4$. Suppose, then, that $n \geqslant 5$. By Theorem 1.3, D has a vertex v^* such that $D-v^*$ is strong. We will show that D has a vertex $u^* \neq v^*$ such that $D-u^*$ is also strong. Let H be a maximal strong proper subdigraph of D, which contains the vertex v^*, and let $H'=D-V(H)$. Since the subdigraph induced by $\{v^*\}$ is strong, such a subdigraph H exists. By Lemma 2.1, H' has a Hamilton path $x_1x_2\ldots x_k$ such that any arc going from $V(H)$ to $V(H')$ enters x_1, any arc going from $V(H')$ to $V(H)$ leaves from x_k, and there is no arc of the form x_ix_j, where $i \leqslant j-2$.

If $k=1$, we are done since $H=D-x_1$ is strong and $x_1 \neq v^*$.

If $k \geqslant 3$, we construct the digraph D' from D by removing the vertex x_1 and adding all the arcs wx_2 where w is an in-neighbor of x_1 in H, that is,

$$D'=(D-x_1)+\{wx_2 : wx_1 \in A(D) \quad \text{and} \quad w \in V(H)\}.$$

Clearly, D' is strong and the arcs lost in constructing D' are x_1x_2 and those from $\{x_3,\ldots,x_k\}$ to x_1. So D' has at least

$$\binom{n-1}{2}+3-(k-1) \geqslant \binom{n-1}{2}+3-(n-2)=\binom{n-2}{2}+3$$

arcs. By the induction hypothesis, we may choose a vertex $u \in V(D')\backslash\{v^*\}$ such that $D'-u$ is strong. Since $D'-x_2$ is not strong, we have $u \neq x_2$. Moreover, there exists an arc wx_2 with $w \in V(H)$ and $w \neq u$. It is now clear that $D-u$ is also strong.

Next, we suppose that $k=2$. Then $H=D-\{x_1,x_2\}$. The arcs lost in removing the vertices x_1, x_2 from D are those from $V(H)$ to x_1 and those from x_2 to $V(H)$ apart from x_1x_2.

Let

$$I_1=\{wx_1 : w \in V(H) \ \text{ and } \ wx_1 \notin A(D)\}$$

and

$$I_2=\{x_2w : w \in V(H) \ \text{ and } \ x_2w \notin A(D)\}.$$

Suppose that $|I_1|+|I_2| \geqslant 2$. If $n=5$, then H is a strong oriented graph with 3 vertices and so H is a directed cycle of length 3. It follows that

$$\begin{aligned}|A(D)| &= |A(H')|+|A(H)|+2(n-2)-(|I_1|+|I_2|)\\ &\leqslant 1+3+2\times 3-2\\ &= 8<9\\ &= \binom{n-1}{2}+3,\end{aligned}$$

a contradiction. This implies that $n \geqslant 6$. So we have that

$$|V(H)| = n - 2 \geqslant 4$$

and

$$\begin{aligned}|A(H)| &= |A(D)| - |A(H')| - (2(n-2) - (|I_1| + |I_2|)) \\ &\geqslant \binom{n-1}{2} + 3 - 1 - (2(n-2) - 2) \\ &= \binom{n-3}{2} + 3.\end{aligned}$$

Apply the induction hypothesis to H, we can find a vertex u of H such that $u \neq v^*$ and $H - u$ is strong. If $D - u$ is strong, we are done. If $D - u$ is not strong, then u is either the sole in-neighbor of x_1 in H or the sole out-neighbor of x_2 in H, but not both. In fact, if u is the sole in-neighbor of x_1 in H and is the sole out-neighbor of x_2 in H, then

$$|A(H)| = |A(D)| - 3 \geqslant \binom{n-1}{2} > \binom{n-2}{2},$$

contradicting the fact that D is an oriented graph. So, by duality, we may assume that u is the sole out-neighbor of x_2, but not the sole in-neighbor of x_1. Let $A_1 = \{x_2w : uw \in A(D) \text{ and } w \in V(H)\}$ and let $D' = (D - u) + A_1$. Clearly, D' is strong and the possible arcs lost in constructing D' are ux_1, x_2u and those from $V(H)$ to u. Since H is a strong digraph, there exists at least one out-neighbor v of u in $V(H)$. Combining this with the fact that H is an oriented graph, we have $vu \notin A(D)$. So D' has at least

$$\binom{n-1}{2} + 3 - (2 + (n-4)) = \binom{n-2}{2} + 3$$

arcs. By the induction hypothesis, we may choose a vertex $u' \in V(D') \setminus \{v^*\}$ such that $D' - u'$ is strong. Clearly, $u' \neq x_2$. It follows that $D - u'$ is strong.

Suppose that $|I_1| + |I_2| \leqslant 1$. Then, by duality, we may assume that x_1 is dominated by every vertex of H, and x_2 dominates every vertex of H but at most one vertex. If x_2 dominates v^*, let T be a spanning out-arborescence of H rooted at v^* and let u be a leaf of T. Then $u \neq v^*$ and the subdigraph $D - u$ is strong. If x_2 does not dominate v^*, then x_2 dominates every vertex of $V(H) \setminus \{v^*\}$. Choose an arc $wv^* \in A(H)$ and let u be an arbitrary vertex in $V(H) \setminus \{w, v^*\}$. Then $D - u$ is strong. The proof is complete.

Next we introduce a class of oriented graphs D_n that will show that the bound on the number of arcs in Theorem 2.1 is sharp.

Example 2.1 For an integer $n \geqslant 4$, let D_n be the oriented graph obtained by starting with a directed cycle $x_1x_2 \ldots x_nx_1$ and adding a set of arcs $\{x_jx_i : 1 \leqslant i \leqslant j - 2 \leqslant n - 3\}$ to the cycle. The oriented graph D_6 is shown in Figure 1. The oriented graph D_n has $\binom{n-1}{2} + 2$ arcs, is strong, and x_n is the unique vertex whose removal does not destroy strongly connectivity.

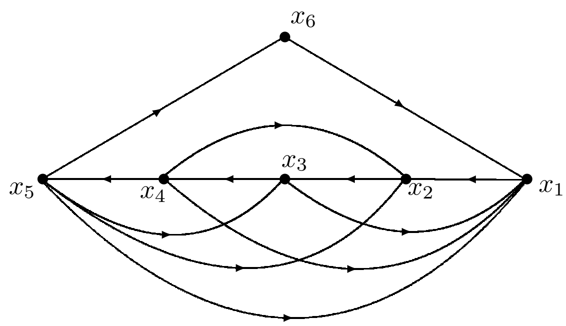

Figure 1. The oriented graph D_6

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