On a discrete Davey-Stewartson system

Gegenhasi[1,2*] Xing-Biao Hu[1†] Decio Levi[3‡]
[1]Institute of Computational Mathematics and
Scientific Engineering Computing, AMSS, Academia Sinica,
P.O. Box 2719, Beijing 100080, P.R. CHINA
[2]Graduate School of the Chinese Academy of Sciences, Beijing, P.R. CHINA
[3]Dipartimento di Ingegneria Elettronica Universitá degli Studi Roma Tre
Via della Vasca Navale, 84 I00146 Roma Italy

Abstract

We propose a differential difference equation in $\mathcal{R}^1 \times \mathcal{Z}^2$ and study it by Hirota's bilinear method. This equation has a singular continuum limit into a system which admits the reduction to the Davey-Stewartson equation. The solutions of this discrete DS system are characterized by Casorati and Grammian determinants. Based on the bilinear form of this discrete DS system, we construct the bilinear Bäcklund transformation which enables us to obtain its Lax pair.

KEYWORDS: Discrete DS equation, Casorati determinant, Grammian determinant, Bäcklund transformation, Lax pair

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1 Introduction

The nonlinear Schrödinger equation (NLS)

$$q_t + (\mathbf{v}\bigtriangledown)q - i\sum_{\alpha,\beta}\frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta}\frac{\partial^2 q}{\partial x_\alpha \partial x_\beta} - i\omega_1|q|^2 q = 0 \tag{1.1}$$

is the simplest universal model for the slow evolution of the envelope $q(\mathbf{r}, t)$ of an almost monochromatic wavetrain $\exp(i\mathbf{k_0}\mathbf{r} - i\omega(k_0)t)$ in a weakly nonlinear medium of nonlinear dispersion relation

$$\omega(k) = \omega_0(k) + \omega_1(k)|q|^2 + \cdots \tag{1.2}$$

In a $d + 1$ dimensional space this equation has $d + 1$ canonical forms. For $d = 1$ they are the "self-focusing NLS"

$$q_t + i(q_{xx} + |q|^2 q) = 0, \qquad if \quad \omega_0''\omega_1 > 0 \tag{1.3}$$

*gegen@amss.ac.cn
†hxb@lsec.cc.ac.cn
‡levi@fis.uniroma3.it

and the "self-defocusing NLS"

$$q_t + i(q_{xx} - |q|^2 q) = 0, \qquad if \quad \omega_0'' \omega_1 < 0. \tag{1.4}$$

In the literature, one can find many results on eqs. (1.3, 1.4) and their discrete versions (see, e.g. [1, 12] and references therein).

There are several important generalizations of the NLS's (1.3, 1.4). The best known example is the Davey-Stewartson (DS) equation [2, 4], a partial differential equation in $\mathcal{R}^3$ given by:

$$iq_t + \sigma^2(q_{xx} + \sigma^2 q_{yy}) = -\alpha |q|^2 q + q\phi, \tag{1.5}$$
$$\phi_{xx} - \sigma^2 \phi_{yy} = 2\alpha(|q|^2)_{xx}, \tag{1.6}$$

where $\alpha^2 = 1, \sigma^2 = \pm 1$. In the hyperbolic case $\sigma^2 = 1$, the system (1.5, 1.6) is called DSI while DSII corresponds to the elliptic case $\sigma^2 = -1$. In the following we will focus on the DSI equation. The DSI equation (1.5, 1.6) is a reduction of the system

$$iq_t + (q_{xx} + q_{yy}) = -\alpha q^2 r + q\phi, \tag{1.7}$$
$$-ir_t + (r_{xx} + r_{yy}) = -\alpha q r^2 + r\phi, \tag{1.8}$$
$$\phi_{xx} - \phi_{yy} = 2\alpha(qr)_{xx}, \tag{1.9}$$

obtained by requiring that $r = q^*$. By the variable transformation $\partial_x = \frac{1}{\sqrt{2}}(\partial_X + \partial_Y), \partial_y = \frac{1}{\sqrt{2}}(\partial_X - \partial_Y), \phi = \alpha qr + \alpha\psi$, the system (1.7–1.9) may be transformed into

$$iq_t + (q_{XX} + q_{YY}) = \alpha q \psi, \tag{1.10}$$
$$-ir_t + (r_{XX} + r_{YY}) = \alpha r \psi, \tag{1.11}$$
$$\psi_{XY} = \frac{1}{2}(\partial_X^2 + \partial_Y^2)(qr). \tag{1.12}$$

Integrable discrete versions for the DS equations have not been much studied yet although much work has been done on the discrete NLS equation [1]. A few partial results have been presented by Nijhoff and Konopelchenko in [10, 11, 15]. The purpose of this paper is to propose a new discrete integrable system of equations which can be considered as a discrete version for the DSI system (1.10–1.12).

Let us consider the following system

$$iv_t + \alpha_1 e^{u_{n-1}+u_{n+1}-2u} v_{n-1} + \alpha_2 e^{u_{k-1}+u_{k+1}-2u} v_{k+1} - (\alpha_1 + \alpha_2) v = 0, \tag{1.13}$$
$$-iw_t + \alpha_1 e^{u_{n-1}+u_{n+1}-2u} w_{n+1} + \alpha_2 e^{u_{k-1}+u_{k+1}-2u} w_{k-1} - (\alpha_1 + \alpha_2) w = 0, \tag{1.14}$$
$$z_1 - z_1 e^{u_{n+1,k+1}+u-u_{k+1}-u_{n+1}} + z_2 v_{k+1} w_{n+1} = 0, \tag{1.15}$$

where α_1, α_2, z_1 and z_2 are constants. In eqs. (1.13–1.15) and in the following we always use a simplified notation for $f(n, k, t)$. We write explicitly a discrete independent variable only when it is shifted from its position. For example,

$$f \equiv f(n,k,t), \quad f_{n+1} \equiv f(n+1,k,t), \quad f_{k+1} \equiv f(n,k+1,t), \quad f_{n+1,k-1} \equiv f(n+1,k-1,t).$$

Let us now show that eqs. (1.13–1.15) may be thought of as a discrete version of the DSI system (1.10–1.12). Let us set

$$\alpha_1 = \frac{2}{\epsilon^2}, \quad \alpha_2 = \frac{2}{\delta^2}, \quad z_1 = \frac{1}{\epsilon\delta}, \quad z_2 = -\frac{1}{4}\alpha, \quad \epsilon n = X, \quad \delta k = Y,$$

and expand the dependent fields in power series around $\delta = 0$ and $\epsilon = 0$,

$$
\begin{aligned}
&v_{k+1} = v(n,k) \equiv q(n\epsilon,(k+1)\delta) = q(X, Y+\delta) = q + \delta q_Y + \frac{\delta^2}{2} q_{YY} + \cdots \\
&w_{n+1} = w(n+1,k) \equiv r((n+1)\epsilon, k\delta) = r(X+\epsilon, Y) = r + \epsilon r_X + \frac{\epsilon^2}{2} r_{XX} + \cdots \\
&u_{n+1} = u + \epsilon u_X + \frac{\epsilon^2}{2} u_{XX} + \cdots \\
&u_{n+1,k+1} = u + \epsilon u_X + \delta u_Y + \frac{\epsilon^2}{2} u_{XX} + \epsilon\delta u_{XY} + \frac{\delta^2}{2} u_{YY} + \cdots \\
&\qquad\qquad \cdots\cdots
\end{aligned}
$$

Then the continuous limit of eqs. (1.13–1.15) gives

$$iq_T + (q_{XX} + q_{YY}) = -2q(\partial_X^2 + \partial_Y^2)u, \tag{1.16}$$

$$-ir_T + (r_{XX} + r_{YY}) = -2r(\partial_X^2 + \partial_Y^2)u, \tag{1.17}$$

$$u_{XY} = -\frac{1}{4}\alpha qr, \tag{1.18}$$

where $i\partial_T = i\partial_t - \frac{2}{\epsilon}\partial_X + \frac{2}{\delta}\partial_Y$. Under the transformation $\psi = -2\alpha(\partial_X^2 + \partial_Y^2)u, T \longrightarrow t$, the system (1.16–1.18) reduces to the DSI system (1.10–1.12) .

In the following we will study eqs. (1.13–1.15) using Hirota bilinear method. By the dependent variable transformation

$$u = \ln F, \qquad v = e^{-i(\alpha_1+\alpha_2)t} G/F, \qquad w = e^{i(\alpha_1+\alpha_2)t} H/F \tag{1.19}$$

eqs. (1.13-1.15) are transformed into the bilinear form

$$[iD_t + \alpha_1 e^{-D_n} + \alpha_2 e^{D_k}] G \cdot F = 0, \tag{1.20}$$

$$[iD_t + \alpha_1 e^{-D_n} + \alpha_2 e^{D_k}] F \cdot H = 0, \tag{1.21}$$

$$z_1[e^{1/2(D_n-D_k)} - e^{1/2(D_n+D_k)}] F \cdot F + z_2 e^{1/2(D_k-D_n)} G \cdot H = 0, \tag{1.22}$$

where, as usual, the bilinear operators D_t and $\exp(\delta D_n)$[9] are defined as:

$$
\begin{aligned}
&D_t^m a \cdot b \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(t) b(t')\bigg|_{t'=t}, \\
&\exp(\delta D_n)\ a \cdot b \equiv a(n+\delta) b(n-\delta).
\end{aligned}
$$

In section 2, we present the double-Casorati determinant solutions to the differential–difference system (1.13-1.15). Its Grammian determinant solutions are presented in section 3, while in section 4, a bilinear Bäcklund transformation and Lax pair are derived. Section 5 is devoted to the conclusions and a discussion of the result obtained.

2 Double-Casorati determinant solutions to the discrete DS system

It is well–known that the continuous DS equation (1.7–1.9) has solutions expressed in terms of double-Wronskians of the solutions of the Spectral Problem [3, 5, 16, 8, 6]. In this section, we present the double-Casorati determinant solutions for the discrete DS system (1.20–1.22). An example of double-Casorati determinant solution for eqs (1.13–1.15) is the one-soliton solution given later in Fig. 1.

Let us introduce the following double-Casorati determinant:

$$
\begin{aligned}
&|0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'| = \\
&= \begin{vmatrix} \phi_1(n) & \cdots & \phi_1(n+m-1); \psi_1(k) & \cdots & \psi_1(k+2N-m-1) \\ \phi_2(n) & \cdots & \phi_2(n+m-1); \psi_2(k) & \cdots & \psi_2(k+2N-m-1) \\ \vdots & & \vdots \quad\quad \vdots & & \vdots \\ \phi_{2N}(n) & \cdots & \phi_{2N}(n+m-1); \psi_{2N}(k) & \cdots & \psi_{2N}(k+2N-m-1) \end{vmatrix}, \quad (2.1)
\end{aligned}
$$

where $\phi_r(n,t)$ and $\psi_r(k,t)(r=1,2,\cdots,2N)$ satisfy the equations

$$i\frac{\partial}{\partial t}\varphi_r(n) = -\alpha_1\varphi_r(n-1), \quad (2.2)$$

$$i\frac{\partial}{\partial t}\psi_r(k) = \alpha_2\psi_r(k-1). \quad (2.3)$$

Taking into account eq. (2.1) we can state the following Proposition:

Proposition 1 *The following double-Casorati determinants*

$$F = |0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'|, \quad (2.4)$$

$$G = z_1|0,1,\cdots,m;\ 0',1',\cdots,(2N-m-2)'|, \quad (2.5)$$

$$H = \frac{1}{z_2}|0,1,\cdots,m-2;\ 0',1',\cdots,(2N-m)'|, \quad (2.6)$$

provide solutions to eqs. (1.20–1.22).

Proof: From eqs. (2.4–2.6) for any integer number i and j we have

$$F_{n+i,m+j} = |1,2,\cdots,m-1+i;\ 0',1',\cdots,(2N-m-1+j)'|, \quad (2.7)$$

$$G_{n+i,k+j} = z_1|0,1,\cdots,m+i;\ 1',2',\cdots,(2N-m-2+j)'|, \quad (2.8)$$

$$H_{n+i,k+j} = \frac{1}{z_2}|1,2,\cdots,m-2+i;\ 0',1',\cdots,(2N-m+j)'|. \quad (2.9)$$

From the equations (2.2, 2.3) we get

$$
\begin{aligned}
iF_t = -\alpha_1|-1,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'| \\
+\alpha_2|0,1,\cdots,m-1;\ (-1)',1',\cdots,(2N-m-1)'|, \quad (2.10) \\
iG_t = z_1(-\alpha_1|-1,1,\cdots,m;\ 0',1',\cdots,(2N-m-2)'| \\
+\alpha_2|0,1,\cdots,m;\ (-1)',1',\cdots,(2N-m-2)'|), \quad (2.11) \\
iH_t = \frac{1}{z_2}(-\alpha_1|-1,1,\cdots,m-2;\ 0',1',\cdots,(2N-m)'| \\
+\alpha_2|0,1,\cdots,m-2;\ (-1)',1',\cdots,(2N-m)'|). \quad (2.12)
\end{aligned}
$$

Introducing eqs. (2.7–2.12) into eq. (1.20) we get the determinant identity [9]:

$$
\begin{aligned}
&|1,2,\cdots,m;\ 0',1',\cdots,(2N-m-1)'||0,1,\cdots,m-1;\ 1',2',\cdots,(2N-m)'| \\
&-|1,2,\cdots,m;\ 1',2',\cdots,(2N-m)'||0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'| \\
&+|0,1,\cdots,m;\ 1',2',\cdots,(2N-m-1)'||1,2,\cdots,m-1;\ 0',1',\cdots,(2N-m)'| = 0. \quad (2.13)
\end{aligned}
$$

Substituting eqs. (2.7–2.11) into eqs. (1.21, 1.22) we get the equations

$$\begin{aligned}
&\alpha_1(|-1,0,\cdots,m-2;\ 0',1',\cdots,(2N-m-1)'||1,2,\cdots,m-1;\ 0',1',\cdots,(2N-m)'| \\
&\quad -|-1,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'||0,1,\cdots,m-2;\ 0',1',\cdots,(2N-m)'| \\
&\quad\ +|-1,1,\cdots,m-2;\ 0',1',\cdots,(2N-m)'||0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'|) \\
&+\alpha_2(|0,1,\cdots,m-1;\ 1',2',\cdots,(2N-m)'||0,1,\cdots,m-2;\ (-1)',0',\cdots,(2N-m-1)'| \\
&\quad -|0,1,\cdots,m-2;\ (-1)',1',\cdots,(2N-m)'||0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'| \\
&\quad\ +|0,1,\cdots,m-1;\ (-1)',1',\cdots,(2N-m-1)'||0,1,\cdots,m-2;\ 0',1',\cdots,(2N-m)'|)=0,
\end{aligned}$$

$$\begin{aligned}
&\alpha_1(|-1,0,\cdots,m-1;\ 0',1',\cdots,(2N-m-2)'||1,2,\cdots,m;\ 0',1',\cdots,(2N-m-1)'| \\
&\quad -|-1,1,\cdots,m;\ 0',1',\cdots,(2N-m-2)'||0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'| \\
&\quad\ +|-1,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'||0,1,\cdots,m;\ 0',1',\cdots,(2N-m-2)'|) \\
&+\alpha_2(|0,1,\cdots,m;\ 1',2',\cdots,(2N-m-1)'||0,1,\cdots,m-1;\ (-1)',0',\cdots,(2N-m-2)'| \\
&\quad -|0,1,\cdots,m-1;\ (-1)',1',\cdots,(2N-m-1)'||0,1,\cdots,m;\ 0',1',\cdots,(2N-m-2)'| \\
&\quad\ +|0,1,\cdots,m;\ (-1)',1',\cdots,(2N-m-2)'||0,1,\cdots,m-1;\ 0',1',\cdots,(2N-m-1)'|)=0,
\end{aligned}$$

which are identically satisfied when we take into account the determinant identities (2.13). In this way Proposition 1 is proved.

To construct the soliton solution, we choose a simple solution of eqs. (2.2, 2.3)

$$\phi_r(n,t)=\sum_{l=1}^{2N} a_{rl}p_l^{-n}e^{i\alpha_1 p_l t}, \qquad \psi_r(k,t)=\sum_{l=1}^{2N} b_{rl}q_l^{-k}e^{-i\alpha_2 q_l t}, \qquad r=1,2,\cdots,2N \tag{2.14}$$

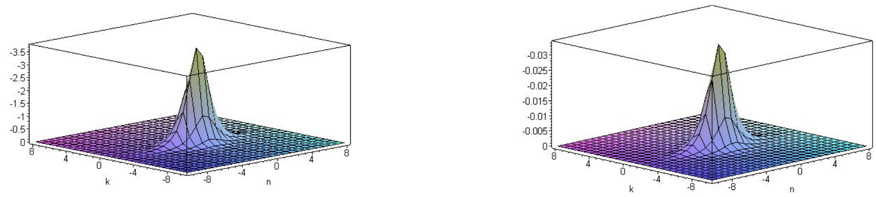
where p_r, q_r, a_r, b_r are arbitrary constants. Than the one-dromion solution of the discrete DS system is obtained by setting N=1 in eq. (2.1) and choosing $\phi_r(n), \psi_r(k)$ $(r=1,2)$ given by eq. (2.14) with, for example, $\alpha_1=i,\ \alpha_2=-i,\ z_1=z_2=1$. In such a case we have

$$\begin{aligned}
F&=(a_{11}b_{21}-a_{21}b_{11})p_1^{-n}q_1^{-k}e^{-(p_1+q_1)t}+(a_{12}b_{22}-a_{22}b_{12})p_2^{-n}q_2^{-k}e^{-(p_2+q_2)t} \\
&\quad +(a_{11}b_{22}-a_{21}b_{12})p_1^{-n}q_2^{-k}e^{-(p_1+q_2)t}+(a_{12}b_{21}-a_{22}b_{11})p_2^{-n}q_1^{-k}e^{-(p_2+q_1)t} &(2.15)\\
G&=(a_{11}a_{22}-a_{21}a_{12})p_1^{-n}p_2^{-n}(p_2^{-1}-p_1^{-1})e^{-(p_1+p_2)t} &(2.16)\\
H&=(b_{11}b_{22}-b_{21}b_{12})q_1^{-k}q_2^{-k}(q_2^{-1}-q_1^{-1})e^{-(q_1+q_2)t}. &(2.17)
\end{aligned}$$

In Fig. 1, we plot the 1-dromion solution $|v|=\frac{|G|}{F},\ |w|=\frac{|H|}{F}$ in the nk–plane with $a_{11}=a_{22}=\frac{1}{2},\ a_{12}=0,\ a_{21}=1,\ b_{11}=\frac{3}{4},\ b_{12}=\frac{1}{4},\ b_{21}=-\frac{1}{4},\ b_{22}=0,\ p_1=e,\ p_2=e^{-1},\ q_1=e^2,\ q_2=e^{-2}$, at the time $t=1$.

3 Grammian determinant solutions to the discrete DS equation

The Grammian technique was first used by Nakamura for constructing the solutions expressed in terms of the special functions for the two-dimensional Toda lattice equation and the KP equation [13, 14]. In [7] we can find a Grammian determinant solution for the continuous DS system. In this section, we present solutions of the discrete DS system written down in terms of Grammian determinants. At the end we show that by a proper choice of parameters the double-Casorati determinant solution and Grammian determinant solution give the same 1-soliton solution.



(a) (b)

Figure 1: The 1-dromion solution: (a) $|v|-$ field , (b) $|w|-$ field

Proposition 2 *The functions*

$$F=|C+\Omega|=|\boldsymbol{F}|, \tag{3.1}$$

$$G=\frac{z_1}{\alpha_1}\begin{vmatrix}\boldsymbol{F} & \Phi(n+1)\\ \Psi'(-k+1)^T & 0\end{vmatrix}, \qquad H=\frac{1}{z_2\alpha_2}\begin{vmatrix}\boldsymbol{F} & \Psi(k+1)\\ \Phi'(-n+1)^T & 0\end{vmatrix}, \tag{3.2}$$

where $\boldsymbol{F}$ *is a* $(M+N)\times(M+N)$ *matrix,* $C=(c_{\mu\nu})$ *is a* $(M+N)\times(M+N)$ *constant matrix,* Ω *is a* $(M+N)\times(M+N)$ *block diagonal matrix*

$$\Omega=\begin{pmatrix}\int_{-\infty}^{t}\varphi_r(n)\varphi'_j(-n)dt & \\ & \int_t^{+\infty}\psi_s(k)\psi'_l(-k)dt\end{pmatrix},$$

and Φ,Φ',Ψ,Ψ' *are* $M+N$ *column vectors*

$$\begin{aligned}
\Phi(n)&=(\varphi_1(n),\cdots,\varphi_M(n);\ 0,\cdots,0)^T,\\
\Phi'(-n)&=(\varphi'_1(-n),\cdots,\varphi'_M(-n);\ 0,\cdots,0)^T,\\
\Psi(k)&=(0,\cdots,0;\ \psi_1(k),\cdots,\psi_N(k))^T,\\
\Psi'(-k)&=(0,\cdots,0;\ \psi'_1(-k),\cdots,\psi'_N(-k))^T,
\end{aligned}$$

with $\varphi_r(n,t),\varphi'_j(n,t),\psi_s(k,t),\psi'_l(k,t)$, $r,j\in\{1,\cdots,M\}$, $s,l\in\{1,\cdots,N\}$, *satisfying the following equations:*

$$i\frac{\partial\varphi_r(n)}{\partial t}=-\alpha_1\varphi_r(n-1), \qquad i\frac{\partial\varphi'_j(-n)}{\partial t}=\alpha_1\varphi_j(-n-1), \tag{3.3}$$

$$i\frac{\partial\psi_s(k)}{\partial t}=\alpha_2\psi_s(k-1), \qquad i\frac{\partial\psi'_l(-k)}{\partial t}=-\alpha_2\psi'_l(-k-1), \tag{3.4}$$

solve the equations (1.20–1.22).

Proof: Using eqs. (3.3, 3.4), we are able to express, after some calculations, the functions appearing in eqs. (1.20–1.22) in terms of the Grammian determinants

$$F_{n+1}=F-\frac{i}{\alpha_1}\begin{vmatrix}\mathbf{F} & \Phi(n+1)\\ \Phi'(-n)^T & 0\end{vmatrix}, \qquad F_{k+1}=F-\frac{i}{\alpha_2}\begin{vmatrix}\mathbf{F} & \Psi(k+1)\\ \Psi'(-k)^T & 0\end{vmatrix}, \tag{3.5}$$

$$F_{k-1} = F + \frac{i}{\alpha_2}\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Psi'(-k+1)^T & 0 \end{vmatrix}, \quad F_{n-1} = F + \frac{i}{\alpha_1}\begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Phi'(-n+1)^T & 0 \end{vmatrix}, \tag{3.6}$$

$$F_{n+1,k+1} = F - \frac{i}{\alpha_2}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Psi'(-k)^T & 0 \end{vmatrix} - \frac{i}{\alpha_1}\begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Phi'(-n)^T & 0 \end{vmatrix} - \frac{1}{\alpha_1\alpha_2}\begin{vmatrix} \mathbf{F} & \Phi(n+1) & \Psi(k+1) \\ \Phi'(-n)^T & 0 & 0 \\ \Psi'(-k)^T & 0 & 0 \end{vmatrix}, \tag{3.7}$$

$$G_{k+1} = \frac{z_1}{\alpha_1}\begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Psi'(-k)^T & 0 \end{vmatrix}, \qquad H_{n+1} = \frac{1}{z_2\alpha_2}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Phi'(-n)^T & 0 \end{vmatrix}, \tag{3.8}$$

$$H_{k-1} = \frac{1}{z_2\alpha_2}\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Phi'(-n+1)^T & 0 \end{vmatrix}, \qquad G_{n-1} = \frac{z_1}{\alpha_1}\begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Psi'(-k+1)^T & 0 \end{vmatrix}, \tag{3.9}$$

$$F_t = \begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Psi'(-k)^T & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Phi'(-n)^T & 0 \end{vmatrix}, \tag{3.10}$$

$$iG_t = \frac{z_1}{\alpha_1}\{-\alpha_2\begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Psi'(-k)^T & 0 \end{vmatrix} - \alpha_1\begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Psi'(-k+1)^T & 0 \end{vmatrix} - i\begin{vmatrix} \mathbf{F} & \Phi(n+1) & \Phi(n) \\ \Psi'(-k+1)^T & 0 & 0 \\ \Phi'(-n)^T & 0 & 0 \end{vmatrix} + i\begin{vmatrix} \mathbf{F} & \Phi(n+1) & \Psi(k) \\ \Psi'(-k+1)^T & 0 & 0 \\ \Psi'(-k)^T & 0 & 0 \end{vmatrix}\}, \tag{3.11}$$

$$iH_t = \frac{1}{z_2\alpha_2}\{\alpha_1\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Phi'(-n)^T & 0 \end{vmatrix} + \alpha_2\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Phi'(-n+1)^T & 0 \end{vmatrix} + i\begin{vmatrix} \mathbf{F} & \Psi(k+1) & \Psi(k) \\ \Phi'(-n+1)^T & 0 & 0 \\ \Psi'(-k)^T & 0 & 0 \end{vmatrix} - i\begin{vmatrix} \mathbf{F} & \Psi(k+1) & \Phi(n) \\ \Phi'(-n+1)^T & 0 & 0 \\ \Phi'(-n)^T & 0 & 0 \end{vmatrix}\}. \tag{3.12}$$

We can thus prove that the functions F, G and H given by eqs. (3.1, 3.2) effectively satisfy the discrete DS system as, by substituting eqs. (3.5–3.6) into eqs. (1.20–1.22) we get the following three Jacobi identities for the determinants

$$\begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Phi'(-n)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Psi'(-k)^T & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{F} \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(n+1) & \Psi(k+1) \\ \Phi'(-n)^T & 0 & 0 \\ \Psi'(-k)^T & 0 & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Psi'(-k)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Phi'(-n)^T & 0 \end{vmatrix} = 0, \tag{3.13}$$

$$
\begin{aligned}
&\{\begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Phi'(-n+1)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Phi'(-n)^T & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{F} \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) & \Phi(n) \\ \Phi'(-n+1)^T & 0 & 0 \\ \Phi'(-n)^T & 0 & 0 \end{vmatrix} \\
&\qquad - \begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Phi'(-n)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Phi'(-n+1)^T & 0 \end{vmatrix}\} \\
&+\{\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Psi'(-k)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Phi'(-n+1)^T & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{F} \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k+1) & \Psi(k) \\ \Phi'(-n+1)^T & 0 & 0 \\ \Psi'(-k)^T & 0 & 0 \end{vmatrix} \\
&\qquad - \begin{vmatrix} \mathbf{F} & \Psi(k+1) \\ \Psi'(-k)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Phi'(-n+1)^T & 0 \end{vmatrix}\} = 0. \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
&\{\begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Psi'(-k)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Psi'(-k+1)^T & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{F} \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(n+1) & \Psi(k) \\ \Psi'(-k+1)^T & 0 & 0 \\ \Psi'(-k)^T & 0 & 0 \end{vmatrix} \\
&\qquad - \begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Psi'(-k+1)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Psi(k) \\ \Psi'(-k)^T & 0 \end{vmatrix}\} \\
&+\{\begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Psi'(-k+1)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Phi'(-n)^T & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{F} \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(n+1) & \Phi(n) \\ \Psi'(-k+1)^T & 0 & 0 \\ \Phi'(-n)^T & 0 & 0 \end{vmatrix} \\
&\qquad - \begin{vmatrix} \mathbf{F} & \Phi(n+1) \\ \Phi'(-n)^T & 0 \end{vmatrix}\begin{vmatrix} \mathbf{F} & \Phi(n) \\ \Psi'(-k+1)^T & 0 \end{vmatrix}\} = 0. \qquad (3.15)
\end{aligned}
$$

The simplest soliton solution for the discrete DS system (1.13–1.15) is obtained by taking the simplest possible choice for the functions $\varphi_r, \varphi'_j, \psi_s, \psi'_l$ satisfying eqs. (3.3, 3.4), i.e. an exponential

$$
\begin{aligned}
\varphi_r(n) &= k_r^n e^{i\alpha_1 k_r^{-1} t}, \qquad & \varphi'_j(-n) &= \bar{k}_j^{-n} e^{-i\alpha_1 \bar{k}_j^{-1} t}, \\
\psi_s(k) &= \omega_s^k e^{-i\alpha_2 \omega_s^{-1} t}, \qquad & \psi'_l(-k) &= \bar{\omega}_l^{-k} e^{i\alpha_2 \bar{\omega}_l^{-1} t},
\end{aligned}
$$

where $k_i, \bar{k}_j, \omega_s, \bar{\omega}_l$ are arbitrary constants.
When $N = 1$, if we take

$$
C = \begin{pmatrix} 0 & -\frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{pmatrix}, \quad k_1 = \omega_1 = 2, \quad \bar{k}_1 = (\frac{1}{2} + i)^{-1}, \quad \bar{\omega}_1 = (\frac{1}{2} + i)^{-1},
$$

we have the following 1-soliton solution for equations (1.13)-(1.15):

$$
u = \ln(\frac{1}{\alpha_1 \alpha_2}[(1+2i)^n (1+2i)^k e^{(\alpha_1 - \alpha_2)t} - 1]), \tag{3.16}
$$

$$
v = \alpha_2 z_1 \frac{2^{n+1}(\frac{1}{2}+i)^{k-1} e^{it[-\frac{1}{2}\alpha_1 + (i-\frac{1}{2})\alpha_2]}}{(1+2i)^n (1+2i)^k e^{(\alpha_1-\alpha_2)t} - 1}, \tag{3.17}
$$

$$
w = \frac{\alpha_1}{z_2} \frac{2^{k+1}(\frac{1}{2}+i)^{n-1} e^{it[(\frac{1}{2}-i)\alpha_1 + \frac{1}{2}\alpha_2]}}{(1+2i)^n (1+2i)^k e^{(\alpha_1-\alpha_2)t} - 1}. \tag{3.18}
$$

This same solution is obtained by considering the double-Casorati determinant solution (2.4–2.6) with $N = 1$ and

$$\phi_1(n,t) = (\frac{1}{2}+i)^{-n} e^{(\frac{1}{2}+i)i\alpha_1 t}, \qquad \phi_2(n,t) = (\frac{1}{2})^{-n} e^{\frac{1}{2}i\alpha_1 t}$$
$$\psi_1(k,t) = (\frac{1}{2})^{-k} e^{-\frac{1}{2}i\alpha_2 t}, \qquad \psi_2(k,t) = (\frac{1}{2}+i)^{-k} e^{-(\frac{1}{2}+i)i\alpha_2 t}.$$

4 Bilinear Bäcklund transformation and Lax pair

In this section we construct a bilinear Bäcklund transformation for the bilinear equations (1.20–1.22), and then we derive from it a Lax pair for the discrete DS system (1.13–1.15).

To do so, let us redefine the functions F, G and H in term of one function f depending on an additional discrete variable m

$$F(n,k;t) = f(m,n,k;t), \qquad G(n,k;t) = f(m+1,n,k;t), \qquad H(n,k;t) = f(m-1,n,k;t),$$

Then eqs. (1.20–1.22) can be written as:

$$[iD_t e^{1/2D_m} + \alpha_1 e^{D_n - 1/2D_m} + \alpha_2 e^{D_k + 1/2D_m}] f \cdot f = 0, \tag{4.1}$$
$$[z_1 e^{1/2(D_n - D_k)} + z_2 e^{1/2(D_k - D_n) + D_m} + z_3 e^{1/2(D_n + D_k)}] f \cdot f = 0. \tag{4.2}$$

We can now state the following proposition:

Proposition 3 *The bilinear system (4.1, 4.2) has the Bäcklund transformation*

$$[\beta_1 e^{1/2D_n} - e^{-1/2D_n} - \mu_1 e^{D_m - 1/2D_n}] f \cdot g = 0, \tag{4.3}$$
$$[\beta_2 e^{1/2(D_m + D_k)} - e^{-1/2(D_m + D_k)} - \mu_2 e^{1/2(D_m - D_k)}] f \cdot g = 0, \tag{4.4}$$
$$[iD_t - \alpha_1 \frac{\mu_1}{\beta_1} e^{D_m - D_n} - \alpha_2 \frac{\mu_2}{\beta_2} e^{-D_k}] f \cdot g = 0, \tag{4.5}$$

where β_1, β_2, μ_1, μ_2 are arbitrary constants, with μ_1, μ_2 satisfying the constraint

$$\mu_1 z_1 + \mu_2 z_2 = 0. \tag{4.6}$$

Proof: Let f be a solution of equations (4.1, 4.2). Using eqs. (4.3–4.6), we can by straightforward calculations show that eqs. (4.1, 4.2) are satisfied for $g(m,n,k;t)$

$$\begin{aligned}
&-[e^{1/2D_m}f\cdot f][iD_te^{1/2D_m}+\alpha_1e^{D_n-1/2D_m}+\alpha_2e^{D_k+1/2D_m}]g\cdot g\\
&\quad\equiv\{[iD_te^{1/2D_m}+\alpha_1e^{D_n-1/2D_m}+\alpha_2e^{D_k+1/2D_m}]f\cdot f\}[e^{1/2D_m}g\cdot g]\\
&\qquad-\{[iD_te^{1/2D_m}+\alpha_1e^{D_n-1/2D_m}+\alpha_2e^{D_k+1/2D_m}]g\cdot g\}[e^{1/2D_m}f\cdot f]\\
&\quad=2\sinh(1/2D_m)(iD_tf\cdot g)\cdot fg+2\alpha_1\sinh(1/2(D_n-D_m))(e^{1/2D_m}f\cdot g)\cdot(e^{-1/2D_m}f\cdot g)\\
&\qquad+2\alpha_2\sinh(1/2D_k)(e^{1/2(D_k+D_m)}f\cdot g)\cdot(e^{-1/2(D_k+D_m)}f\cdot g\\
&\quad=2\sinh(1/2D_m)(iD_tf\cdot g)\cdot fg+2\alpha_1\sinh(1/2(D_n-D_m))(\frac{\mu_1}{\beta_1}e^{D_m-1/2D_n}f\cdot g)\cdot(e^{-1/2D_m}f\cdot g)\\
&\qquad+2\alpha_2\sinh(1/2D_k)(\frac{\mu_2}{\beta_2}e^{1/2(D_m-D_k)}f\cdot g)\cdot(e^{-1/2(D_k+D_m)}f\cdot g\\
&\quad=2\sinh(1/2D_m)(iD_tf\cdot g)\cdot fg-2\alpha_1\frac{\mu_1}{\beta_1}\sinh(1/2D_m))(e^{D_m-D_n}f\cdot g)\cdot fg\\
&\qquad-2\alpha_2\frac{\mu_2}{\beta_2}\sinh(1/2D_m))(e^{-D_k}f\cdot g)\cdot fg=0,\\
&-[e^{1/2(D_n+D_k)}f\cdot f][z_1e^{1/2(D_n-D_k)}+z_2e^{1/2(D_k-D_n)+D_m}+z_3e^{1/2(D_n+D_k)}]g\cdot g\\
&\quad\equiv\{[z_1e^{1/2(D_n-D_k)}+z_2e^{1/2(D_k-D_n)+D_m}+z_3e^{1/2(D_n+D_k)}]f\cdot f\}[e^{1/2(D_n+D_k)}g\cdot g]\\
&\qquad-\{[z_1e^{1/2(D_n-D_k)}+z_2e^{1/2(D_k-D_n)+D_m}+z_3e^{1/2(D_n+D_k)}]g\cdot g\}[e^{1/2(D_n+D_k)}f\cdot f]\\
&\quad=2z_1\sinh(-1/2D_k)(e^{1/2D_n}f\cdot g)\cdot(e^{-1/2D_n}f\cdot g)\\
&\qquad+2z_2\sinh(1/2(D_m-D_n))(e^{1/2(D_m+D_k)}f\cdot g)\cdot(e^{-1/2(D_m+D_k)}f\cdot g)\\
&\quad=-2z_1\mu_1\sinh(1/2D_k)(e^{1/2D_n}f\cdot g)\cdot(e^{D_m-1/2D_n}f\cdot g)\\
&\qquad+2z_2\mu_2\sinh(1/2(D_m-D_n)(e^{1/2(D_m+D_k)}f\cdot g)\cdot(e^{1/2(D_m-D_k)}f\cdot g)\\
&\quad=-2z_1\mu_1\sinh(1/2D_k)(e^{1/2D_n}f\cdot g)\cdot(e^{D_m-1/2D_n}f\cdot g)\\
&\qquad-2z_2\mu_2\sinh(1/2D_k)(e^{1/2D_n}f\cdot g)\cdot(e^{D_m-1/2D_n}f\cdot g)=0
\end{aligned}$$

In this way, Proposition 3 is satisfied and eqs. (4.3–4.6) constitute a BT for (4.1, 4.2).

From the bilinear Bäcklund transformation (4.3-4.6), we can derive a Lax pair for the discrete DS system (1.13-1.15).

Let us set

$$u=\ln f,\quad v=\frac{f_{m+1}}{f},\quad w=\frac{f_{m-1}}{f},\quad \phi=\frac{g}{f}. \tag{4.7}$$

Under the dependent variable transformation (4.7), the bilinear BT (4.3-4.5) become the nonlinear equations:

$$\beta_1\phi-\phi_{n+1}-\mu_1vw_{n+1}\phi_{m-1,n+1}=0, \tag{4.8}$$

$$\beta_2w_{k-1}\phi_{m-1,k-1}-w_{k-1}\phi-\mu_2w\phi_{m-1}=0, \tag{4.9}$$

$$i\phi_t+\alpha_1\frac{\mu_1}{\beta_1}v_{n-1}w_{n+1}e^{u_{n+1}+u_{n-1}-2u}\phi_{m-1.n+1}+\alpha_2\frac{\mu_2}{\beta_2}e^{u_{k+1}+u_{k-1}-2u}\phi_{k+1}=0, \tag{4.10}$$

where β_1, β_2, μ_1, μ_2 are arbitrary constants satisfying the constraint (4.6). Eliminating $\phi_{m-1,n+1}$, $\phi_{m-1,k-1}$, ϕ_{m-1} from eqs. (4.8–4.10), we obtain the following Lax pair for the differential–difference DS system (1.13–1.15)

$$\beta_2(\frac{\beta_1\phi_{n-1,k-1}-\phi_{k-1}}{\mu_1v_{n-1,k-1}})-\mu_2(\frac{\beta_1\phi_{n-1}-\phi}{\mu_1v_{n-1}})-\phi w_{k-1}=0, \tag{4.11}$$

$$i\phi_t+\frac{\alpha_1}{\beta_1}v_{n-1}e^{u_{n-1}+u_{n+1}-2u}\frac{\beta_1\phi-\phi_{n+1}}{v}+\frac{\alpha_2}{\beta_2}\mu_2e^{u_{k-1}+u_{k+1}-2u}\phi_{k+1}=0. \tag{4.12}$$

By imposing the compatibility of eqs. (4.11, 4.12) we obtain the discrete Davey-Stewartson system (1.13–1.15). In fact, from eq. (4.11), we can derive

$$\beta_1\phi_{n-1,k-1} = \phi_{k-1} + \frac{\mu_1}{\beta_2} v_{n-1,k-1}[\phi w_{k-1} + \frac{\mu_2}{\mu_1 v_{n-1}}(\beta_1\phi_{n-1} - \phi)], \tag{4.13}$$

$$\phi_{n+1,k-1} = \beta_1\phi_{k-1} - \frac{\mu_1}{\beta_2} v_{k-1}[w_{n+1,k-1}\phi_{n+1} + \frac{\mu_2}{\mu_1 v}(\beta_1\phi - \phi_{n+1})], \tag{4.14}$$

$$\beta_1\phi_{n-1,k+1} = \phi_{k+1} + \frac{\mu_1}{\mu_2} v_{n-1,k+1}[\frac{\beta_2}{\mu_1 v_{n-1}}(\beta_1\phi_{n-1} - \phi) - \phi_{k+1} w], \tag{4.15}$$

the expressions of $\phi_{n-1,k-1}, \phi_{n+1,k-1}, \phi_{n-1,k+1}$ in terms of $\phi, \phi_{n-1}, \phi_{k-1}, \phi_{n+1}, \phi_{k+1}$. By differentiating eq. (4.11) with respect to t and substituting into it eqs. (4.13–4.15) we obtain an expression in terms of just ϕ, ϕ_{n-1}, ϕ_{k-1}, ϕ_{n+1} and ϕ_{k+1}. Equating to zero the coefficients of ϕ , ϕ_{n-1}, ϕ_{k-1}, ϕ_{n+1}, and ϕ_{k+1} we derive that the coefficient of ϕ_{n-1} gives eq. (1.13), the coefficient of ϕ gives eq. (1.14), both the coefficients of ϕ_{n+1} and ϕ_{k+1} give eq. (1.15) and the coefficient of ϕ_{k-1} vanishes.

5 Conclusion.

A discrete version of the Davey-Stewartson (DSI) system is proposed and investigated using the bilinear method. This DSI system exhibits N-soliton solutions expressed in terms of determinants of two different types, double–Casorati and Grammians. Moreover, we have constructed the bilinear Bäcklund transformation and derived from it its Lax pair.

A few problems are still open. Among them the most significant is surely to find the proper reduction which gives the Davey-Stewartson equation from the system. Moreover, since in the continuous case we have two physically interesting cases, the DSI and DSII equations, it would also be interesting to find the discrete version of the DSII equation.

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