Construction of dKP and BKP equations with self-consistent sources

X B Hu[1] and H Y Wang[1,2]

[1]Institute of Computational Mathematics and Scientific Engineering Computing, AMSS, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, P.R. CHINA
[2]Graduate School of the Chinese Academy of Sciences, Beijing, P.R. CHINA

E-mail: hxb@lsec.cc.ac.cn and wanghy@lsec.cc.ac.cn

Abstract. A new procedure is firstly proposed to construct soliton equations with self-consistent sources(SESCSs) in bilinear forms, starting from the Gram-type determinant solution or Gram-type pfaffian solution of soliton equations without sources, then soliton solutions of SESCSs can be given. This procedure is applied to the 2D Toda lattice equation, the discrete KP equation and the BKP equation.

1. Introduction

While integrable systems have been under active investigation since the discovery of solitons, only a relatively small portion of the literature was devoted to the subject of multi-dimensional integrable systems. However, increasing attention has been paid to the study of (2+1)-dimensional integrable systems in recent years [1, 2, 3]. An important area of this research is to search for new (2+1)-dimensional soliton equations. Several celebrated examples of multi-dimensional integrable systems have been found in fields ranging from fluid dynamics, nonlinear optics, particle physics and general relativity to differential and algebraic geometry, topology, and etc. The special significance of integrable systems is that they combine tractability with nonlinearity. Hence these systems enable one to explore nonlinear phenomena in multi-dimensions while working with explicit solutions. In the literature, there exist several approaches to search for new candidates of (2+1)-dimensional integrable systems. One of effective ways to do so is to find integrable extensions of the known (2+1)-dimensional integrable systems. For example, for the well-known KP equation, two coupled KP equations have been found along two different lines of research. One is the so-called KP equation with self-consistent sources [4] while the other is generated through the procedure of what we now call pfaffianization [5, 6]. Following pioneering work by Melnikov [4, 7, 8, 9, 10, 11, 12], a number of interesting contributions have been made to the study of soliton equations with self-consistent sources (SESCSs) via Inverse Scattering Method, Darboux transformations and Hirota bilinear method etc. [13]-[38]. Unfortunately, most results in this direction have been achieved just in *continuous* case. Less work has been done on semi-discrete soliton equations with a source and self-consistent sources. In [39], a differential-difference version to the KdV equation with a source was investigated. In [40], Toda lattice hierarchy with self-consistent sources are constructed and studied by means of the Darboux transformation. In [41], the authors presented 2D Toda lattice equation with self-consistent sources and showed its integrability. As for the study of SESCSs in the fully discrete case, the situation is even worse than differential-difference case as it still remains as an open problem how to find SESCSs in fully discrete case. One obvious cause for this is that a unified algebraic method to construct both continuous and discrete SESCSs is still missing. On the other hand, it is noted that pfaffianization method has been successfully applied to the fully discrete case: the Hirota-Miwa discrete KP equation [43]. Based on this observation, it is natural to expect that an algebraic method similar to pfaffianization method, would be found, which enable one to produce both continuous and discrete SESCSs in a systematic way.

The purpose of this paper is to give such a new algebraic method which provides a unified way to generate SESCSs both in continuous and discrete cases. We call it "source generalization" method. As an application of "source generalization" method, the discrete KP equation with self-consistent sources are found. Besides, as a bonus, this method also enables one to produce B-type KP equation with self-consistent sources which has not been known yet before.

The paper is organized as follows. In section 2, we will present so-called source generalization method to produce SESCSs by taking 2D Toda Lattice equation as an illustrative example. Then as an application of source generalization method, the discrete KP equation with self-consistent sources and the BKP equation with self-consistent sources are found in section 3 and section 4 respectively. It turns out that resulting SESCSs possess bilinear Backlund transformations and soliton solutions. Finally, conclusion and discussions are given in section 5.

2. 2D Toda Lattice equation with self-consistent sources

In this section, we will present a new procedure of producing SESCSs which is similar to the procedure of Pfaffianization. In order to do so, let us first remind you the procedure of Pfaffianization. The key points behind Pfaffianization method are to first express N-soliton solutions of an 'un-Pfaffianized' equation in the form of Wronskian, Casorati or Grammian type determinant, then to construct a pfaffian with elements satisfying the Pfaffianized form of the dispersion relation given in the determinant solutions of the 'un-Pfaffianized' equation and finally to seek coupled bilinear equations whose solutions are these pfaffians.

We now briefly describe our procedure of producing SESCSs. There are three steps involved in the procedure:

1. to express N-soliton solutions of a soliton equation without sources in the form of determinant or pfaffian with some arbitrary constants, say $c_{i,j}$.

2. to generalize the determinant or pfaffian in step 1 by replacing arbitrary constants

with arbitrary functions of one variable, e.g. $c_{i,j}(t)$.

3. to seek coupled bilinear equations whose solutions are these generalized determinants or pfaffians.

Compared with Pfaffianization method, the above procedure for producing SESCSs can be applicable to more types of soliton equations as in step 1 there is no restriction that N-soliton solutions should be in determinant form. As for step 2 in our procedure, the idea involved is quite natural. We recall that historically, one often applies a similar technique to solve inhomogeneous differential equations by using solutions to the corresponding homogeneous differential equations. In the following, we will illustrate our procedure for producing SESCSs in more detail by considering a concrete example.

The 2D Toda lattice equation is written as [6]

$$\frac{\partial^2 Q_n}{\partial t \partial x} = V_{n+1} + V_{n-1} - 2V_n, \tag{1}$$

$$Q_n = \ln(1 + V_n). \tag{2}$$

Through the dependent variable transformation

$$V_n = \frac{\partial^2}{\partial t \partial x} \ln \tau_n,$$

equations (1)-(2) can be transformed into the bilinear form

$$(D_x D_t - 2e^{D_n} + 2)\tau_n \cdot \tau_n = 0, \tag{3}$$

where D is the Hirota bilinear operator [6]

$$D_x^m D_t^n f(x,t) \cdot g(x,t) = \frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial s^n} f(x+y, t+s) g(x-y, t-s)|_{s=0,y=0}, \quad m, n = 0, 1, 2, \cdots$$

$$\exp(D_n) f_n \cdot g_n = f_{n+1} g_{n-1}.$$

We have the Gram-type determinant solution of the bilinear 2D Toda lattice equation (3):

$$\tau_n = \det(c_{ij} + (-1)^n \int_{-\infty}^{x} \varphi_i^{(n)} \bar{\varphi}_j^{(-n)} \mathrm{d}x)_{1 \leqslant i,j \leqslant N}, \quad c_{ij} = \text{constant} \tag{4}$$

where each $\varphi_i^{(n)}$, $\bar{\varphi}_j^{(-n)}$ satisfy the linear differential equations:

$$\frac{\partial \varphi_i^{(n)}}{\partial x} = \varphi_i^{(n+1)} - \varphi_i^{(n)}, \quad \frac{\partial \bar{\varphi}_j^{(-n)}}{\partial x} = \bar{\varphi}_j^{(-n+1)} + \bar{\varphi}_j^{(-n)}, \tag{5}$$

$$\frac{\partial \varphi_i^{(n)}}{\partial t} = -\varphi_i^{(n-1)}, \quad \frac{\partial \bar{\varphi}_j^{(-n)}}{\partial t} = -\bar{\varphi}_j^{(-n-1)}. \tag{6}$$

and here we assume that $\varphi_i^{(n)}$, $\bar{\varphi}_j^{(-n)}$ and their derivatives tend to zero when $x \to -\infty$. Let us express τ_n by a special kind of pfaffian:

$$\tau_n = \mathrm{pf}(1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*)_n, \tag{7}$$

where the pfaffian elements are defined as follows:

$$\mathrm{pf}(i,j^*)_n = c_{ij} + (-1)^n \int_{-\infty}^{x} \varphi_i^{(n)} \bar{\varphi}_j^{(-n)} \mathrm{d}x,$$
$$\mathrm{pf}(i,j)_n = \mathrm{pf}(i^*,j^*)_n = 0, \quad i,j = 1,2,\cdots,N$$

and other new pfaffian elements are given by

$$\mathrm{pf}(d_m^*, i)_n = \varphi_i^{(m)}, \qquad \mathrm{pf}(d_{-m}, j^*)_n = (-1)^m \bar{\varphi}_j^{(-m)},$$

$$\mathrm{pf}(d_{-m}, d_k^*)_n = \mathrm{pf}(d_{-m}, d_k)_n = \mathrm{pf}(d_{-m}^*, d_k^*)_n = 0.$$

Then the equation (3) can be reduced to an identity of determinants[6]:

$$\begin{aligned}
&\mathrm{pf}(d_{-n-1}, d_n^*, 1,2,\cdots,N,N^*,\cdots,1^*)_n \mathrm{pf}(d_{n-1}^*, d_{-n}, 1,2,\cdots,N,N^*,\cdots,1^*)_n \\
&+\mathrm{pf}(d_{-n-1}, d_{n-1}^*, 1,2,\cdots,N,N^*,\cdots,1^*)_n \mathrm{pf}(d_{-n}, d_n^*, 1,2,\cdots,N,N^*,\cdots,1^*)_n \\
&-\mathrm{pf}(d_{-n-1}, d_{n-1}^*, d_{-n}, d_n^*, 1,2,\cdots,N,N^*,\cdots,1^*)_n \mathrm{pf}(1,2,\cdots,N,N^*,\cdots,1^*)_n = 0.
\end{aligned}$$

In the following, we will construct the Toda lattice equation with self-consistent sources (TodaESCS). We set the function f_n in the following form:

$$\begin{aligned}
f_n &= \det(\gamma_{ij}(t) + (-1)^n \textstyle\int_{-\infty}^{x} \varphi_i^{(n)} \bar{\varphi}_j^{(-n)} \mathrm{d}x)_{1\leqslant i,j\leqslant N} \\
&= (1,2,\cdots,N,N^*,\cdots,2^*,1^*)_n,
\end{aligned} \tag{8}$$

where

$$\gamma_{ij}(t) \equiv \begin{cases} \gamma_i(t), & i=j \text{ and } 1 \leqslant i \leqslant K \leqslant N, \\ c_{ij}, & i \neq j \text{ and } 1 \leqslant i,j \leqslant N. \end{cases}$$

with $\gamma_i(t)$ being an arbitrary function of t and K being a positive integer, and $\varphi_i^{(n)}$, $\bar{\varphi}_j^{(-n)}$ still satisfy relations (5)-(6) and the boundary condition. Here the pfaffian elements are defined by

$$(i,j^*)_n = \gamma_{ij}(t) + (-1)^n \int_{-\infty}^{x} \varphi_i^{(n)} \bar{\varphi}_j^{(-n)} \mathrm{d}x,$$
$$(i,j)_n = (i^*,j^*)_n = 0, \quad i,j = 1,2,\cdots,N.$$

Through the property of determinants [6]:

$$\det(a_{i,j} - x_i y_j)_{1\leqslant i,j\leqslant N} = \det(a_{i,j})_{1\leqslant i,j\leqslant N} - \sum_{i,j=1}^{N} x_i y_j \Delta_{i,j}, \tag{9}$$

where $\Delta_{i,j}$ denotes the algebraic cofactor of $\det(a_{i,j})_{1\leqslant i,j\leqslant N}$, we can calculate f_{n+1}, f_{n-1}, $f_{n,x}$ and $f_{n,t}$ as follows:

$$f_{n,x} = (d_{-n}, d_n^*, 1,\cdots,N,N^*,\cdots,1^*)_n, \tag{10}$$

$$\begin{aligned}
f_{n,t} = &\sum_{j=1}^{K} \dot{\gamma}_j(t)(1,2,\cdots,\hat{j},\cdots,N,N^*,\cdots,\hat{j}^*,\cdots,1^*)_n \\
&+(d_{-n-1}, d_n^*, 1,\cdots,N,N^*,\cdots,1^*)_n,
\end{aligned} \tag{11}$$

$$f_{n+1} = f_n + (d_{-n-1}, d_n^*, 1, \cdots, N, N^*, \cdots, 1^*)_n,$$
$$f_{n-1} = f_n - (d_{-n}, d_{n-1}^*, 1, \cdots, N, N^*, \cdots, 1^*)_n, \tag{12}$$

where $\hat{}$ indicates deletion of the letter under it. Then the function f_n will not satisfy the equation (3) again and it just satisfies the following new equation:

$$(D_x D_t - 2e^{D_n} + 2) f_n \cdot f_n = -\sum_{j=1}^{K} e^{D_n} g_{j,n} \cdot h_{j,n}, \tag{13}$$

and here $g_{j,n}$ and $h_{j,n}$ are given by the following forms:

$$g_{j,n+1} = \sqrt{2\dot{\gamma}_j(t)} (d_n^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n, \tag{14}$$

$$h_{j,n-1} = \sqrt{2\dot{\gamma}_j(t)} (d_{-n}, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, 1^*)_n, \tag{15}$$

where $j = 1, 2, \cdots, K$, and the dot denotes the derivative of $\gamma_j(t)$ with respect to t. We can show that f_n, $g_{j,n}$ and $h_{j,n}$ also satisfy the following bilinear equations:

$$(D_x + e^{-D_n} - 1) f_n \cdot g_{j,n} = 0, \quad j = 1, 2, \cdots, K \tag{16}$$

$$(D_x + e^{-D_n} - 1) h_{j,n} \cdot f_n = 0, \quad j = 1, 2, \cdots, K. \tag{17}$$

In fact, substitution of (10)-(12) and (14)-(15) into (13) leads to the sum of $(N+1)$ pfaffian identities[6]:

$$\begin{aligned}
&(d_{-n-1}, d_n^*, 1, 2, \cdots, N, N^*, \cdots, 1^*)_n (d_{n-1}^*, d_{-n}, 1, 2, \cdots, N, N^*, \cdots, 1^*)_n \\
&+(d_{-n-1}, d_{n-1}^*, 1, 2, \cdots, N, N^*, \cdots, 1^*)_n (d_{-n}, d_n^*, 1, 2, \cdots, N, N^*, \cdots, 1^*)_n \\
&-(d_{-n-1}, d_{n-1}^*, d_{-n}, d_n^*, 1, 2, \cdots, N, N^*, \cdots, 1^*)_n (1, 2, \cdots, N, N^*, \cdots, 1^*)_n \\
&+\sum_{j=1}^{K} \dot{\gamma}_j(t) [(d_{-n}, d_n^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n (1, \cdots, N, N^*, \cdots, 1^*)_n \\
&-(1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n (d_{-n}, d_n^*, 1, \cdots, N, N^*, \cdots, 1^*)_n \\
&+(d_n^*, 1, \cdots, N, N^*, \cdots, 1^*)_n (d_{-n}, 1, \cdots, N, N^*, \cdots, 1^*)_n] = 0.
\end{aligned}$$

Therefore equation (13) holds. On the other hand, we have

$$\begin{aligned}
g_{j,n} &= \sqrt{2\dot{\gamma}_j(t)} (d_{n-1}^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n, \\
g_{j,nx} &= \sqrt{2\dot{\gamma}_j(t)} [(d_n^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n - g_{j,n} \\
&\quad +(d_{n-1}^*, d_{-n}, d_n^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n].
\end{aligned} \tag{18}$$

Substituting (10), (12) and (18) into equation (16), (16) is reduced to the following pfaffian identity [6]:

$$\begin{aligned}
&(d_{-n}, d_n^*, 1, \cdots, N, N^*, \cdots, 1^*)_n (d_{n-1}^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n \\
&-(1, \cdots, N, N^*, \cdots, 1^*)_n (d_{n-1}^*, d_{-n}, d_n^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n \\
&-(d_{-n}, d_{n-1}^*, 1, \cdots, N, N^*, \cdots, 1^*)_n (d_n^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)_n = 0.
\end{aligned}$$

Then equation (16) holds. Much in the same way, we can prove that f_n and $h_{j,n}$ satisfy equation (17). So equations (13), (16) and (17) construct the 2-dimensional

Toda equation with K pairs of self-consistent sources (TodaESCS), and f_n, $g_{j,n}$, $h_{j,n}$ in (8), (14) and (15) are the N-order ($N \geqslant K$) determinant solutions of the TodaESCS. In [41], we have also given the Casorati-type determinant solutions of the TodaESCS, its bilinear Bäcklund transformation and Lax pair, which indicate the integrability of the coupled system (13), (16) and (17).

3. The discrete KP equation with self-consistent sources(dKPESCS)

In this section we will apply source generalization method to a discrete KP equation or Hirota-Miwa equation. The discrete KP equation or Hirota-Miwa equation[42, 43] has the form:

$$\begin{aligned}&\alpha_1(\alpha_2-\alpha_3)\tau(k_1+\alpha_1,k_2,k_3)\tau(k_1,k_2+\alpha_2,k_3+\alpha_3)\\&+\alpha_2(\alpha_3-\alpha_1)\tau(k_1,k_2+\alpha_2,k_3)\tau(k_1+\alpha_1,k_2,k_3+\alpha_3)\\&+\alpha_3(\alpha_1-\alpha_2)\tau(k_1,k_2,k_3+\alpha_3)\tau(k_1+\alpha_1,k_2+\alpha_2,k_3)=0,\end{aligned} \tag{19}$$

where α_1, α_2, α_3 are constants and k_1, k_2, k_3 are discrete variables. It is known that the discrete KP equation has the following discrete Gram-type determinant solution [42]:

$$\tau(k_1,k_2,k_3)=\det(d_{ij}+m_{ij})_{1\leqslant i,j\leqslant N}, \tag{20}$$

where d_{ij} is a constant and the matrix element m_{ij} is a function of k_1, k_2, k_3 satisfying the difference equation:

$$\Delta_{+k_\nu}m_{ij}=\varphi_i(k_\nu+\alpha_\nu;0)\bar{\varphi}_j(0),\quad i,j=1,2,\cdots,N,\quad \nu=1,2,3, \tag{21}$$

where unshifted independent variables are suppressed and φ_i, $\bar{\varphi}_j$ are arbitrary functions of k_1, k_2, k_3 and an integer s, satisfying the dispersion relations:

$$\Delta_{-k_\nu}\varphi_i(k_1,k_2,k_3,s)=\varphi_i(k_1,k_2,k_3,s+1), \tag{22}$$

$$\Delta_{+k_\nu}\bar{\varphi}_i(k_1,k_2,k_3,s)=\bar{\varphi}_i(k_1,k_2,k_3,s+1), \tag{23}$$

where Δ_{-k_ν}, Δ_{+k_ν} are defined by

$$\Delta_{-k_\nu}F(k_\nu)=\frac{F(k_\nu)-F(k_\nu-\alpha_\nu)}{\alpha_\nu}, \tag{24}$$

$$\Delta_{+k_\nu}F(k_\nu)=\frac{F(k_\nu+\alpha_\nu)-F(k_\nu)}{\alpha_\nu},\quad \nu=0,1,2,3. \tag{25}$$

It was proved in [42] that the determinant $\tau=\det(m_{ij})_{1\leqslant i,j\leqslant N}$ has the following difference formula:

$$\tau=|M|, \tag{26}$$

$$\tau(k_\nu+\alpha_\nu)=\alpha_\nu\begin{vmatrix} M & \boldsymbol{\phi}(0_\nu)\\ -\bar{\boldsymbol{\phi}}(0)^T & \alpha_\nu^{-1}\end{vmatrix},\ \mu,\nu=1,2,3 \tag{27}$$

$$\tau(k_\nu+\alpha_\nu,k_\mu+\alpha_\mu)=\frac{(\alpha_\nu\alpha_\mu)^2}{\alpha_\nu-\alpha_\mu}\begin{vmatrix} M & \boldsymbol{\phi}(0_\nu) & \boldsymbol{\phi}(0_\mu)\\ -\bar{\boldsymbol{\phi}}(1)^T & -\alpha_\nu^{-2} & -\alpha_\mu^{-2}\\ -\bar{\boldsymbol{\phi}}(0)^T & \alpha_\nu^{-1} & \alpha_\mu^{-1}\end{vmatrix}, \tag{28}$$

where T denotes the transpose of the matrix, and M, $\boldsymbol{\phi}(s_\nu)$, $\bar{\boldsymbol{\phi}}(s_\nu)$ are $N\times N$, $N\times 1$, $N\times 1$ matrices defined by

$$M = \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \cdots & m_{NN} \end{pmatrix}, \tag{29}$$

$$\boldsymbol{\phi}(s_\nu) = \begin{pmatrix} \varphi_1(k_\nu+\alpha_\nu; s) \\ \varphi_2(k_\nu+\alpha_\nu; s) \\ \vdots \\ \varphi_N(k_\nu+\alpha_\nu; s) \end{pmatrix}, \quad \nu = 1,2,3, \tag{30}$$

$$\bar{\boldsymbol{\phi}}(s)^T = \begin{pmatrix} \bar{\varphi}_1(s) & \bar{\varphi}_2(s) & \cdots & \bar{\varphi}_N(s) \end{pmatrix}. \tag{31}$$

Now we change the solution $\tau(k_1,k_2,k_3)$ into the following form:

$$f(k_1,k_2,k_3) = det(c_{ij}(k_2)+m_{ij})_{1\leqslant i,j\leqslant N}, \tag{32}$$

where m_{ij} still satisfies relations (21)-(23), and $c_{ij}(k_2)$ satisfies

$$c_{ij}(k_2) \equiv \begin{cases} c_i(k_2), & 1\leqslant i\leqslant K\leqslant N \;\text{ and }\; j=1,2,\cdots,K, \quad K\in Z^+, \\ d_{ij}, & \text{otherwise.} \end{cases}$$

Then we have the following difference formula:

$$f(k_\nu+\alpha_\nu) = \alpha_\nu \begin{vmatrix} M & C(k_2) & \boldsymbol{\phi}(0_\nu) \\ \alpha^T & 1 & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_\nu^{-1} \end{vmatrix}, \quad \nu=1,3, \tag{33}$$

$$f(k_2+\alpha_2) = \alpha_2 \begin{vmatrix} M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) \\ \alpha^T & 1 & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_2^{-1} \end{vmatrix}, \tag{34}$$

$$f(k_1+\alpha_1,k_3+\alpha_3) = \frac{(\alpha_1\alpha_3)^2}{\alpha_1-\alpha_3} \begin{vmatrix} M & C(k_2) & \boldsymbol{\phi}(0_1) & \boldsymbol{\phi}(0_3) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_1^{-2} & -\alpha_3^{-2} \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_1^{-1} & \alpha_3^{-1} \end{vmatrix}, \tag{35}$$

$$f(k_2+\alpha_2,k_\nu+\alpha_\nu) = \frac{(\alpha_2\alpha_\nu)^2}{\alpha_2-\alpha_\nu} \begin{vmatrix} M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) & \boldsymbol{\phi}(0_\nu) \\ \alpha^T & 1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_2^{-2} & -\alpha_\nu^{-2} \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_2^{-1} & \alpha_\nu^{-1} \end{vmatrix}, \quad \nu=1,3 \tag{36}$$

where α^T is an $1\times N$ matrix expressed in

$$\alpha^T = (-1,\cdots,-1,0,\cdots,0), \quad \text{number of} -1 = K$$

and $C(k_2)$ is an $N \times 1$ matrix defined by

$$C(k_2) = \begin{pmatrix} c_1(k_2) \\ \vdots \\ c_K(k_2) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

According to the above results, $f(k_1, k_2, k_3)$ will not satisfy the dKP equation (19) again. In fact it satisfies the following equation:

$$\begin{aligned} &\alpha_1(\alpha_2-\alpha_3)\tau(k_1+\alpha_1, k_2, k_3)\tau(k_1, k_2+\alpha_2, k_3+\alpha_3) \\ &+\alpha_2(\alpha_3-\alpha_1)\tau(k_1, k_2+\alpha_2, k_3)\tau(k_1+\alpha_1, k_2, k_3+\alpha_3) \\ &+\alpha_3(\alpha_1-\alpha_2)\tau(k_1, k_2, k_3+\alpha_3)\tau(k_1+\alpha_1, k_2+\alpha_2, k_3) \\ &= \sum_{i,j=1}^{K} h_{ij}(k_1, k_2+\alpha_2, k_3) g_{ij}(k_1+\alpha_1, k_2, k_3+\alpha_3). \end{aligned} \tag{37}$$

In the above equation, h_{ij} and g_{ij} are functions of variables k_1, k_2, k_3 and have the following forms:

$$h_{ij}(k_1, k_2+\alpha_2, k_3) = \alpha_1\alpha_2\alpha_3\sqrt{\Delta c_i(k_2)} \begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_2) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & -\bar{\varphi}_j(0) \end{vmatrix}, \tag{38}$$

$$g_{ij}(k_1+\alpha_1, k_2, k_3+\alpha_3) = \alpha_1\alpha_2\alpha_3\sqrt{\Delta c_i(k_2)} \begin{vmatrix} D_{ij} & \boldsymbol{\phi}_i(0_1) & \boldsymbol{\phi}_i(0_3) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_1^{-1} & \alpha_3^{-1} \\ M_{ij}^T & \varphi_i(0_1) & \varphi_i(0_3) \end{vmatrix}, \tag{39}$$

where D_{ij}, E_{ij} are the $(N-1)-th$ order matrices obtained by eliminating the i-th row and the j-th column from the matrices $(c_{ij}(k_2) + m_{ij}(k_1, k_2, k_3))_{1\leqslant i,j\leqslant N}$ and $(c_{ij}(k_2+\alpha_2) + m_{ij}(k_1, k_2, k_3))_{1\leqslant i,j\leqslant N}$, respectively. In addition, $\boldsymbol{\phi}_i(s_\nu)$ is an $(N-1)\times 1$ matrix by eliminating the i-th row from $\boldsymbol{\phi}(s_\nu)$, and $\bar{\boldsymbol{\phi}}_j(s)^T$ is an $1\times(N-1)$ matrix by eliminating the j-th column from $\bar{\boldsymbol{\phi}}(s)^T$, and

$$\Delta c_i(k_2) = c_i(k_2+\alpha_2) - c_i(k_2),$$

$$N_{ij} = \begin{pmatrix} c_1(k_2+\alpha_2) + m_{1j} \\ \cdots \\ c_{i-1}(k_2+\alpha_2) + m_{i-1,j} \\ c_{i+1}(k_2+\alpha_2) + m_{i+1,j} \\ \cdots \\ c_K(k_2+\alpha_2) + m_{Kj} \\ d_{K+1,j} + m_{K+1,j} \\ \cdots \\ d_{N,j} + m_{N,j} \end{pmatrix}, \tag{40}$$

$$M_{ij}^T = (c_i(k_2) + m_{i1}, \cdots, c_i(k_2) + m_{i,j-1}, c_i(k_2) + m_{i,j+1}, \cdots \\ \cdots, c_i(k_2) + m_{iK}, d_{i,K+1} + m_{i,K+1}, \cdots, d_{i,N} + m_{i,N}). \tag{41}$$

From the expressions (32), (38)-(39), we can also show that f, h_{ij} and g_{ij} satisfy the following equations:

$$\alpha_3 f(k_1,k_2,k_3+\alpha_3)h_{ij}(k_1+\alpha_1,k_2,k_3) - \alpha_1 f(k_1+\alpha_1,k_2,k_3)h_{ij}(k_1,k_2,k_3+\alpha_3) \\ +(\alpha_1-\alpha_3)f(k_1+\alpha_1,k_2,k_3+\alpha_3)h_{ij}(k_1,k_2,k_3) = 0, \tag{42}$$

$$\alpha_3 g_{ij}(k_1,k_2,k_3+\alpha_3)f(k_1+\alpha_1,k_2,k_3) - \alpha_1 g_{ij}(k_1+\alpha_1,k_2,k_3)f(k_1,k_2,k_3+\alpha_3) \\ +(\alpha_1-\alpha_3)g_{ij}(k_1+\alpha_1,k_2,k_3+\alpha_3)f(k_1,k_2,k_3) = 0. \tag{43}$$

In fact equations (37), (42) and (43) can be verified through Laplace expansion theorem. Firstly, we show that f, g_{ij} and h_{ij} defined by (32), (38) and (39) satisfy the equation (37). Substitution of (32), (38) and (39) into (37) yields the following determinant identity:

$$\begin{vmatrix} M & C(k_2) & \boldsymbol{\phi}(0_1) \\ -1 & 1 & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_1^{-1} \end{vmatrix} \begin{vmatrix} M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) & \boldsymbol{\phi}(0_3) \\ -1 & 1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_2^{-2} & -\alpha_3^{-2} \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_2^{-1} & \alpha_3^{-1} \end{vmatrix}$$
$$- \begin{vmatrix} M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) \\ -1 & 1 & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_2^{-1} \end{vmatrix} \begin{vmatrix} M & C(k_2) & \boldsymbol{\phi}(0_1) & \boldsymbol{\phi}(0_3) \\ -1 & 1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_1^{-2} & -\alpha_3^{-2} \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_1^{-1} & \alpha_3^{-1} \end{vmatrix}$$
$$- \begin{vmatrix} M & C(k_2) & \boldsymbol{\phi}(0_3) \\ -1 & 1 & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_3^{-1} \end{vmatrix} \begin{vmatrix} M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) & \boldsymbol{\phi}(0_1) \\ -1 & 1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_2^{-2} & -\alpha_1^{-2} \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_2^{-1} & \alpha_1^{-1} \end{vmatrix} \tag{44}$$
$$+ \begin{vmatrix} M & \boldsymbol{\phi}(0_1) & \boldsymbol{\phi}(0_3) \\ -1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & \alpha_1^{-1} & \alpha_3^{-1} \end{vmatrix} \begin{vmatrix} M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) & C(k_2) \\ -1 & 1 & 0 & 1 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_2^{-2} & 0 \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & \alpha_2^{-1} & 0 \end{vmatrix} = 0.$$

Now we show that equation (44) holds. Let us introduce the following $2(N+3)\times 2(N+3)$ determinant which is equal to zero:

$$\begin{vmatrix} M & 0 & 0 & 0 & 0 & C(k_2) & \boldsymbol{\phi}(0_1) & \boldsymbol{\phi}(0_3) \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\bar{\boldsymbol{\phi}}(1)^T & 0 & 0 & 0 & 1 & 0 & -\alpha_1^{-2} & -\alpha_3^{-2} \\ -\bar{\boldsymbol{\phi}}(0)^T & 0 & 0 & 0 & 0 & 0 & \alpha_1^{-1} & \alpha_3^{-1} \\ 0 & M & C(k_2+\alpha_2) & \boldsymbol{\phi}(0_2) & 0 & C(k_2) & \boldsymbol{\phi}(0_1) & \boldsymbol{\phi}(0_3) \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\bar{\boldsymbol{\phi}}(1)^T & 0 & -\alpha_2^{-2} & 1 & 0 & -\alpha_1^{-2} & -\alpha_3^{-2} \\ 0 & -\bar{\boldsymbol{\phi}}(1)^T & 0 & \alpha_2^{-1} & 1 & 0 & \alpha_1^{-1} & \alpha_3^{-1} \end{vmatrix} = 0. \tag{45}$$

Applying the Laplace expansion in $(N+3)\times(N+3)$ minors to the left-hand side of equation (45), we obtain the determinant identity (44). So the discrete Gram-type determinants f, h_{ij}, g_{ij} are solutions of equation (37). In the same way, substituting f and h_{ij} into equation (42) gives the determinant identity:

$$\begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_2) & \boldsymbol{\phi}_i(0_3) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\alpha_3^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & \alpha_3^{-1} & -\bar{\varphi}_j(0) \\ Q_{ij}^T & \varphi_i(0_2) & \varphi_i(0_3) & m_{ij}+c_i(k_2+\alpha_2) \end{vmatrix} \begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_2) & \boldsymbol{\phi}_i(0_1) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(2)^T & \alpha_2^{-3} & \alpha_1^{-3} & -\bar{\varphi}_j(2) \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\alpha_1^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & \alpha_1^{-1} & -\bar{\varphi}_j(0) \end{vmatrix}$$
$$+\begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_1) & \boldsymbol{\phi}_i(0_2) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_1^{-2} & -\alpha_2^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_1^{-1} & \alpha_2^{-1} & -\bar{\varphi}_j(0) \\ Q_{ij}^T & \varphi_i(0_1) & \varphi_i(0_2) & m_{ij}+c_i(k_2+\alpha_2) \end{vmatrix} \begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_2) & \boldsymbol{\phi}_i(0_3) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(2)^T & \alpha_2^{-3} & \alpha_3^{-3} & -\bar{\varphi}_j(2) \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\alpha_3^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & \alpha_3^{-1} & -\bar{\varphi}_j(0) \end{vmatrix}$$
$$+\begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_2) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & -\bar{\varphi}_j(0) \end{vmatrix} \begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_1) & \boldsymbol{\phi}_i(0_2) & \boldsymbol{\phi}_i(0_3) & N_{ij} \\ -\bar{\boldsymbol{\phi}}_j(2)^T & \alpha_1^{-3} & \alpha_2^{-3} & \alpha_3^{-3} & -\bar{\varphi}_j(2) \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_1^{-2} & -\alpha_2^{-2} & -\alpha_3^{-2} & -\bar{\varphi}_j(1) \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_1^{-1} & \alpha_2^{-1} & \alpha_3^{-1} & -\bar{\varphi}_j(0) \\ Q_{ij}^T & \varphi_i(0_1) & \varphi_i(0_2) & \varphi_i(0_3) & m_{ij}+c_i(k_2+\alpha_2) \end{vmatrix} = 0.$$

The above determinant identity can be also proved through the Laplace expansion of the $2(N+3)\times 2(N+3)$ determinant which is equal to zero:

$$\begin{vmatrix} E_{ij} & \boldsymbol{\phi}_i(0_2) & N_{ij} & 0 & 0 & 0 & 0 & 0 & \boldsymbol{\phi}_i(0_1) & \boldsymbol{\phi}_i(0_3) \\ -\bar{\boldsymbol{\phi}}_j(2)^T & \alpha_2^{-3} & -\bar{\varphi}_j(2) & 0 & 0 & 0 & 0 & 1 & \alpha_1^{-3} & \alpha_3^{-3} \\ -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\bar{\varphi}_j(1) & 0 & 0 & 0 & 0 & 0 & -\alpha_1^{-2} & -\alpha_3^{-2} \\ -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & -\bar{\varphi}_j(0) & 0 & 0 & 0 & 0 & 0 & \alpha_1^{-1} & \alpha_3^{-1} \\ Q_{ij}^T & \varphi_i(0_2) & \bar{m}_{ij} & 0 & 0 & 0 & 0 & 0 & \varphi_i(0_1) & \varphi_i(0_3) \\ 0 & 0 & 0 & E_{ij} & \boldsymbol{\phi}_i(0_2) & N_{ij} & 0 & 0 & \boldsymbol{\phi}_i(0_1) & \boldsymbol{\phi}_i(0_3) \\ 0 & 0 & 0 & -\bar{\boldsymbol{\phi}}_j(2)^T & \alpha_2^{-3} & -\bar{\varphi}_j(2) & 0 & 1 & \alpha_1^{-3} & \alpha_3^{-3} \\ 0 & 0 & 0 & -\bar{\boldsymbol{\phi}}_j(1)^T & -\alpha_2^{-2} & -\bar{\varphi}_j(1) & 0 & 0 & -\alpha_1^{-2} & -\alpha_3^{-2} \\ 0 & 0 & 0 & -\bar{\boldsymbol{\phi}}_j(0)^T & \alpha_2^{-1} & -\bar{\varphi}_j(0) & 0 & 0 & \alpha_1^{-1} & \alpha_3^{-1} \\ 0 & 0 & 0 & Q_{ij}^T & \varphi_i(0_2) & \bar{m}_{ij} & 1 & 0 & \varphi_i(0_1) & \varphi_i(0_3) \end{vmatrix} = 0,$$

where $\bar{m}_{ij}=c_{ij}(k_2+\alpha_2)+m_{ij}$ and Q_{ij} denotes the following $(N-1)\times 1$ matrix:

$$\begin{pmatrix} c_i(k_2+\alpha_2)+m_{i1} \\ \vdots \\ c_i(k_2+\alpha_2)+m_{i,j-1} \\ c_i(k_2+\alpha_2)+m_{i,j+1} \\ \vdots \\ c_i(k_2+\alpha_2)+m_{iK} \\ d_{i,K+1}+m_{i,K+1} \\ \vdots \\ d_{i,N}+m_{i,N}). \end{pmatrix}.$$

Similarly, we can show that equation (43) holds for f, g_{ij} in (32) and (39). So the system of equations (37), (42) and (43) constructs the discrete KP equation with K pairs of self-consistent sources(dKPESCS), and f, h_{ij}, g_{ij} expressed by (32), (38) and (39) are the N-order ($N \geqslant K$)determinant solutions of the system.

For the dKPESCS (37), (42) and (43), we can also give its bilinear Bäcklund transformation. To this end, we express the system as the following bilinear forms:

$$[z_1 e^{\frac{1}{2}(-D_{k_1}+D_{k_2}+D_{k_3})} + z_2 e^{\frac{1}{2}(D_{k_1}-D_{k_2}+D_{k_3})} + z_3 e^{\frac{1}{2}(-D_{k_1}-D_{k_2}+D_{k_3})}] f \cdot f = \sum_{i,j=1}^{K} e^{\frac{1}{2}(D_{k_1}-D_{k_2}+D_{k_3})} g_{i,j} \cdot h_{i,j}, \tag{46}$$

$$\left[\alpha_3 e^{\frac{1}{2}(D_{k_3}-D_{k_1})} - \alpha_1 e^{\frac{1}{2}(D_{k_1}-D_{k_3})} + (\alpha_1-\alpha_3) e^{\frac{1}{2}(D_{k_1}+D_{k_3})}\right] f \cdot h_{i,j} = 0, \tag{47}$$

$$\left[\alpha_3 e^{\frac{1}{2}(D_{k_3}-D_{k_1})} - \alpha_1 e^{\frac{1}{2}(D_{k_1}-D_{k_3})} + (\alpha_1-\alpha_3) e^{\frac{1}{2}(D_{k_1}+D_{k_3})}\right] g_{i,j} \cdot f = 0, \tag{48}$$

where $z_1 = \alpha_1(\alpha_2-\alpha_3)$, $z_2 = \alpha_2(\alpha_3-\alpha_1)$ and $z_3 = \alpha_3(\alpha_1-\alpha_2)$. If we set

$$D_1 = \frac{1}{2}(-D_{k_1}+D_{k_2}+D_{k_3}),\ D_2 = \frac{1}{2}(D_{k_1}-D_{k_2}+D_{k_3}),\ D_3 = \frac{1}{2}(-D_{k_1}-D_{k_2}+D_{k_3}),$$

Then the bilinear Bäcklund transformation for the system (46)-(48) are as follows:

Proposition 1. *The system (46)-(48) has the bilinear Bäcklund transformation:*

$$\left[e^{\frac{1}{2}D_1+\frac{1}{2}D_3} - \beta_1 e^{-\frac{1}{2}D_1-\frac{1}{2}D_3} - \lambda_1 e^{D_2+\frac{1}{2}D_1-\frac{1}{2}D_3}\right] f \cdot f' = 0, \tag{49}$$

$$\left[e^{\frac{1}{2}D_1+\frac{1}{2}D_3} - \beta_1 e^{-\frac{1}{2}D_1-\frac{1}{2}D_3} - \lambda_1 e^{D_2+\frac{1}{2}D_1-\frac{1}{2}D_3}\right] g_{i,j} \cdot g'_{i,j} = 0, \tag{50}$$

$$\left[e^{\frac{1}{2}D_1+\frac{1}{2}D_3} - \beta_1 e^{-\frac{1}{2}D_1-\frac{1}{2}D_3} - \lambda_1 e^{D_2+\frac{1}{2}D_1-\frac{1}{2}D_3}\right] h_{i,j} \cdot h'_{i,j} = 0, \tag{51}$$

$$e^{\frac{1}{2}D_1+\frac{1}{2}D_2} g_{i,j} \cdot f' = \left(\beta_2 e^{-\frac{1}{2}D_1-\frac{1}{2}D_2} + \lambda_2 e^{\frac{1}{2}D_1+\frac{1}{2}D_2}\right) f \cdot g'_{i,j}, \tag{52}$$

$$e^{\frac{1}{2}D_1+\frac{1}{2}D_2} f \cdot h'_{i,j} = \left(\beta_2 e^{-\frac{1}{2}D_1-\frac{1}{2}D_2} + \lambda_2 e^{\frac{1}{2}D_1+\frac{1}{2}D_2}\right) h_{i,j} \cdot f', \tag{53}$$

$$\left(z_2 e^{\frac{1}{2}D_1-\frac{1}{2}D_2} + \lambda_1 z_3 e^{\frac{1}{2}D_1+\frac{1}{2}D_2-D_3} + \gamma e^{\frac{1}{2}D_2-\frac{1}{2}D_1}\right) f \cdot f' = \lambda_2 \sum_{i,j=1}^{K} e^{\frac{1}{2}D_1-\frac{1}{2}D_2} h_{i,j} \cdot g'_{i,j}, \tag{54}$$

where γ is an arbitrary constant and β_1, β_2, λ_1, λ_2 are constants satisfying $\lambda_1\beta_2\alpha_1 = \lambda_2\beta_1(\alpha_1-\alpha_3)$.

Proof. Let f, $g_{i,j}$, $h_{i,j}$ be solutions of the system (46)-(48), what we need to prove is that f', $g'_{i,j}$ and $h'_{i,j}$ in (49)-(54) are also solutions of equations (46)-(48). In fact, according to Appendix A and relations (49)-(54), we have

$$\begin{aligned}
P &= \{(z_1e^{D_1}+z_2e^{D_2}+z_3e^{D_3})f\cdot f-\sum_{i,j=1}^{N}e^{D_2}g_{i,j}\cdot h_{i,j}\}(e^{D_1}f'\cdot f')\\
&\quad -\{(z_1e^{D_1}+z_2e^{D_2}+z_3e^{D_3})f'\cdot f'-\sum_{i,j=1}^{K}e^{D_2}g'_{i,j}\cdot h'_{i,j}\}(e^{D_1}f\cdot f)\\
&= 2z_2\sinh\tfrac{D_2+D_1}{2}(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\cdot(e^{\frac{1}{2}(D_1-D_2)}f\cdot f')\\
&\quad +2z_3\sinh\tfrac{D_3-D_1}{2}(e^{\frac{1}{2}(D_3+D_1)}f\cdot f')\cdot(e^{-\frac{1}{2}(D_1+D_3)}f\cdot f')\\
&\quad -\sum_{i,j=1}^{K}\{e^{\frac{1}{2}(D_2-D_1)}(e^{\frac{1}{2}(D_2+D_1)}g_{i,j}\cdot f')\cdot(e^{-\frac{1}{2}(D_2+D_1)}h_{i,j}\cdot f')\}\\
&\quad +\sum_{i,j=1}^{K}\{e^{\frac{1}{2}(D_1-D_2)}(e^{\frac{1}{2}(D_2+D_1)}f\cdot h'_{i,j})\cdot(e^{-\frac{1}{2}(D_2+D_1)}f\cdot g'_{i,j})\}\\
&= 2z_2\sinh\tfrac{D_2+D_1}{2}(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\cdot(e^{\frac{1}{2}(D_1-D_2)}f\cdot f')\\
&\quad +2z_3\lambda_1\sinh\tfrac{D_3-D_1}{2}(e^{\frac{1}{2}(D_1+2D_2-D_3)}f\cdot f')\cdot(e^{-\frac{1}{2}(D_1+D_3)}f\cdot f')\\
&\quad -\lambda_2\sum_{i,j=1}^{K}\{e^{\frac{1}{2}(D_2-D_1)}(e^{\frac{1}{2}(D_2+D_1)}f\cdot g'_{i,j})\cdot(e^{-\frac{1}{2}(D_2+D_1)}h_{i,j}\cdot f')\}\\
&\quad +\lambda_2\sum_{i,j=1}^{K}\{e^{\frac{1}{2}(D_1-D_2)}(e^{\frac{1}{2}(D_2+D_1)}h_{i,j}\cdot f')\cdot(e^{-\frac{1}{2}(D_2+D_1)}f\cdot g'_{i,j})\}\\
&= 2z_2\sinh\tfrac{D_2+D_1}{2}(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\cdot(e^{\frac{1}{2}(D_1-D_2)}f\cdot f')\\
&\quad +2z_3\lambda_1\sinh\tfrac{D_2+D_1}{2}(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\cdot(e^{\frac{1}{2}(D_1+D_2-2D_3)}f\cdot f')\\
&\quad -2\lambda_2\sinh\tfrac{D_2+D_1}{2}(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\cdot(e^{\frac{1}{2}(D_1-D_2)}h_{i,j}\cdot g'_{i,j})\\
&= -2\gamma\sinh\tfrac{D_2+D_1}{2}(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\cdot(e^{\frac{1}{2}(D_2-D_1)}f\cdot f')\\
&\equiv 0.
\end{aligned}$$

The above results indicate f', $h'_{i,j}$ and $g'_{i,j}$ satisfy equation (46). Similarly, we can prove f', $h'_{i,j}$ and $g'_{i,j}$ satisfy (47)-(48). So f', $h'_{i,j}$ and $g'_{i,j}$ are solutions of equations (46)-(48). Then we have completed the proof of the proposition. □

4. a BKP-type equation with self-consistent sources (BKPESCS)

The BKP hierarchy (KP hierarchy of the B-type) was introduced by Date, Jimbo, Kashiwara and Miwa [44, 45]. Here take (2+1)-dimensional SK equation as an example. The equation in bilinear form is:

$$(D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5)\tau\cdot\tau=0, \tag{55}$$

where $D_i = D_{x_i}$ and $x_1 = x$. In [46], N-soliton solution of the equation (55) was expressed by the pfaffians

$$\tau=(1,2,\cdots,2N), \tag{56}$$

where pfaffian entries (i,j) are defined by

$$(i,j)=C_{ij}+\int_{-\infty}^{x}D_x\phi_i\cdot\phi_j\mathrm{d}x. \tag{57}$$

In the above expression, each function $\phi_i \equiv \phi_i(x, x_3, x_5)$ satisfies the linear equations:

$$\frac{\partial}{\partial x_m}\phi_i = \frac{\partial^m}{\partial x^m}\phi_i, \quad m = 1, 3, 5, \tag{58}$$

with the boundary condition $\phi_i = 0$ and $C_{ij} = -C_{ji}$ being constant. It was proved in [46] that τ has the following differential formulas:

$$\frac{\partial}{\partial x}\tau = (d_0, d_1, 1, \cdots, 2N),$$

$$\frac{\partial}{\partial x_3}\tau = (d_0, d_3, 1, \cdots, 2N) - 2(d_1, d_2, 1, \cdots, 2N),$$

$$\frac{\partial}{\partial x_5}\tau = (d_0, d_5, 1, \cdots, 2N) - 2(d_1, d_4, 1, \cdots, 2N) + 2(d_2, d_3, 1, \cdots, 2N),$$

$$\frac{\partial^2}{\partial x \partial x_5}\tau = -(d_1, d_5, 1, \cdots, 2N) + (d_0, d_6, 1, \cdots, 2N) + 2(d_0, d_1, d_2, d_3, 1, \cdots, 2N),$$

where d_m is defined by

$$(d_m, j) = \frac{\partial^m}{\partial x^m}\phi_j, \quad m = 0, 1, \cdots \tag{59}$$

$$(d_m, d_n) = 0. \quad n, m = 0, 1, \cdots \tag{60}$$

Now we change the solution τ into the following form:

$$f = (1, 2, \cdots, 2N)_1 = (\bullet)_1, \tag{61}$$

whose pfaffian elements are defined as follows

$$(i, j)_1 = C_{ij}(x_5) + \int_{-\infty}^{x} D_x\phi_i \cdot \phi_j \mathrm{d}x, \quad i, j = 1, 2, \cdots, 2N$$

where $C_{ij}(x_5) = -C_{ji}(x_5)$ satisfying

$$C_{ij}(x_5) = \begin{cases} C_k(x_5), & i = 2k-1, j = 2k, \quad k = 1, 2, \cdots, K \\ C_{ij}, & \text{otherwise.} \end{cases}$$

Then we have the following differential formula:

$$\frac{\partial}{\partial x_5}f = (d_0, d_5, \bullet)_1 - 2(d_1, d_4, \bullet)_1 + 2(d_2, d_3, \bullet)_1 \\ + \sum_{1 \leqslant i < j \leqslant 2N} (-1)^{i+j-1}\dot{C}_{ij}(x_5)(1, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, 2N)_1, \tag{62}$$

$$\frac{\partial^2}{\partial x \partial x_5}f = -(d_1, d_5, \bullet)_1 + (d_0, d_6, \bullet)_1 + 2(d_0, d_1, d_2, d_3, \bullet)_1 \\ + \sum_{1 \leqslant i < j \leqslant 2N} (-1)^{i+j-1}\dot{C}_{ij}(x_5)(d_0, d_1, 1, \cdots, \hat{\imath}, \cdots, \hat{\jmath}, \cdots, 2N)_1, \tag{63}$$

where d_m is defined as

$$(d_m, j)_1 = \frac{\partial^m}{\partial x^m}\phi_j, \quad (d_m, d_n)_1 = 0, \quad n, m = 0, 1, \cdots$$

Then f in (61) satisfies the following new equation:

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5) f \cdot f = \sum_{k=1}^{K} D_1 g_k \cdot h_k, \tag{64}$$

where g_k and h_k are defined by

$$g_k = 3\sqrt{2\dot{C}_k(x_5)}(d_0, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1, \tag{65}$$

$$h_k = 3\sqrt{2\dot{C}_k(x_5)}(d_0, 1, \cdots, \hat{2k}, \cdots, 2N)_1. \tag{66}$$

From the expression (65)-(66), we can show that f, g_k and h_k satisfy the following equations at the same time:

$$(D_3 - D_1^3) f \cdot g_k = 0, \quad k = 1, 2, \cdots, K \tag{67}$$

$$(D_3 - D_1^3) f \cdot h_k = 0, \quad k = 1, 2, \cdots, K \tag{68}$$

In fact, substitution of (61), (65)-(66) into equation (64) yields the following pfaffian identities:

$$\begin{aligned}
&\sum_{k=1}^{K} \dot{C}_k(x_5)[(d_0, d_1, 1, \cdots, \widehat{2k-1}, \hat{2k}, \cdots, 2N)_1(\bullet)_1 \\
&-(1, \cdots, \widehat{2k-1}, \hat{2k}, \cdots, 2N)_1(d_0, d_1, \bullet)_1 \\
&-(d_0, 1, \cdots, \hat{2k}, \cdots, 2N)_1(d_1, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1 \\
&+(d_0, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1(d_1, 1, \cdots, \hat{2k}, \cdots, 2N)_1] = 0.
\end{aligned} \tag{69}$$

Then the equation (64) holds for f, g_k and h_k in (61), (65)-(66). On the other hand, we have

$$\frac{\partial g_k}{\partial x} = 3\sqrt{2\dot{C}_k(x_5)}(d_1, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1,$$

$$\frac{\partial^2 g_k}{\partial x^2} = 3\sqrt{2\dot{C}_k(x_5)}(d_2, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1,$$

$$\frac{\partial^3 g_k}{\partial x^3} = 3\sqrt{2\dot{C}_k(x_5)}[(d_3, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1 - (d_0, d_1, d_2, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1],$$

$$\frac{\partial g_k}{\partial x_3} = 3\sqrt{2\dot{C}_k(x_5)}[(d_3, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1 - 2(d_0, d_1, d_2, 1, \cdots, \widehat{2k-1}, \cdots, 2N)_1].$$

Substituting above results into (67), we obtain the following identities:

$$(d_1, d_2, \bullet)_1(d_0, \star)_1 - (\bullet)_1(d_0, d_1, d_2, \star)_1$$

$$-(d_0, d_2, \bullet)_1(d_1, \star)_1 + (d_0, d_1, \bullet)_1(d_2, \star)_1 = 0,$$

where $\star$ denotes $\{1, \cdots, \widehat{2k-1}, \cdots, 2N\}$. That shows that f, g_k satisfy the equation (67). Equally, f, h_k satisfy the equation (68). So equations (64), (67) and (68) constitute a coupled system with K pairs of self-consistent sources which can be viewed as the BKPESCS. And f, g_k and h_k defined by (61), (65) and (66) are pfaffian solutions of the system. To explain the integrability of the system, we give a bilinear Bäcklund transformation of the system (64), (67) and (68).

Proposition 2. *The bilinear system (64), (67)-(68) has the bilinear Bäcklund transformation:*

$$(D_1-\mu)g_k\cdot f^{'}=\lambda(D_1-\mu)f\cdot g_k^{'}, \tag{70}$$

$$(D_1-\mu)f\cdot h_k^{'}=\lambda(D_1-\mu)h_k\cdot f^{'}, \tag{71}$$

$$(D_3-D_1^3+3\mu D_1^2-3\mu^2D_1-\gamma)f\cdot f^{'}=0, \tag{72}$$

$$(D_3-D_1^3+3\mu D_1^2-3\mu^2D_1-\gamma)g_k\cdot g_k^{'}=0, \tag{73}$$

$$(D_3-D_1^3+3\mu D_1^2-3\mu^2D_1-\gamma)h_k\cdot h_k^{'}=0, \tag{74}$$

$$\begin{aligned}&(-D_1^5+5\gamma D_1^2-5D_1^2D_3+5\mu D_1^4-5\mu^2D_1^3-10\mu D_3\\&\quad-10\gamma\mu D_1+10\mu D_1D_3+6D_5+\theta)f\cdot f^{'}+\tfrac{1}{3}\sum_{i=1}^{K}(\lambda g_i^{'}h_i-\lambda^{-1}g_ih_i^{'})=0,\end{aligned} \tag{75}$$

where γ, μ, λ and θ are arbitrary constants and $1\leqslant k\leqslant K$.

Proof. Let f, g_k and h_k be solutions of (64), (67)-(68). What we only need to prove is that $f^{'}$, $g_k^{'}$ and $h_k^{'}$ in (70)-(75) satisfy equations (64), (67) and (68). In fact, utilizing relations (70)-(75) and bilinear operator identities in Appendix A , we have

$$\begin{aligned}
P\ =\ &\{(D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5)f\cdot f-\sum_{i=1}^{K}D_1g_i\cdot h_i\}f^{'}f^{'}\\
&-\{(D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5)f^{'}\cdot f^{'}-\sum_{k=1}^{K}D_1g_i^{'}\cdot h_i^{'}\}f^2\\
=\ &3D_1[ff^{'}\cdot(D_1^5f\cdot f^{'})+5(D_1^3f\cdot f^{'})\cdot(D_1^2f\cdot f^{'})]+5D_1^3(D_1^3f\cdot f^{'})\cdot ff^{'}\\
&-10D_3(D_1^3f\cdot f^{'})\cdot ff^{'}-30D_1(D_1f\cdot f^{'})\cdot(D_1D_3f\cdot f^{'})\\
&-10D_3(D_3f\cdot f^{'})\cdot ff^{'}+18D_1(D_5f\cdot f^{'})\cdot ff^{'}\\
&-\sum_{i=1}^{K}[(D_1g_i\cdot f^{'})h_if^{'}-g_if^{'}(D_1h_i\cdot f^{'})-(D_1f\cdot h_i^{'})fg_i^{'}+h_i^{'}f(D_1f\cdot g_i^{'})]\\
=\ &3D_1[(-D_1^5+6D_5)f\cdot f^{'}]\cdot ff^{'}-5D_1[(-D_3+3\mu^2D_1+\gamma)f\cdot f^{'}]\cdot(D_1^2f\cdot f^{'})\\
&-5D_1^3[(-D_3+3\mu^2D_1-3\mu D_1^2)f\cdot f^{'}]\cdot ff^{'}\\
&-10D_3(D_1^3f\cdot f^{'})\cdot ff^{'}-10D_3(D_3f\cdot f^{'})\cdot ff^{'}\\
&-30D_1(D_1f\cdot f^{'})\cdot(D_1D_3f\cdot f^{'})-\sum_{k=1}^{K}\lambda[h_if^{'}(D_1f\cdot g_i^{'})-fg_i^{'}D_1h_i\cdot f^{'}]\\
&-\sum_{k=1}^{K}\lambda^{-1}[fh_i^{'}(D_1g_i\cdot f^{'})-f^{'}g_iD_1f\cdot h_i^{'}]\\
=\ &3D_1[(-D_1^5+5\gamma D_1^2-5D_1^2D_3+5\mu D_1^4-5\mu^2D_1^3-10\mu D_3-10\gamma\mu D_1\\
&+10\mu D_1D_3+6D_5)f\cdot f^{'}+\sum_{i=1}^{K}(\tfrac{1}{3}\lambda g_i^{'}h_i-\tfrac{1}{3}\lambda^{-1}g_ih_i^{'})]\cdot ff^{'}\\
=\ &-3\theta D_1ff^{'}\cdot ff^{'}\equiv 0;
\end{aligned}$$

$$\begin{aligned}
P_k &= \{(D_3-D_1^3)f\cdot h_k\}f'h_k' - \{(D_3-D_1^3)f'\cdot h_k'\}fh_k \\
&= (D_3f\cdot f')h_kh_k' - ff'(D_3h_k\cdot h_k') - (D_1^3f\cdot f')h_kh_k' \\
&\quad +ff'(D_1^3h_kh_k') + 3D_1(D_1f\cdot h_k')\cdot(D_1h_k\cdot f') \\
&= [(D_3-D_1^3)f\cdot f']h_kh_k' - ff'[(D_3-D_1^3)h_k\cdot h_k'] \\
&\quad +3\mu D_1fh_k'\cdot(D_1h_k\cdot f') - \lambda\mu D_1h_kf'\cdot(D_1h_k\cdot f') \\
&= [(D_3-D_1^3+3\mu D_1^2)f\cdot f']h_kh_k' - ff'[(D_3-D_1^3+3\mu D_1^2)h_k\cdot h_k'] \\
&\quad -3\mu^2D_1fh_k'\cdot h_kf' \\
&= [(D_3-D_1^3+3\mu D_1^2-3\mu^2D_1)f\cdot f']h_kh_k' \\
&\quad -ff'[(D_3-D_1^3+3\mu D_1^2-3\mu^2D_1)h_k\cdot h_k'] - 3\mu^2D_1fh_k'\cdot h_kf' \\
&= \gamma ff'h_kh_k' - \gamma ff'h_kh_k' \equiv 0.
\end{aligned}$$

The above results indicate that f', h_k' satisfy equations (64) and (68). Similarly, it can be shown that f', g_k' satisfy equation (67). So f', h_k' and g_k' are solutions of the system (64), (67)-(68). Then we complete the proof. □

5. Conclusion and Discussions

In the paper, we have proposed a new method to construct soliton equations with self-consistent sources. One of the advantages of this approach is that SESCSs and their soliton solutions can be generated simultaneously from the procedure. This procedure has been successfully applied to the 2D Toda equation, discrete KP equation and a $(2+1)$-dimensional BKP equation. In addition, we have derived the bilinear Bäcklund transformations for the dKPESCS and BKPESCS and thus showed the integrability of dKPESCS and BKPESCS. If we let the arbitrary functions $\gamma_j(t)$, $c_j(k_2)$ and $C_j(x_5)$ in solutions of these SESCSs be constants, respectively, these SESCSs come to the initial equations without sources, and the solutions of SESCSs will be reduced to the original solutions of equations without sources. So the SESCSs are a kind of generalization of equations without sources, and solutions of SESCSs obtained in the procedure are also generalization of Gram-type determinant or pfaffian solutions of original equations. We know many soliton equations possess determinant or pfaffian solutions with some arbitrary constants. For example, the semi-discrete Toda equation has such kind of determinant solutions which can also be expressed by means of pfaffian. As for BKP equations, we can only find pfafffian solutions. So we believe that this approach can be applicable to a variety of soliton equations, both continuous and discrete, such as the semi-discrete Toda equation, Leznov lattice equation, DKP-type equations and semi-discrete BKP-type equations. The work in this direction is in progress.

Finally, we believe that it would be quite interesting to consider the reduction of the soliton equations with self-consistent sources, say the discrete KP equation with self-consistent sources. It is noted that in [47] a variety of (1+1)-dimensional famous soliton equations have been derived from the reductions of the discrete KP equation. Therefore it is natural for us to expect that many (1+1)-dimensional SESCSs may be derived from the reductions of the discrete KP equation with self-consistent sources obtained in this paper.

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Appendix A.. Hirota's bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions a, b, c and d.

$$D_z ab \cdot cd = (D_z a \cdot d)cb - ad(D_z c \cdot b); \tag{A1}$$

$$(D_z a \cdot b)c^2 = (D_z a \cdot c)bc - ac(D_z b \cdot c); \tag{A2}$$

$$(D_z a \cdot b)cd - ab(D_z c \cdot d) = (D_z a \cdot c)bd - ac(D_z b \cdot d); \tag{A3}$$

$$(D_z^3 a \cdot b)cd - ab(D_z^3 c \cdot d) = (D_z^3 a \cdot c)bd - ac(D_z^3 b \cdot d) - 3D_z(D_z a \cdot d) \cdot (D_z b \cdot c); \tag{A4}$$

$$(D_z^2 a \cdot b)cd - ab(D_z^2 c \cdot d) = D_z[(D_z a \cdot d) \cdot cb + ad(D_z c \cdot b)]; \tag{A5}$$

$$(e^{D_1} a \cdot b)(e^{D_2} c \cdot c) = e^{\frac{D_1+D_2}{2}}(e^{\frac{D_1+D_2}{2}} a \cdot c) \cdot (e^{-\frac{D_1+D_2}{2}} b \cdot c); \tag{A6}$$

$$\begin{aligned} 2\sinh(\frac{D_1 - D_2}{2})&(e^{\frac{D_1+D_2}{2}} a \cdot b) \cdot (e^{-\frac{D_1+D_2}{2}} a \cdot b) \\ &= (e^{D_1} a \cdot a)(e^{D_2} b \cdot b) - (e^{D_2} a \cdot a)(e^{D_1} b \cdot b); \end{aligned} \tag{A7}$$

$$\begin{aligned} (e^{D_1} a \cdot a)&(e^{D_2} b \cdot b) - (e^{D_2} a \cdot a)(e^{D_1} b \cdot b) \\ &= 2\sinh(\frac{D_1 + D_2}{2})(e^{\frac{D_1-D_2}{2}} a \cdot b) \cdot (e^{\frac{D_2-D_1}{2}} a \cdot b); \end{aligned} \tag{A8}$$

$$\begin{aligned} (e^{D_1} a \cdot a)&(e^{D_2} b \cdot b) - (e^{D_2} a \cdot a)(e^{D_1} b \cdot b) \\ &= 2\sinh(\frac{D_1 - D_2}{2})(e^{\frac{D_1+D_2}{2}} a \cdot b) \cdot (e^{-\frac{(D_1+D_2)}{2}} a \cdot b). \end{aligned} \tag{A9}$$

where z ia a variable and D_1, D_2 are linear combination of D_{k_1}, D_{k_2} and D_{k_3}.

References

[1] Ablowitz M J and Clarkson M J 1991 Solitons, nonlinear evolution equations and inverse scattering London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge

[2] Konopelchenko B G 1992 Introduction to multidimensional integrable equations. The inverse spectral transform in $2+1$ dimensions Plenum Press, New York

[3] Konopelchenko B G 1993 Solitons in multidimensions. Inverse spectral transform method. World Scientific Publishing Co., Inc., River Edge, NJ

[4] Mel'nikov V K 1989 Interaction of solitary waves in the system described by the Kadomtsev-Petviashvili equation with a self-consistent source Commun. Math.Phys. 126 201-215

[5] Hirota R and Ohta Y 1991 Hierarchies of coupled soliton equations. I. J. Phys. Soc. Japan 60 798-809

[6] R. Hirota, *Direct method in soliton theory (In English)*, (Edited and Translated by Nagai, A., Nimmo, J. and Gilson, C., Cambridge University Press, 2004.6)

[7] Mel'nikov V K 1983 On equations for wave interactions. Lett. Math. Phys. 7 129-136

[8] Mel'nikov V K 1988 Exact solutions of the Korteweg-de Vries equation with a self-consistent source, Phys. Lett. A **128** 488-492

[9] Mel'nikov V K 1989 Capture and confinement of solitons in nonlinear integrable systems, Commun. Math. Phys. **120** 451-468

[10] Mel'nikov V K 1990 Integration of the Korteweg-de Vries equation with a source, Inverse Problems **6** 233-246

[11] Mel'nikov V K 1992 Integration of the nonlinear Schrödinger equation with a source. Inverse Problems **8** 133-147

[12] Mel'nikov V K 1987 A direct method for deriving a multi-soliton solution for the problem of interaction of waves on the x,y plane, Commun. Math.Phys. **112**1987 639-652

[13] Zeng Y B, Ma W X and Lin R L 2000 Integration of the soliton hierarchy with self-consistent sources, J. Math. Phys. **41** 5453-5489

[14] Zeng Y B, Ma W X and Shao Y J 2001 Two binary Darboux transformations for the KdV hierarchy with self-consistent sources, J. Math. Phys. **42** 2113-2128

[15] Antonowicz M and Rauch-Wojciechowski S 1993 Soliton hierarchies with sources and Lax representation for restricted flows, Inverse Problems **9** 201-215

[16] Leon J and Latifi A 1990 Solution of an initial-boundary value problem for coupled nonlinear waves, J. Phys. A **23** 1385-1403

[17] Deng S F, Chen D Y and Zhang D J 2003 The multisoliton solutions of the KP equation with self-consistent sources, J. Phys. Soc. Jpn. **72** 2184-2192

[18] Ma W X and Strampp W 1994 An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, Phys. Lett. A **185** 277-286

[19] Ma W X and Zhou Z X 2001 Binary symmetry constraints of N-wave interaction equations in $1+1$ and $2+1$ dimensions, J. Math. Phys. **42** 4345-4382

[20] Zeng Y B, Ma W X and Lin R L 2000 Integration of the soliton hierarchy with self-consistent sources, J. Math. Phys. **41** 5453-5489

[21] Lin R L, Zeng Y B and Ma W X 2001 Solving the KdV hierarchy with self-consistent sources by inverse scattering method, Physica A **291** 287-298

[22] Urasboev G U and Khasanov A B 2001 Integration of the Korteweg-de Vries equation with a self-consistent source in the presence of "steplike" initial data, Theor. Math. Phys. **129** 1341-1356

[23] Ye S and Zeng Y B 2002 Integration of the modified Korteweg-de Vries hierarchy with an integral type of source, J. Phys. A **35** L283-L291

[24] Zeng Y B 1994 New factorization of the Kaup-Newell hierarchy, Physica D **73** 171-188

[25] Zeng Y B and Li Y S 1996 The Lax representation and Darboux transformation for constrained flows of the AKNS hierarchy, Acta Math. Sin., New Ser. **12** 217-224

[26] Zeng Y B, Shao Y J and Ma W X 2002 Integrable-type Darboux transformation for the mKdV hierarchy with self-consistent sources, Commun. Theor. Phys. (Beijing) **38** 641-648

[27] Ma W X 2003 Soliton, Positon and Negaton Solutions to a Schrödinger Self-consistent source equation, J. Phys. Soc. Jpn. **72** 3017-3019

[28] Xiao T and Zeng Y B 2004 Generalized Darboux transformations for the KP equation with self-consistent sources, J. Phys. A. **37** 7143-7162

[29] T. Xiao, Y. B. Zeng, Bäcklund transformations for the constrained dispersionless hierarchies and dispersionless hierarchies with self-consistent sources, Inverse Problems 22(3)(2006),869-880

[30] Xiao T and Zeng Y B 2005 A new constrained mKP hierarchy and the generalized Darboux transformation for the mKP equation with self-consistent sources, Physica A. **353** 38-60

[31] Hase Y, Hirota R, Ohta Y and Satsuma J 1989 Soliton solutions of the Mel'nikov equations, J.

Phys. Soc. Japan **58** (8), 2713-2720

[32] Matsuno Y 1991 Bilinear Bäcklund transformation for the KdV equation with a source, J. Phys. A. **24** L273-L277

[33] Hu X B 1991 Nonlinear superposition formula of the KdV equation with a source, J. Phys. A **24** 5489-5497

[34] Hu X B 1996 The higher-order KdV equation with a source and nonlinear superposition formula, Chaos Solitons Fractals **7** 211-215

[35] Matsuno Y 1990 KP equation with a source and its soliton solutions, J. Phys. A. **23** L1235-L1239

[36] Zhang D J 2002 The N-soliton solutions for the modified KdV equation with self-consistent sources, J. Phys. Soc. Jpn. **71** 2649-2656

[37] Zhang D J and Chen D Y 2003 The N-soliton solutions of the sine-Gordon equation with self-consistent sources, Physica A **321** 467-481

[38] Zhang D J 2003 The N-soliton solutions of some soliton equations with self-consistent sources, Chaos, Solitons and Fractals **18** 31-43

[39] Gegenhasi and Hu X B 2006 On a integrable differential-difference equation with a source, J. Nonlinear Math. Phys. 13(2), 183-192

[40] Liu X J and Zeng Y B 2005 On the Toda lattice equation with self-consistent sources. J. Phys. A **38**8951-8965

[41] Wang H Y, Hu X B and Gegenhasi, 2006 2D Toda lattice equation with self-consistent sources: Casoratian type solution, bilinear Bäcklund transformation and Lax pair, Special Issue of J. Comp.Appl.Math. in press

[42] Ohta Y, Hirota R and Tsujimoto S 1993 Casorati and discrete Gram Type determinant represenation of solutions to the discrete KP hierarchy. J. Phys. Soc. Japan. 62 1872-1886

[43] Gilson C R, Nimmo J J C and Tsujimoto S 2001 Pfaffianization of the discrete KP equation, J. Phys. A: Math. Gen. 34 10569-10575

[44] Date E, Jimbo M, Kashiwara M and Miwa T 1981 Transformation groups for soliton equations. IV. A new hierarchy of soliton equations of KP-type. Phys. D 82 343–365

[45] Date E, Jimbo M, Kashiwara M and Miwa T 1981 Transformation groups for soliton equations. VI. KP hierarchies of orthogonal and symplectic type. J. Phys. Spc. Jpn. 50 3813

[46] Hirota R 1989 Soliton solutions to the BKP equations. I. The pfaffian techniqus. J. Phys. Spc. Jpn. 58 2285-2296

[47] Hirota R 1981 J. Phys. Soc. Japan 50 3785.