2011 年 3 月 运筹学学报 第 15 卷第 1 期
March, 2011 OR TRANSACTIONS Vol.15 No.1

A Generalization for Directed Scale-Free Graphs*

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Abstract We study a dynamically evolving directed random graph which randomly adds vertices and directed edges using preferential attachment and prove that its vertice degree obey power law and has elaborate power law exponents.

Keywords Operations research, power law graphs, degree distributions, scale free networks, random graph models
Subject Classification (GB/T13745-92) 110.74

关于有向无标度图的一个推广模型

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摘要 研究了一个动态的有向随机图演化模型：每个时间步模型随机的加入一个顶点及随机数目条依出、入度择优连接的有向边．证明了该模型出、入度分布服从幂律且具有对称的幂律指数．

关键词 运筹学，幂率图，度分布，无标度网络，随机图模型
学科分类号(GB/T13745-92) 110.74

1 Introduction

In the past few years, there has been much interest in understanding the properties of real-world large-scale networks such as the structure of the Internet and the World Wide Web. It has been observed that many such networks have a so-called power law degree distribution: the proportion of nodes of degree k is approximately $\frac{1}{k^\gamma}$, where $\gamma > 1$ is a fixed real number. Such graphs are sometimes called *scale-free* in the literature. A graph is called a *power law graph* if the fraction of vertices with degree k is proportional to $\frac{1}{k^\gamma}$ for some constant $\gamma > 0$. The standard models of random graphs introduced by Erdős and Rényi[12] and Gilbert[13] are not appropriate for

收稿日期： 2008 年 3 月 28 日．
∗ The research was supported by NSFC (Grant NO. 60872060), Shanghai Citylevel Finance Departmental Budget Project (1138IA0005/1139IA0013) and the Innovation Program of Shanghai Municipal Education Commission (11yz241).

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studying these networks, since they generate graphs which, with high probability, have binomial degree distributions. A large number of power law random graph models [1-2,5,9-10,15] have been proposed now. Many models have been suggested to explain this and other features of the graphs studied. One of the basic ideas is the combination of growth with 'preferential attachment'; the graph grows one vertex at a time, and edges are added, perhaps only from the new vertex to old vertices, or perhaps also between old vertices, where the old vertices involved are chosen with probabilities proportional to their degrees. One of the simplest and earliest models is that outlined by Barabási and Albert in [5], made precise in [9]. The degree sequence of this model was analyzed heuristically in [5, 6], and rigorously in [9]. Many generalizations have been suggested and studied heuristically, see [3]; a few have been analyzed precisely, see [16]. In [10] Cooper and Frieze have analyzed rigorously a very general version of the model allowing for (finite) distributions of out-degrees and mixtures of uniform and preferential attachment.

The models mentioned above essentially describe undirected graphs. The only exception is [10], where the authors treat either in-degrees or out-degrees, but not both simultaneously; a full treatment of directed graphs was announced there, but has not yet appeared. Bollobás et al. in [8] and Wang in [17] introduce a directed scale-free graph model that grow with preferential attachment depending on the in- and out-degrees. However, in their model when a new vertex is added it can't be viewed as both origin and destination simultaneously since only a single directed edge is added to it. In [9] Bollobás and Riordan introduce a model that multi-edge is added when a new vertex is added, but they don't differentiate between the in-degree and out-degree while the graphs grow with preferential attachment depending on degrees of the vertices. In the model mentioned in this paper, the number of edges added at every time step is a random number, which generalize the models and some result in [8].

2 The model

We consider a directed graph which grows by adding single edges at discrete time steps. At each such step a vertex may or may not be added. For simplicity we allow multiple edges and loops. More precisely, let α, δ_{in} and δ_{out} be non-negative real numbers. Let $G(0)$ be any fixed initial directed graph, for example a single vertex without edges. At time t the graph $G(t)$ has random e_t edges and n_t vertices. In what follows, to *choose* a vertex v of $G(t)$ *according to* $d_{out}+\delta_{out}$ means to choose v so that $\mathbf{Pr}(v=v_i)$ is proportional to $d_{out}(v_i)+\delta_{out}$, i.e., so that

$$\mathbf{Pr}(v=v_i)=(d_{out}(v_i)+\delta_{out})/(e_t+\delta_{out}n_t);$$

to *choose* a vertex v *according to* $d_{in}+\delta_{in}$ means to choose v so that $\mathbf{Pr}(v=v_i)$ is proportional to $d_{in}(v_i)+\delta_{in}$, i.e., so that

$$\mathbf{Pr}(v=v_i)=(d_{in}(v_i)+\delta_{in})/(e_t+\delta_{in}n_t).$$

Here $d_{out}(v_i)$ and $d_{in}(v_i)$ are the out-degree and in-degree of v_i, measured in the graph $G(t)$.

For $t\geqslant 0$ we form $G(t+1)$ from $G(t)$ according to the following rules:

(A) With probability α, add a new vertex v together with random number ξ_{t+1} edges (v,w) from v to ξ_{t+1} vertices w's of $G(t)$ and random number η_{t+1} edges (w,v) from η_{t+1} vertices w's of $G(t)$ to v, where the former ξ_{t+1} w's is chosen according to $d_{in}+\delta_{in}$ and the later η_{t+1} w's according to $d_{out}+\delta_{out}$, and the $\xi_{t+1}+\eta_{t+1}$ w's is chosen independently.

(B) With probability β, add random number ζ_{t+1} edges (v,w) from ζ_{t+1} existing vertex v's to ζ_{t+1} existing vertex w's, where each v's is chosen according to $d_{out}+\delta_{out}$ and each w's according to $d_{in}+\delta_{in}$, and the v's and w's are chosen independently.

Let ξ has probability distribution $(a_i:i\geqslant 0)$, η has distribution $(b_i:i\geqslant 0)$ and ζ has distribution $(c_i:i\geqslant 0)$, where the probability that ξ be i is a_i, etc. We write $\bar{\xi}=\mathbf{E}\xi,\bar{\eta}=\mathbf{E}\eta,\bar{\zeta}=\mathbf{E}\zeta$. In (A) and (B), $\{\xi_t,t>0\},\{\eta_t,t>0\},\{\zeta_t;t>0\}$ are three sequences of independent random variable which respectively have the same probability distributions with ξ,η,ζ for every t.

The model we study here require $\alpha>0$. Depending on the parameters, we may have to assume $e_0\geqslant 1, a_0=\mathbf{Pr}(\xi=0)=0, b_0=\mathbf{Pr}(\eta=0)=0$ or $c_0=\mathbf{Pr}(\zeta=0)=0$ for our process to make sense. Additionally, it is convenient to assume a *finiteness* condition for the distribution of ξ,η,ζ. This means that there exist a constant m such that $a_i=b_i=c_i=0, i>m$.

Now define the sequences $(p_{-1},p_0,p_1,...,p_k,...);(q_{-1},q_0,q_1,...,q_k,...)$ by $p_{-1}=q_{-1}=0$, and for $i\geqslant 0$

$$p_i=\frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(i-1+\delta_{in})p_{i-1}}{\alpha(\bar{\xi}+\bar{\eta})+(1-\alpha)\bar{\zeta}+\alpha\delta_{in}}-\frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(i+\delta_{in})p_i}{\alpha(\bar{\xi}+\bar{\eta})+(1-\alpha)\bar{\zeta}+\alpha\delta_{in}}+\alpha b_i \tag{1}$$

$$q_i = \frac{(\alpha\bar{\eta} + (1-\alpha)\bar{\zeta})(i-1+\delta_{out})q_{i-1}}{\alpha(\bar{\xi}+\bar{\eta}) + (1-\alpha)\bar{\zeta} + \alpha\delta_{out}} - \frac{(\alpha\bar{\eta} + (1-\alpha)\bar{\zeta})(i+\delta_{out})q_i}{\alpha(\bar{\xi}+\bar{\eta}) + (1-\alpha)\bar{\zeta} + \alpha\delta_{out}} + \alpha a_i$$

For simplicity we set

$$\gamma_{in} = 2 + \frac{\alpha(\bar{\eta} + \delta_{in})}{\alpha\bar{\xi} + (1-\alpha)\bar{\zeta}} \quad \text{and} \quad \gamma_{out} = 2 + \frac{\alpha(\bar{\xi} + \delta_{out})}{\alpha\bar{\eta} + (1-\alpha)\bar{\zeta}}.$$

(The definitions above we may have to assume $\alpha\bar{\xi}+(1-\alpha)\bar{\zeta} > 0$ or $\alpha\bar{\eta}+(1-\alpha)\bar{\zeta} > 0$ to make sense.) And we write $x_i(t)$ for the number of vertices of G_t with in-degree i, and $y_i(t)$ for the number of vertices of G_t with out-degree i.

3 Main results

The main results of this paper are

Theorem 1 *There exists a constant $M > 0$ such that almost surely for $t, k \geqslant 0$,*

$$|\mathbf{E}x_k(t) - tp_k| \leqslant Mt^{1/2}\log t; \quad |\mathbf{E}y_k(t) - tq_k| \leqslant Mt^{1/2}\log t.$$

Theorem 2 *For $k = O(\log t)$, there exists some sufficiently large constant M,*

$$\mathbf{Pr}(|x_k(t) - \mathbf{E}(x_k(t))| \geqslant Mt^{2/3}\log t) \leqslant t^{-\Omega(\log t)};$$
$$\mathbf{Pr}(|y_k(t) - \mathbf{E}(y_k(t))| \geqslant Mt^{2/3}\log t) \leqslant t^{-\Omega(\log t)}.$$

The next theorem show that p_i and q_i asymptotically have the form of pow law functions.

Theorem 3 *(i) If $\alpha\bar{\xi} + (1-\alpha)\bar{\zeta} > 0$, then as $i \to \infty$ we have $p_i \sim C_{in}i^{-\gamma_{in}}$, where C_{in} is a positive constant.*

(ii) If $\alpha\bar{\eta} + (1-\alpha)\bar{\zeta} > 0$, then as $i \to \infty$ we have $q_i \sim C_{out}i^{-\gamma_{out}}$, where C_{out} is a positive constant.

In the statements above, $g(t) = O(f(t))$ means there exist constants T and $M > 0$ such that for all $t \geqslant T$, $|g(t)/f(t)| \leqslant M$; $g(t) = \Omega(f(t))$ means there exist constants T and $M_2 \geqslant M_1 > 0$ such that for all $t \geqslant T$, $M_1 \leqslant g(t)/f(t) \leqslant M_2$; $g(i) \sim f(i)$ means $g(i)/f(i) \to 1$ as $i \to \infty$.

4 Proof of theorems

We prove all three theorems just considering the state of in-degrees, for out-degrees proofs is exactly the same after interchanging the roles of ξ_t and η_t and of δ_{in} and δ_{out}. We have mainly used the methods of that used in [10] and [8] for our proof.

By (1), the probability that there's at least a vertex of in-degree j in G_t which gets l, $l \geqslant 2$ in-incidences and becomes a vertex of in-degree $j+l$ at time step $t+1$ is at most

$$m!x_j(t)\left(\frac{j+\delta_{in}}{e_t+\delta_{in}n_t}\right)^l \leqslant m!x_j(t)\left(\frac{j+\delta_{in}}{e_t+\delta_{in}n_t}\right)^2 = O\left(\frac{j}{e_t+\delta_{in}n_t}\right), \tag{2}$$

since $r_j(t) := x_j(t)(j+\delta_{in})/(e_t+\delta_{in}n_t) \leqslant 1$. With this effect, we have

$$x_i(t+1) = \begin{cases} x_i(t) + B_{t+1}(1,\alpha)B(\xi_{t+1}, r_{i-1}(t)) + (1-B_{t+1}(1,\alpha))B(\zeta_{t+1}, r_{i-1}(t)) \\ -(B_{t+1}(1,\alpha)B(\xi_{t+1}, r_i(t)) + (1-B_{t+1}(1,\alpha))B(\zeta_{t+1}, r_i(t))) \\ +\mathbf{1}_{\{B_{t+1}(1,\alpha)=1, \eta_{t+1}=i\}} & \text{w.p. } 1-O(\frac{i}{e_t+\delta_{in}n_t}); \\ x_i(t)+u, |u| \leqslant m & \text{w.p. } O(\frac{i}{e_t+\delta_{in}n_t}). \end{cases} \tag{3}$$

where $\{B_{t+1}(1,\alpha); t \geqslant 0\}$ is a sequence of independent random variables which have the same distribution with 0,1 random variable $B(1,\alpha)$, where $\mathbf{Pr}(B(1,\alpha)=1)=\alpha$; $\{B(\xi_{t+1}, r_{i-1}(t))$;
$t \geqslant 0\}$ is a sequence of random variables which have binomial distribution $B(l,p)$ on condition that $\xi_{t+1}=l, r_j(t)=p$, and $B_{t+1}(1,\alpha)$ is independent with $B(\xi_{t+1}, r_j(t))$ for every $t \geqslant 0$; $\mathbf{1}_{\mathcal{D}}$ is the indicator function which is 1 if the event $\mathcal{D}$ holds and 0 otherwise; u is an error term that there's some vertices have added more than 1 in-incidences to generate $x_i(t+1)$ in-degree i vertices at step t, which has probability of $O(\frac{i}{e_t+\delta_{in}n_t})$ by (2). We abbreviate 'with probability' by 'w.p.'.

At first we establishes an upper bound on p_k given in by (1). Denote

$$\theta = \mathbf{E}(B_i(1,\alpha)(\xi_i+\eta_i) + (1-B_i(1,\alpha))\zeta_i) = \alpha(\bar{\xi}+\bar{\eta}) + (1-\alpha)\bar{\zeta}.$$

Lemma 1 *For $k \geqslant 1$ the solution of (1) satisfies $p_k \leqslant C/k$.*

Proof We assume that $k \geqslant m$, and thus $b_k = 0$. Smaller values of k can be dealt with by adjusting C. We proceed by induction on k. By (1),

$$p_k = \frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k-1+\delta_{in})p_{k-1}}{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k+\delta_{in})+\theta+\delta_{in}\alpha}$$

Let $p_{k-1} \leqslant \frac{C}{k-1}$, then we just need to prove $p_k \leqslant \frac{C}{k}$, which only need to prove

$$\frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k-1+\delta_{in})p_{k-1}}{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k+\delta_{in})+\theta+\delta_{in}\alpha} \leqslant \frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k-1+\delta_{in})C/(k-1)}{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k+\delta_{in})+\theta+\delta_{in}\alpha} \leqslant \frac{C}{k}$$

The right inequality of above is equivalent to

$$k(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k-1+\delta_{in}) \leqslant (k-1)((\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(k+\delta_{in})+\theta+\delta_{in}\alpha)$$

That is

$$(k-1)(\theta+\delta_{in}\alpha) \geqslant (\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})\delta_{in}.$$

A large k guaranty above holds clearly. This completes the proof of Lemma 1.

Proof of Theorem 1 When $k \geqslant t^{1/2}$ it is trivial by lemma 1 since $e_t \leqslant 2mt$. Now we prove when $k \leqslant t^{1/2}$ the first part of Theorem 1 holds. Since

$$n_t = n_0 + \sum_{\tau=1}^{t} B_\tau(1,\alpha),$$
$$e_t = e_0 + \sum_{\tau=1}^{t} (B_\tau(1,\alpha)(\xi_\tau+\eta_\tau) + (1-B_\tau(1,\alpha))\zeta_\tau),$$

the random variables n_t, e_t are sharply concentrated provided $t \to \infty$. Indeed by Azuma-Hoeffding's theorem([4,7,14]),

$$\mathbf{Pr}(|n_t-\alpha t| \geqslant t^{\frac{1}{2}}\log t) \leqslant 2e^{-(\log t)^2/(2m^2)}; \tag{4}$$
$$\mathbf{Pr}(|e_t-\theta t| \geqslant t^{\frac{1}{2}}\log t) \leqslant 2e^{-(\log t)^2/(2m^2)}. \tag{5}$$

Take the expectation of (3) we have

$$\mathbf{E}x_i(t+1) = \mathbf{E}x_i(t) + (\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(\mathbf{E}r_{i-1}(t)-\mathbf{E}r_i(t)) + \alpha b_i + O(\frac{i}{e_t+\delta_{in}n_t}). \tag{6}$$

It follows from (4-6) that

$$\begin{aligned}\mathbf{E}x_k(t+1) =& \mathbf{E}x_k(t) + \frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})}{(\theta+\delta_{in}\alpha)t}((k-1+\delta_{in})\mathbf{E}x_{k-1}(t) - (k+\delta_{in})\mathbf{E}\mathbf{x_k}(\mathbf{t}))\\ &+ \alpha b_k + O(t^{-1/2}\log t),\end{aligned}$$

Let $\Delta_k(t) = \mathbf{E}x_k(t) - tp_k$, by (1) and above

$$\begin{aligned}\Delta_k(t+1) =& \Delta_k(t) + \frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})}{(\theta+\delta_{in}\alpha)t}((k-1+\delta_{in})\Delta_{k-1}(t) - (k+\delta_{in})\Delta_k(t))\\ &+ O(t^{-1/2}\log t)\end{aligned} \tag{7}$$

To prove the first part of theorem 1 we must show exactly $|\Delta_i(t)| \leqslant Mt^{1/2}\log t$ for all $k \leqslant t^{1/2}$. We do this by induction on k; suppose that $k \geqslant 0$ and $|\Delta_{k-1}(t)| \leqslant Mt^{1/2}\log t$, noting that $\Delta_{-1}(t) = 0$ and we can adjust M to deal with small values of t for each k, so the induction starts. Let L denote the hidden constant in $O(t^{-1/2}\log t)$ of (7). Assume inductively that $|\Delta_\kappa(\tau)| \leqslant M\tau^{1/2}\log\tau$ for all $\kappa \leqslant k, \tau \leqslant t$. It follows from (7) that

$$\begin{aligned}|\Delta_k(t+1)| &\leqslant Mt^{1/2}\log t + \frac{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}}{(\theta+\delta_{in}\alpha)t}((k-1+\delta_{in})Mt^{1/2}\log t \\ &\quad -(k+\delta_{in})Mt^{1/2}\log t) + Lt^{-1/2}\log t \\ &= Mt^{1/2}\log t - \frac{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}}{\theta+\delta_{in}\alpha}Mt^{-1/2}\log t + Lt^{-1/2}\log t \\ &\leqslant Mt^{1/2}\log t + Lt^{-1/2}\log t \\ &\leqslant M(t+1)^{1/2}\log(t+1)\end{aligned}$$

provided $M \geqslant 2L$. Above the first inequation holds because on the right side of (7) the coefficient of the first $\Delta_k(t)$ is larger than that of the second since $k \leqslant t^{1/2}$. This completes the proof by induction.

Let us choose at each step which of operation (A) or (B) to perform and the number of edges added. Let $\mathcal{A}$ be an event corresponding to one (infinite) sequence of such choices and set $\widetilde{\mathcal{A}}$ is composed by all such events $\mathcal{A}$. Given $\mathcal{A}$ define $\alpha_{\mathcal{A}} = \alpha_{\mathcal{A}}(\tau) = \mathbf{1}_{\mathbf{E}}(B_\tau|\mathcal{A}) = 1\}$ to be the indicator for an new vertex to generate at time τ, define $\xi_{\mathcal{A}} = \xi_{\mathcal{A}}(\tau) = \sum\limits_{i=1}^{m} i\mathbf{1}_{\{\mathbf{E}(B_\tau|\mathcal{A})=i\}}$, similarly we define $\eta_{\mathcal{A}}, \zeta_{\mathcal{A}}$. Denote

$$\epsilon_{\mathcal{A}}(t) = \sum_{\tau=1}^{t}(\alpha_{\mathcal{A}}(\tau)\xi_{\mathcal{A}}(\tau) + (1-\alpha_{\mathcal{A}}(\tau))\zeta_{\mathcal{A}}(\tau))$$

which counts the number of in-incidence vertices that until time t the added edges will choose randomly to join to condition on $\mathcal{A}$.

The following lemma help us to prove Theorem 2.

Lemma 2 *For any $u > 0$,*

$$\mathbf{Pr}(|x_k(t) - \mathbf{E}(x_k(t)|\mathcal{A})| \geqslant u|\mathcal{A}) \leqslant \exp\Big\{-\frac{u^2}{4\epsilon_{\mathcal{A}}(t)}\Big\}.$$

Proof Given $\mathcal{A}$, let $Y_1, Y_2, ..., Y_{\epsilon_{\mathcal{A}}(t)}$ be the sequence of single choices of edges created. We let

$$Z_i = \mathbf{E}(x_k(t)|Y_1, Y_2, ..., Y_i, \mathcal{A}) - \mathbf{E}(x_k(t)|Y_1, Y_2, ..., Y_{i-1}, \mathcal{A})$$

and prove that $|Z_i| \leqslant 2$. The Azuma-Hoeffding martingale inequality then implies that lemma 2 holds.

For each sequence of edges choice $\mathbf{Y} = Y_1, Y_2, ..., Y_{\epsilon_{\mathcal{A}}(t)}$ denote $Y_i = x_i\vec{v}_i, \widehat{Y}_i = x_i\vec{v}_i$, then the choice of Y_i can be viewed as such a program to perform: Either choose an edge and take its endvertice as Y_i's one, or choose Y_i's endvertice otherwise. Now we define $\widehat{\mathbf{Y}} = \widehat{Y}_1, \widehat{Y}_2, ..., \widehat{Y}_{i-1}, \widehat{Y}_i, ..., \widehat{Y}_{\epsilon_{\mathcal{A}}(t)} := Y_1, Y_2, ..., Y_{i-1}, \widehat{Y}_i, ..., \widehat{Y}_{\epsilon_{\mathcal{A}}(t)}$ where $\widehat{Y}_i$'s endvertice $\hat{v}_i$ is a vertex randomly chosen according to $d_{in} + \delta_{in}$, for $j > i, \widehat{Y}_j$ is obtained from $Y_j = x_j\vec{v}_j$ as follows: At step j, if the choice of Y_j's endvertice is decided by choosing $Y_{j'} = x_{j'}\vec{v}_{j'}, j' < j$ and takeing its endvertice $v_{j'}$ as v_j, then $\widehat{Y}_j$ choose $\widehat{Y}_{j'}$ and take $\widehat{Y}_{j'}$'s endvertice $\hat{v}_{j'}$ as endvertice $\hat{v}_j$. Otherwise $\widehat{Y}_j = Y_j$.

The map $\mathbf{Y} \to \widehat{\mathbf{Y}}$ is measure preserving and in going from $\mathbf{Y}$ to $\widehat{\mathbf{Y}}$ only the degree of vertex v_i and $\hat{v}_i$ change and so the number of vertices of degree k changes by at most 2 and lemma 2 holds.

Proof of Theorem 2 Going back to (3) we can write

$$\begin{aligned}\bar{x}_k^{\mathcal{A}}(t+1) =& \bar{x}_k^{\mathcal{A}}(t) + (\alpha_{\mathcal{A},t+1}\xi_{\mathcal{A},t+1} + (1-\alpha_{\mathcal{A},t+1})\zeta_{\mathcal{A},t+1})(\bar{r}_{k-1}^{\mathcal{A}}(t) - \bar{r}_k^{\mathcal{A}}(t)) \\ &+ \alpha_{\mathcal{A},t+1}\mathbf{1}_{\{\eta_{\mathcal{A},t+1}=k\}} + O\Big(\frac{k}{e_t^{\mathcal{A}} + \delta_{in} n_t^{\mathcal{A}}}\Big)\end{aligned}$$

where

$$\bar{x}_k^{\mathcal{A}}(t) = \mathbf{E}(x_k(t)|\mathcal{A}), \quad \bar{r}_{k-1}^{\mathcal{A}}(t) - \bar{r}_k^{\mathcal{A}}(t) = \frac{(k-1+\delta_{in})\bar{x}_{k-1}^{\mathcal{A}} - (k+\delta_{in})\bar{x}_k^{\mathcal{A}}}{e_t^{\mathcal{A}} + \delta_{in} n_t^{\mathcal{A}}}.$$

Again we let $\Delta_k^{\mathcal{A}}(t) = \bar{x}_k^{\mathcal{A}}(t) - tp_k$ then

$$\begin{aligned}\Delta_k^{\mathcal{A}}(t+1) =& \Delta_k^{\mathcal{A}}(t) + (\alpha_{\mathcal{A}}\xi_{\mathcal{A}} + (1-\alpha_{\mathcal{A}})\zeta_{\mathcal{A}})(\bar{r}_{k-1}^{\mathcal{A}}(t) - \bar{r}_k^{\mathcal{A}}(t)) + \alpha_{\mathcal{A}}\mathbf{1}_{\{\eta_{\mathcal{A}}=k\}} \\ &+ O\Big(\frac{k}{e_t^{\mathcal{A}} + \delta_{in} n_t^{\mathcal{A}}}\Big) - p_k\end{aligned} \tag{8}$$

Let $s = t^{2/3}$, then by Azuma-Hoeffding's theorem,

$$\mathbf{Pr}\Big(\Big|\sum_{\tau=t-s}^{t} B_\tau\eta_\tau - (s+1)\alpha\bar{\eta}\Big| \geqslant s^{\frac{1}{2}}\log s\Big) \leqslant 2e^{-(\log s)^2/(2m^2)} = t^{-\Omega(\log t)} \tag{9}$$

Let

$$\mathcal{A}_1 = \{\mathcal{A} \in \widetilde{\mathcal{A}} : |n_t - \alpha t| \leqslant t^{\frac{1}{2}}\log t\}; \quad \mathcal{A}_2 = \{\mathcal{A} \in \widetilde{\mathcal{A}} : |e_t - \theta t| \leqslant t^{\frac{1}{2}}\log t\};$$

$$\mathcal{A}_3 = \Big\{\mathcal{A} \in \widetilde{\mathcal{A}} : \Big|\sum_{\tau=t-s}^{t} B_\tau\eta_\tau - (s+1)\alpha\bar{\eta}\Big| \leqslant s^{\frac{1}{2}}\log s\Big\}; \quad \widehat{\mathcal{A}} = \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3.$$

It follows easily from (4)(5) and (9) that

$$\mathbf{Pr}(\mathcal{A} \notin \widehat{\mathcal{A}}) = t^{-\Omega(\log t)}. \tag{10}$$

Assuming that $\mathcal{A} \in \widehat{\mathcal{A}}$ and set

$$\Sigma_k^{\mathcal{A}}(t) = \frac{1}{s+1} \sum_{l=0}^{s} \Delta_k^{\mathcal{A}}(t-l).$$

We can deduce from (1) and (8) that

$$\begin{aligned}\Sigma_k^{\mathcal{A}}(t+1) = \Sigma_k^{\mathcal{A}}(t) &+ \frac{(\alpha\bar{\xi} + (1-\alpha)\bar{\zeta})}{(\theta + \delta_{in}\alpha)t}((k-1+\delta_{in})\Sigma_{k-1}^{\mathcal{A}}(t) - (k+\delta_{in})\Sigma_k^{\mathcal{A}}(t)) \\ &+ O\Big(\frac{ks}{(\theta+\delta_{in}\alpha)t + O(t^{2/3})}\Big) + O(s^{-1/2}\log s) \\ &= \Sigma_k^{\mathcal{A}}(t) + \frac{(\alpha\bar{\xi} + (1-\alpha)\bar{\zeta})}{(\theta + \delta_{in}\alpha)t}((k-1+\delta_{in})\Sigma_{k-1}^{\mathcal{A}}(t) - (k+\delta_{in})\Sigma_k^{\mathcal{A}}(t)) \\ &+ O(t^{-1/3}\log t) \qquad (11)\end{aligned}$$

since $k = O(\log t)$, $s = t^{2/3}$. (We leave out the straightforward but somewhat technical details.) We inductively prove the following inequalities for $\kappa \leqslant \Omega(\log t)$ and some sufficiently large M :

$$|\Sigma_\kappa^{\mathcal{A}}(t)| \leqslant Mt^{2/3}\log t. \qquad (12)$$

Let $\bar{x}_{-1}^{\mathcal{A}}(\tau) = 0$ for every τ, then $\Sigma_{-1}^{\mathcal{A}}(\tau) = 0$. Again for small t this holds trivially so the induction starts. Let L denote the hidden constant of the term $O(t^{-1/3}\log t)$ in (11). Let $k \leqslant \Omega(\log t)$ and some sufficiently large M, for any $\kappa \leqslant k$,

$$|\Sigma_\kappa^{\mathcal{A}}(t)| \leqslant Mt^{2/3}\log t,$$

then by the last part of (11)

$$|\Sigma_k^{\mathcal{A}}(t+1)| \leqslant Mt^{2/3}\log t + Lt^{-1/3}\log t \leqslant M(t+1)^{2/3}\log(t+1),$$

provided $M > 3L/2$. This completes the induction. Noting that

$$|\Delta_k^{\mathcal{A}}(t) - \Sigma_k^{\mathcal{A}}(t)| \leqslant s = t^{2/3},$$

this together with (12) we get $\Delta_k^{\mathcal{A}}(t) = O(t^{2/3}\log t)$, that is

$$\bar{x}_k^{\mathcal{A}}(t) = tp_k + O(t^{2/3}\log t), \mathcal{A} \in \widehat{\mathcal{A}}. \qquad (13)$$

In lemma 2, if $\mathcal{A} \in \widehat{\mathcal{A}}$ then $\epsilon_{\mathcal{A}}(t) = \Omega(t)$. Take $u = t^{2/3}$ we get

$$\mathbf{Pr}(|x_k(t) - \bar{x}_k^{\mathcal{A}}(t)|\mathcal{A})| \geqslant t^{2/3}|\mathcal{A}) \leqslant \exp\{-\Omega(t^{1/3})\}.$$

Associate (10)(13) with above, with probability

$$\begin{aligned}(1-\exp\{-\Omega(t^{1/3})\})P(\widehat{\mathcal{A}}) &= (1-\exp\{-\Omega(t^{1/3})\})(1-t^{-\Omega(\log t)})\\ &= 1-t^{-\Omega(\log t)}\end{aligned}$$

we have $x_k(t) = tp_k + O(t^{2/3}\log t)$

Now we turn to the more substantial part of the result, determining the behavior of the quantities p_i defined by (1).

Proof of Theorem 3 For $i > m$, by finiteness condition $b_i = 0$, we have

$$\begin{aligned} p_i &= \frac{(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(i-1+\delta_{in})p_{i-1}}{\theta+\delta_{in}\alpha+(\alpha\bar{\xi}+(1-\alpha)\bar{\zeta})(i+\delta_{in})}\\ &= \frac{(i-1+\delta_{in})p_{i-1}}{\frac{\theta+\delta_{in}\alpha}{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}}+i+\delta_{in}}\\ &= \frac{(i-1+\delta_{in})_{i-m}}{(\frac{\theta+\delta_{in}\alpha}{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}}+i+\delta_{in})_{i-m}}p_m\\ &= \frac{\Gamma(i+\delta_{in})}{\Gamma(\frac{\theta+\delta_{in}\alpha}{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}}+i+\delta_{in}+1)}\frac{\Gamma(\frac{\theta+\delta_{in}\alpha}{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}}+m+\delta_{in}+1)}{\Gamma(m+\delta_{in})!}p_m. \end{aligned}$$

We skip some detail in the derivations, as equations such as (1) clearly have unique solutions, and it is straightforward to check that the formulae we obtain do indeed give solutions. One can check that, as expected, $\sum_{i=1}^{\infty} p_i = \alpha$; there are $(\alpha+o(1))t$ vertices at large times t. Now using the fact that $\Gamma(x) = \sqrt{2\pi}e^{-x}x^{x-1/2}(1+O(x^{-1}))$ we see that as $i \to \infty$ we have $p_i \sim C_{in}i^{-\gamma_{in}}$ with

$$\gamma_{in} = 2 + \frac{\alpha(\bar{\eta}+\delta_{in})}{\alpha\bar{\xi}+(1-\alpha)\bar{\zeta}},$$

where C_{in} is a constant.

5 Remark

In this paper, based on the result of [8,10] we introduce a general model for directed scale-free graphs that grow with preferential attachment depending on the in- and out-degrees, we show that the resulting in- and out-degree distributions are power laws with exponent $2+\alpha_1,\ 2+\alpha_2$, respectively. Noticing that substitute $\alpha+\gamma$ for α, β for $1-\alpha$ and μ_1, μ_2, μ_3 for $\frac{\alpha}{\alpha+\gamma}, \frac{\gamma}{\alpha+\gamma}, 1$, respectively, $\gamma_{in}, \gamma_{out}$ is just the form of the power law exponent in [8], it is easy to see that our result is a generalization of Theorem 1 that in [8].

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