The Exact Implied Volatility Smile for Exponential Lévy Models

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Abstract

For any exponential Lévy model whose diffusion component is nonzero, we provide an exact series representation for the implied volatility of a European call option. Numerical examples are provided.

Keywords: Implied Volatility, Exponential Lévy.

1 Introduction

Various approaches have been taken to studying the implied volatility surface induced by a given class of equity models. Most of these approaches explore asymptotic regimes of strikes and maturities (long expiries, short expiries, large strikes, small strikes, etc.) or a specific feature of the implied volatility surface, such as the at-the-money skew. An exhaustive review of the implied volatility literature would be prohibitive. But, we mention a few papers that deal with exponential Lévy processes in particular. The short maturity volatility smile is studied in Figueroa-López and Forde (2012), and the long maturity smile in Figueroa-López, Forde, and Jacquier (2011). The model-free results of Lee (2004); Gao and Lee (2011) take a particularly simple form for exponential Lévy models and, as such, are useful for studying extreme strike behavior (large and small) of implied volatility. For a review of results on asymptotics for implied volatilities in exponential Lévy models we refer the reader to Tankov (2011); Andersen and Lipton (2012).

Our approach to studying implied volatility is quite different from the above-mentioned works. Rather than exploit a particular maturity and/or strike regime, we exploit the simple structure of exponential Lévy models. In doing so, we obtain an exact formula (written as an infinite series) for the implied volatility of a given call option. As far as we are aware, this is the first time a formula for the exact implied volatility has been given in any framework – exponential Lévy or otherwise. We also mention that our formula is extremely

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simple to derive. While previous authors have used advanced mathematical techniques (e.g., saddle-point methods, moment analysis, large-deviation principle, etc.) to derive asymptotic implied volatility results, our exact result requires only basic calculus.

The rest of this paper proceeds as follows. In section 2 we introduce the class of exponential Lévy models. In section 3, we review how European options may be valued in an exponential Lévy setting using generalized Fourier transforms. Finally, in section 4 we define implied volatility and – for a given call option and exponential Lévy model – derive a formula for the corresponding implied volatility. The main result of our work is summarized in Theorem 7. Numerical examples are provided at the conclusion of this paper.

2 Exponential Lévy Models

In this section we review the class of exponential Lévy models. A detailed development can be found in Cont and Tankov (2004); Øksendal and Sulem (2005). We assume a frictionless market, no arbitrage and take an equivalent martingale measure $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$. All processes defined below live on this space. Let S represent the price process of a risky asset. The main assumptions of this paper is that S can be modeled as an exponential Lévy process

$$S_t = e^{X_t}, \qquad dX_t = \gamma\, dt + \sigma\, dW_t + \int_{|z|<R} z\, d\widetilde{N}_t(dz) + \int_{|z|\geq R} z\, dN_t(dz), \qquad X_0 = x.$$

Here, $R \in [0, \infty]$, the volatility satisfies $\sigma > 0$, W is a Brownian motion and N is a Poisson random measure characterized by Lévy measure ν

$$\mathbb{E}\, N_t(dz) = \nu(dz)\, dt, \qquad d\widetilde{N}_t(dz) = dN_t(dz) - \nu(dz)\, dt.$$

We require that ν satisfy

$$\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty, \qquad \int_{|z|\geq R} e^z \nu(dz) < \infty. \tag{1}$$

The first condition must be satisfied by all Lévy measures. The second condition guarantees that $\mathbb{E}\, S_t < \infty$ for all $t \in \mathbb{R}^+$. Valid choices for R depend on the Lévy measure ν. We can always choose $R = 1$. If $\int_{|z|\geq 1} |z|\nu(dz) < \infty$ then we may choose $R = \infty$. For simplicity, we assume S pays no dividends and the risk-free rate of interest is zero. Thus, S must be a martingale. The martingale condition is satisfied if and only if

$$\gamma = -\frac{1}{2}\sigma^2 - \int_{\mathbb{R}} \nu(dz) \left(e^z - 1 - z\, \mathbb{I}_{\{|z|<R\}}\right). \tag{2}$$

3 European Option Pricing

We consider a European option expiring at time $t > 0$ with payoff $h(X_t)$. Using risk-neutral pricing, the time-zero value of such an option is the $\mathbb{P}$-expectation of the option payoff

$$u(t,x) = \mathbb{E}_x\, h(X_t).$$

Lewis (2001); Lipton (2002) independently show that $u(t,x)$ can be computed using generalized Fourier transforms. We review their method below. For brevity, we do not include any proofs. Let $\phi(\lambda)$ denote the characteristic exponent of X

$$\phi(\lambda) := \log \mathbb{E}\, e^{i\lambda X_1}, \qquad \phi(\lambda) = i\gamma\lambda - \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} \nu(dz) \left(e^{i\lambda z} - 1 - i\lambda z\, \mathbb{I}_{\{|z|<R\}} \right).$$

We assume that ϕ is analytic in an infinite strip Λ^ϕ of the complex plane

$$\Lambda^\phi := \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \in (\lambda_-^\phi, \lambda_+^\phi)\},$$
$$\lambda_-^\phi = \inf\left\{ \lambda < 0 : \int_{-\infty}^{-1} \nu(dz) e^{\lambda z} < \infty \right\}, \qquad \lambda_+^\phi = \sup\left\{ \lambda > 1 : \int_{1}^{\infty} \nu(dz) e^{\lambda z} < \infty \right\}.$$

Let $\widehat{h}(\lambda)$ denote the generalized Fourier transform of $h(x)$

$$\widehat{h}(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx\, e^{-i\lambda x} h(x).$$

We assume $\widehat{h}(\lambda)$ is analytic on an infinite strip of the complex plane of the form $\Lambda^h := \{\lambda \in \Lambda^\phi : \mathrm{Im}(\lambda) \in (\lambda_-^h, \lambda_+^h)\}$. Let $\lambda = \lambda_r + i\lambda_i$ where $\lambda_r, \lambda_i \in \mathbb{R}$ and fix the imaginary component: $\lambda_i \in (\lambda_-^h, \lambda_+^h)$. Then the value of the option $u(t,x)$ is given by

$$u(t,x) = \int_{\mathbb{R}} d\lambda_r\, e^{t\phi(\lambda)} \widehat{h}(\lambda) \psi_\lambda(x), \qquad \psi_\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{i\lambda x}.$$

4 Implied Volatility

In this section we fix (t,x) and a call option payoff $h(x) = (e^x - e^k)^+$. Note that

$$\widehat{h}(\lambda) = \frac{-e^{k - ik\lambda}}{\sqrt{2\pi}\,(i\lambda + \lambda^2)}, \qquad \mathrm{Im}(\lambda) < -1.$$

We also fix $\sigma > 0$ and a Lévy measure $\nu = \varepsilon\, \mu$ where $\varepsilon \geq 0$ and μ is any Lévy measure that satisfies (1). By (2), the parameter γ is fixed by σ and ν. To keep track of their dependence on ε we write the characteristic exponent as $\phi(\lambda)$ and the option price $u(t,x)$ as $\phi^\varepsilon(\lambda)$ and $u^\varepsilon(t,x)$ respectively. We have

$$\begin{aligned}
\phi^\varepsilon(\lambda) &= \phi_0(\lambda) + \varepsilon\,\phi_1(\lambda),\\
\phi_0(\lambda) &= \frac{1}{2}\sigma^2(-\lambda^2 - i\lambda),\\
\phi_1(\lambda) &= -i\lambda\int_{\mathbb{R}}\mu(dz)\left(e^z - 1 - z\,\mathbb{I}_{\{|z|<R\}}\right) + \int_{\mathbb{R}}\mu(dz)\left(e^{i\lambda z} - 1 - i\lambda z\,\mathbb{I}_{\{|z|<R\}}\right),
\end{aligned}$$

and

$$u^\varepsilon = \int_{\mathbb{R}} d\lambda\, e^{t\phi^\varepsilon(\lambda)}\widehat{h}(\lambda)\psi_\lambda. \tag{3}$$

To ease notation, we have dropped the subscript r from $d\lambda_r$. The following definitions will be useful:

Definition 1. The *Black-Scholes Price* $u^{BS} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$u^{BS}(\rho) := \int d\lambda\, e^{t\phi^{BS}(\lambda;\rho)}\widehat{h}(\lambda)\psi_\lambda, \qquad \phi^{BS}(\lambda;\rho) = \frac{1}{2}\rho^2(-\lambda^2 - i\lambda).$$

Definition 2. The *Implied Volatility* is defined implicitly as the unique number $\sigma^\varepsilon \in \mathbb{R}^+$ such that

$$u^{BS}(\sigma^\varepsilon) = u^\varepsilon, \tag{4}$$

where u^ε is given by (3).

Remark 3. For $0 < t < \infty$ the existence and uniqueness of the implied volatility σ^ε can be deduced by using the general arbitrage bounds for call prices and the monotonicity of u^{BS}.

Remark 4. Note that u^{BS} is an invertible analytic function that satisfies $\partial_\rho u^{BS}(\rho) > 0$ for all $\rho > 0$. By the Lagrange inversion theorem, the inverse $[u^{BS}]^{-1}$ of such a function is also analytic.

Our goal is to find an explicit formula for the implied volatility σ^ε. To this end, we note that

$$e^{t\phi^\varepsilon(\lambda)} = e^{t(\phi_0(\lambda)+\varepsilon\phi_1(\lambda))} = e^{t\phi_0(\lambda)}\sum_{n=0}^\infty \frac{1}{n!}\left(t\,\varepsilon\,\phi_1(\lambda)\right)^n. \tag{5}$$

Inserting (5) into (3) we obtain the following series representation [1] for u^ε

$$u^\varepsilon = \sum_{n=0}^\infty \varepsilon^n u_n, \qquad u_n = \frac{t^n}{n!}\int_{\mathbb{R}} d\lambda\, e^{t\phi_0(\lambda)}\left(\phi_1(\lambda)\right)^n\widehat{h}(\lambda)\,\psi_\lambda. \tag{6}$$

Note in particular that $u_0 = u^{BS}(\sigma)$.

From (6), it is clear that u^ε is an analytic function of ε. It is a useful fact that the composition of two analytic functions is also analytic (see Brown and Churchill (1996), section 24, p. 74). Thus, in light

[1] By Fubini's Theorem, exchanging the order of summation and integration is allowed since $\int_{\mathbb{R}} d\lambda\,|e^{t\phi^\varepsilon(\lambda)}\widehat{h}(\lambda)\psi_\lambda| < \infty$.

of Remark 4, we deduce that $\sigma^\varepsilon = [u^{BS}]^{-1}(u^\varepsilon)$ is an analytic function and therefore has a power series expansion in ε. We write this expansion as follows

$$\sigma^\varepsilon = \sigma_0 + \delta^\varepsilon, \qquad \delta^\varepsilon = \sum_{k=1}^{\infty} \varepsilon^k \sigma_k. \tag{7}$$

Taylor expanding u^{BS} about the point σ_0 we have

$$\begin{aligned}
u^{BS}(\sigma^\varepsilon) &= u^{BS}(\sigma_0 + \delta^\varepsilon) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (\delta^\varepsilon \partial_\sigma)^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \varepsilon^k \sigma_k \right)^n \partial_\sigma^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^{\infty} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \varepsilon^k \right] \partial_\sigma^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sigma_k \partial_\sigma + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(\sigma_0).
\end{aligned} \tag{8}$$

Now, we insert expansions (6) and (8) into (4) and collect terms of like order in ε

$$\begin{aligned}
&\mathcal{O}(1): && u_0 = u^{BS}(\sigma_0), \\
&\mathcal{O}(\varepsilon^k): && u_k = \sigma_k \partial_\sigma u^{BS}(\sigma_0) + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(\sigma_0), && k \geq 1.
\end{aligned}$$

Solving the above equations for $\{\sigma_k\}_{k=0}^{\infty}$ we find

$$\begin{aligned}
&\mathcal{O}(1): && \sigma_0 = \sigma, \\
&\mathcal{O}(\varepsilon^k): && \sigma_k = \frac{1}{\partial_\sigma u^{BS}(\sigma)} \left(u_k - \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(\sigma) \right), && k \geq 1.
\end{aligned} \tag{9}$$

Remark 5. The right hand side of (9) involves only σ_j for $j \leq k-1$. Thus, the $\{\sigma_k\}_{k=1}^{\infty}$ can be found recursively.

Remark 6. Note that $\partial_\sigma^n u^{BS}(\sigma)$ is easily computed using

$$\partial_\sigma^n u^{BS}(\sigma) = \int d\lambda \left(\partial_\sigma^n e^{t\phi_0(\lambda)} \right) \widehat{h}(\lambda) \psi_\lambda.$$

Explicitly, up to $\mathcal{O}(\varepsilon^4)$ we have

$$\begin{aligned}
\mathcal{O}(\varepsilon): \qquad & \sigma_1 = \frac{u_1}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^2): \qquad & \sigma_2 = \frac{u_2 - \frac{1}{2!}\sigma_1^2 \partial_\sigma^2 u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^3): \qquad & \sigma_3 = \frac{u_3 - (\sigma_2 \sigma_1 \partial_\sigma^2 + \frac{1}{3!}\sigma_1^3 \partial_\sigma^3) u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^4): \qquad & \sigma_4 = \frac{u_4 - (\sigma_3 \sigma_1 \partial_\sigma^2 + \frac{1}{2}\sigma_2^2 \partial_\sigma^2 + \frac{1}{2}\sigma_2 \sigma_1^2 \partial_\sigma^3 + \frac{1}{24}\sigma_1^4 \partial_\sigma^4) u_0}{\partial_\sigma u_0}.
\end{aligned}$$

We summarize our main result in the following theorem:

Theorem 7. *The implied volatility* σ^ε *defined in* (4) *is given explicitly by* (7) *where* $\sigma_0 = \sigma$ *and* $\{\sigma_k\}_{k=1}^\infty$ *are given by* (9).

Remark 8. We emphasize: we have made no assumption about the size of ε. Theorem 7 is valid for any $\varepsilon \geq 0$. In particular, one can always choose $\varepsilon = 1$.

Remark 9. Everything we have done so far is exact. The accuracy of the implied volatility expansion (7) is limited only by the number of terms one wishes to compute.

Define the $\mathcal{O}(\varepsilon^n)$ approximation of the implied volatility

$$\sigma^{(n)} := \sum_{k=0}^{n} \varepsilon^k \sigma_k.$$

At the end of this document, we provide numerical examples illustrating convergence of $\sigma^{(n)}$ to σ^ε for three well-known exponential Lévy models:

- the Jump-diffusion model of Merton (1976): figure 1,
- the Variance Gamma model of Madan, Carr, and Chang (1998): figure 2,
- the CGMY model of Carr, Geman, Madan, and Yor (2002): figure 3.

We plot implied volatility as a function of the log-moneyness to maturity ratio, LMMR $:= (k - x)/t$. In all three models, we see excellent convergence of $\sigma^{(n)}$ to σ^ε. Convergence is fastest for values of k near x and slows as k moves away from x.

Remark 10. Although our focus has been on exponential Lévy models, the exact implied volatility expansion outlined above will work for *any* model whose European call price can be expanded analytically in ε as

$$u^\varepsilon = u^{BS} + \sum_{k=1}^{\infty} \varepsilon^k u_k,$$

where ε is some model-specific parameter. See, for example, Lorig (2012b,a).

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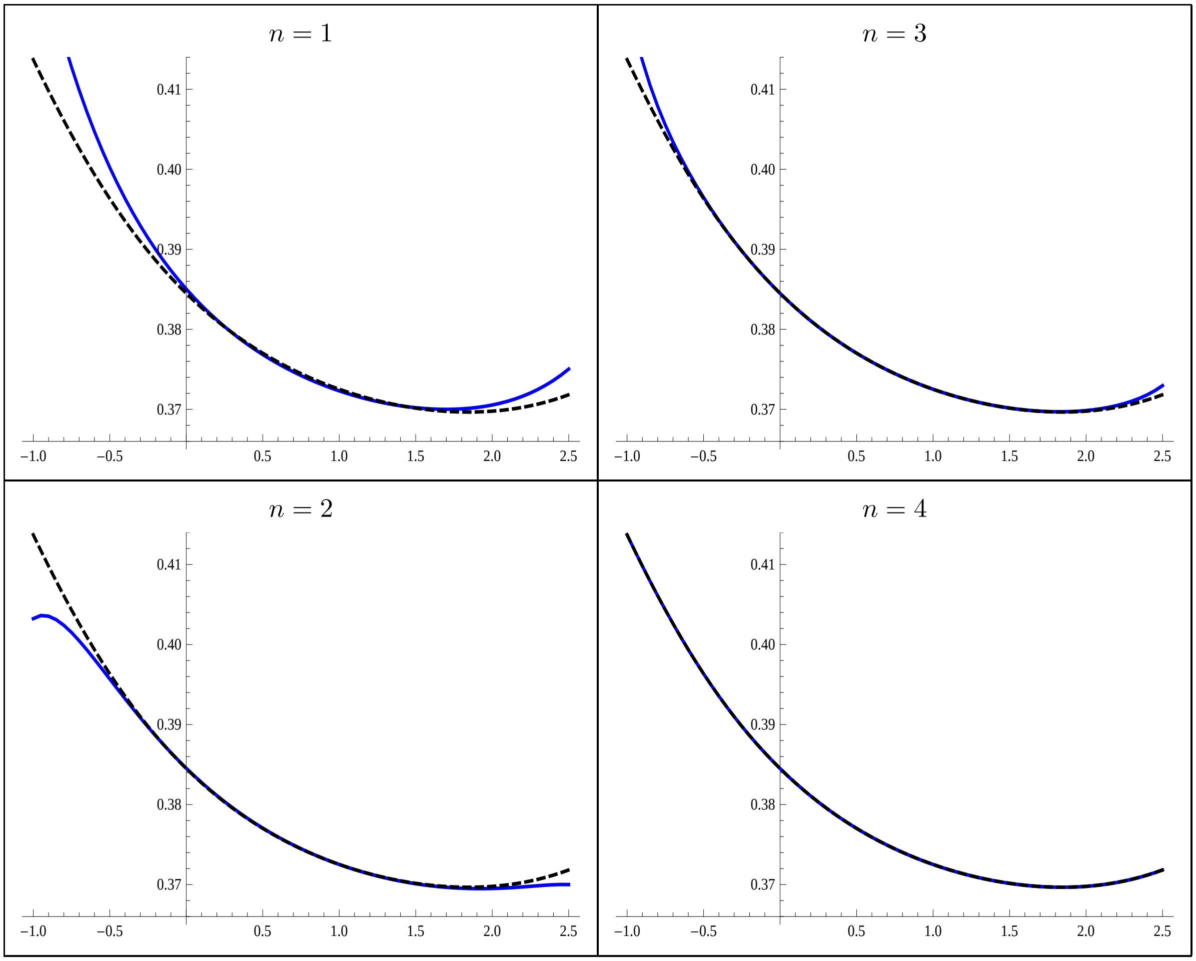

Figure 1: Using the Merton model, we plot $\sigma^{(n)}$ (solid blue) and σ^{ε} (dashed black) as a function of LMMR. The following parameters are used throughout: $s = 0.15$, $m = -0.15$, $\sigma = 0.35$, $\varepsilon = 0.75$, $t = 0.33$.

$$\text{Merton model :} \qquad \nu(dz) = \frac{\varepsilon}{\sqrt{2\pi s^2}} \exp\left(\frac{-(z-m)^2}{2s^2}\right) dz.$$

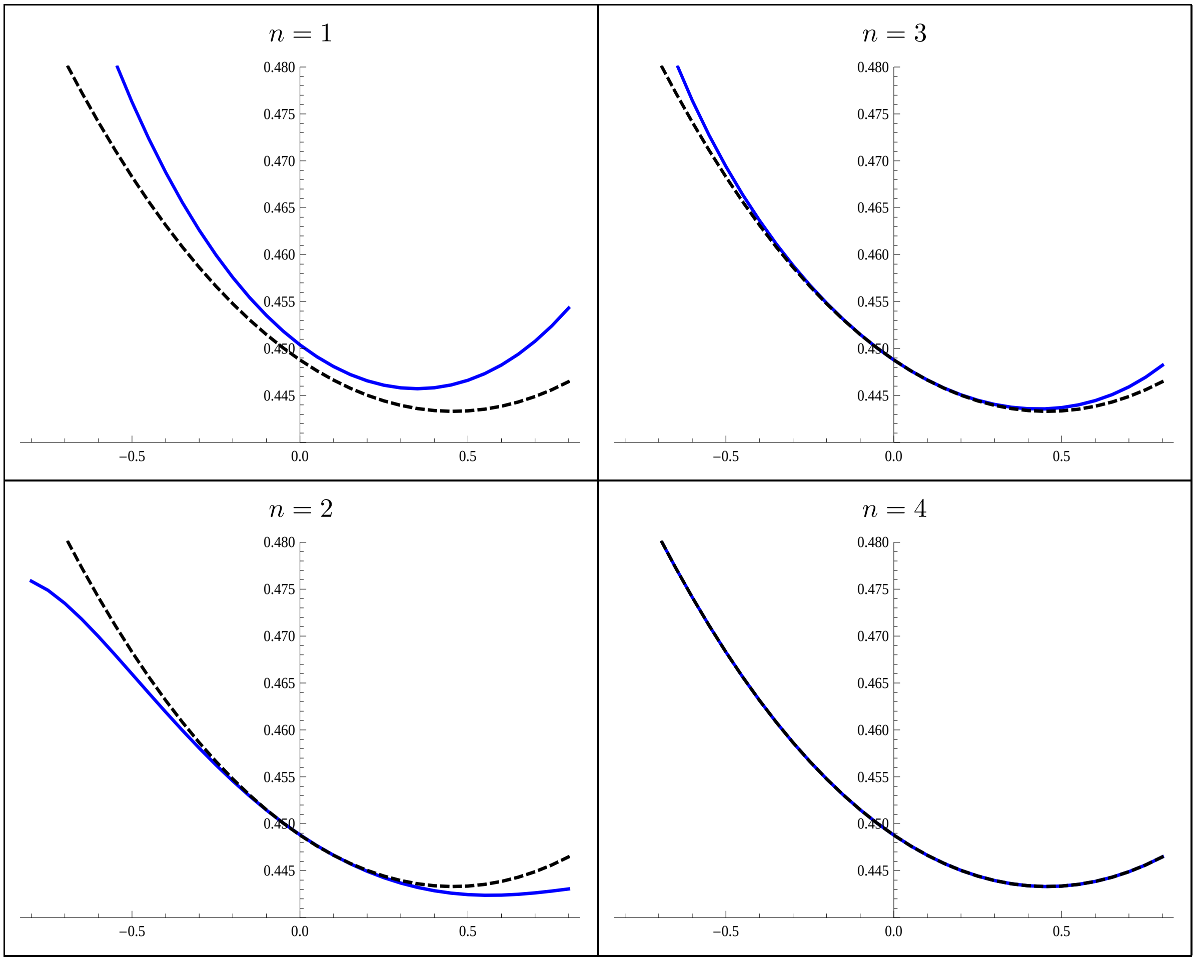

Figure 2: Using the Variance Gamma model, we plot $\sigma^{(n)}$ (solid blue) and σ^{ε} (dashed black) as a function of LMMR. The following parameters are used throughout: $G = 1.0$, $M = 3.0$, $\sigma = 0.35$, $\varepsilon = 0.3$, $t = 0.15$.

$$\text{Variance Gamma model :} \qquad \nu(dz) = \varepsilon \left(\frac{e^{Gz}}{-z} \mathbb{I}_{\{z<0\}} + \frac{e^{-Mz}}{z} \mathbb{I}_{\{z>0\}} \right) dz.$$

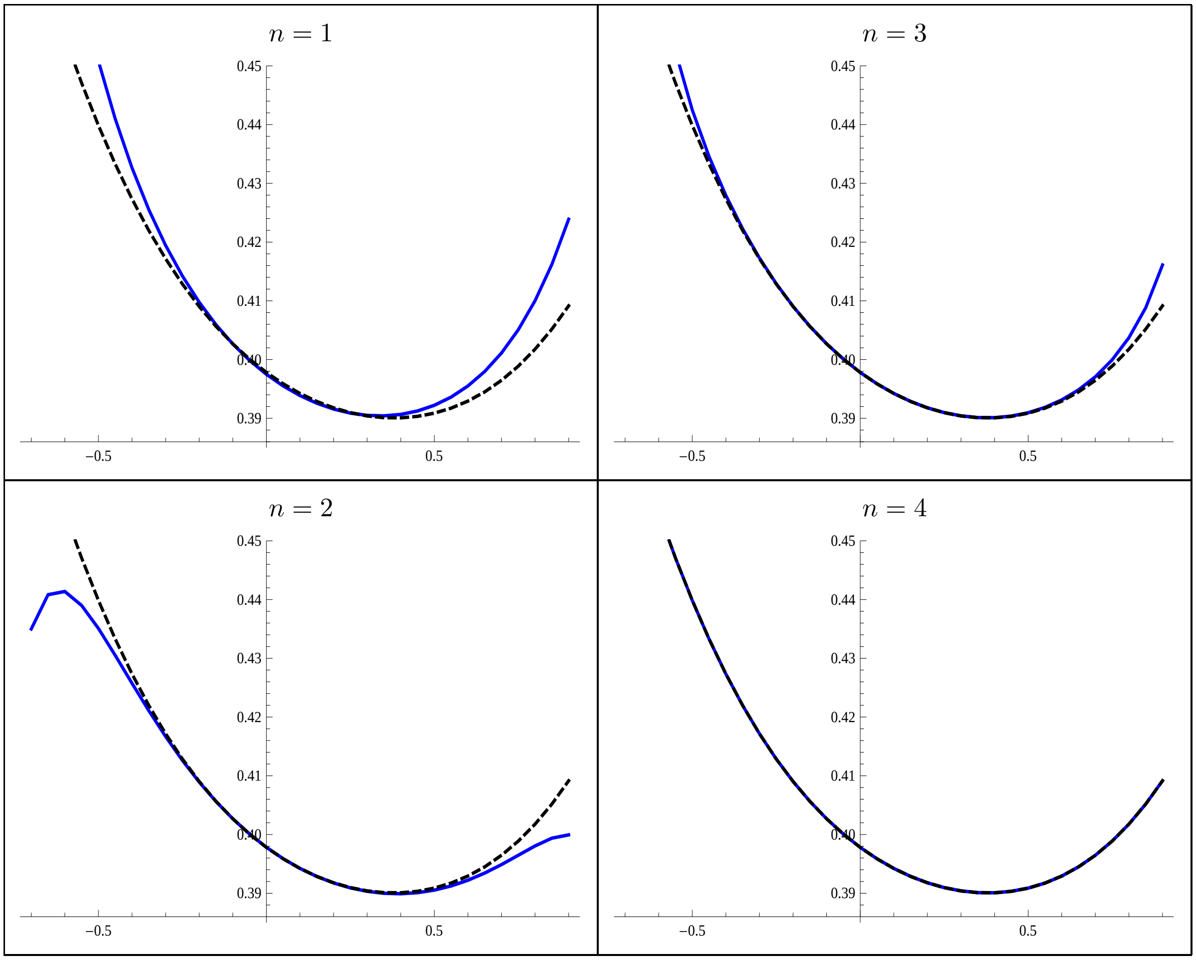

Figure 3: Using the CGMY model, we plot $\sigma^{(n)}$ (solid blue) and σ^{ε} (dashed black) as a function of LMMR. The following parameters are used throughout: $G = 2.0$, $M = 4.0$, $Y = -3.0$, $\sigma = 0.35$, $\varepsilon = 0.3$, $t = 0.5$.

$$\text{CGMY model :} \qquad \nu(dz) = \varepsilon \left(\frac{e^{Gz}}{|z|^{1+Y}} \mathbb{I}_{\{z<0\}} + \frac{e^{-Mz}}{z^{1+Y}} \mathbb{I}_{\{z>0\}} \right) dz.$$