

Robust utility maximization for Lévy processes: Penalization and solvability

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Abstract

In this paper the robust utility maximization problem for a market model based on Lévy processes is analyzed. The interplay between the form of the utility function and the penalization function required to have a well posed problem is studied, and for a large class of utility functions it is proved that the dual problem is solvable as well as the existence of optimal solutions. The class of equivalent local martingale measures is characterized in terms of the parameters of the price process, and the connection with convex risk measures is also presented.

Key words: Convex risk measures, duality, robust utility, Lévy processes.

Mathematical Subject Classification: 91G10, 60G51.

1 Introduction

The progress in portfolio optimization is operationally related to the possibility of solving a complex optimization problem in a typically infinite dimensional space, and the ability to translate the emerging problem to the available optimization methods. Convex duality, stochastic control are among the most used approaches.

The development in optimal portfolio management is conceptually determined by the form of the problems emerging from the prevailing theory of choice under uncertainty. Optimal portfolio selection corresponds in a very abstract form to choose a maximal element X with respect to a *preference order* from a class of admissible elements \mathcal{X} . In a very short form the story might be traced as follows: The axiom system proposed by von Neumann and

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Morgenstern, and Savage lead to a preference representation as the expectation of a utility function U under a fixed probability measure \mathbb{Q} . The paradigm of expected utility became one of the pillars in economics during the last century. Starting from an expected utility problem of the form

$$\mathbb{E}_{\mathbb{Q}}[U(X)] \rightarrow \max, \quad (1.1)$$

Harry Markowitz [21] derived in the early 50s, for the first time, a quantitative solution in form of his celebrated mean-variance analysis [22], and confronted the academic world with the ubiquitous trade-off between profit and risk in a financial market. It is common to refer to (1.1) as the Merton-problem, because a solution to this problem in the context of a continuous time Markovian market model was established in [24] and [25] using stochastic control methods. Harrison and Pliska accomplished in [9] and [10] the connection to stochastic calculus (initiated by Bachelier at the beginning of the last century), what led to the continuous time investment-consumption problems, widely studied in the second half of the last century.

It is merit of Pliska [27] to provide the martingale and duality approach, which is still one of the most influential ideas to solve the expected utility maximization problem. For the application of stochastic control methods in the solution of the dual problem for utility maximization with consumption see [1]. Kramkov and Schachermayer [16] and [17] studied the problem (1.1) in a very general semimartingale setting, for utility functions defined in the positive halfline. A utility function $U : (0, \infty) \rightarrow \mathbb{R}$ will be hereafter a strictly increasing, strictly concave, continuously differentiable real function, which satisfies the Inada conditions (i.e. $U'(0+) = +\infty$ and $U'(\infty-) = 0$). The log-utility $U(x) = \log(x)$ and the power utility $U(x) = \frac{1}{q}x^q$, with $q \in (-\infty, 1) \setminus \{0\}$, satisfy those properties, and are in the group of utility functions that more attention have received in the literature. In [16]-[17] the authors fixed a prior probability measure \mathbb{Q} representing the market measure, and tackle the primal problem in a dynamic setting for a fixed finite time horizon T

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \{\mathbb{E}_{\mathbb{Q}}[U(X_T)]\}, \quad (1.2)$$

over a set of admissible wealth processes $\mathcal{X}(x)$, which will be explained later. The market model should be arbitrage free in the sense that the class of *equivalent local martingale measures* $\mathcal{Q}_{elmm}(\mathbb{Q}) := \{\mathbb{Q}' \approx \mathbb{Q} : \mathcal{X}(1) \subset \mathcal{M}_{loc}(\mathbb{Q}')\}$ is not empty, where $\mathcal{M}_{loc}(\mathbb{Q}')$ denotes the set of local martingales with respect to \mathbb{Q}' . In these papers the analysis was based on the dual formulation, which basic idea is to pass to the convex conjugate V (also known as the Fenchel-Legendre transformation) of the function $-U(-x)$, defined by

$$V(y) = \sup_{x>0} \{U(x) - xy\}, \quad y > 0. \quad (1.3)$$

From the conditions imposed to the utility function U , we have that the conjugate function V is continuously differentiable, decreasing, and strictly convex, satisfying: $V'(0+) = -\infty$, $V'(\infty) = 0$, $V(0+) = U(\infty)$, $V(\infty) = U(0+)$. Further, the biconjugate of U is again U

itself; in other words the bidual relationship holds

$$U(x) = \inf_{y>0} \{V(y) + xy\}, \quad x > 0.$$

Kramkov and Schachermayer [16] formulated the dual problem in the non-robust setting in terms of the value function

$$v_{\mathbb{Q}}(y) := \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \{\mathbb{E}_{\mathbb{Q}}[V(Y_T)]\}, \quad (1.4)$$

where

$$\mathcal{Y}_{\mathbb{Q}}(y) := \{Y \geq 0 : Y_0 = y, YX \text{ } \mathbb{Q}\text{-supermartingale } \forall X \in \mathcal{X}(1)\}. \quad (1.5)$$

Observe that any $Y \in \mathcal{Y}_{\mathbb{Q}}(y)$ is a \mathbb{Q} -supermartingale, since $X \equiv 1 \in \mathcal{X}(1)$. The authors introduced also in [16] the concept of *asymptotic elasticity* $AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)}$ and proved that when the utility function U has asymptotic elasticity strictly less than one $AE(U) < 1$, then:

- (i) There is always a unique solution for $x > 0$, i.e. there exists a unique $\widehat{X} \in \mathcal{X}(x)$ such that $u_{\mathbb{Q}}(x) = \mathbb{E}_{\mathbb{Q}}[U(\widehat{X}_T)]$.
- (ii) The value function $u_{\mathbb{Q}}(x)$ is a utility function i.e. strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions ($u'(0+) = +\infty$ and $u'(\infty-) = 0$).
- (iii) The dual problem satisfies $v_{\mathbb{Q}}(y) < \infty$, $\forall y > 0$, and it can be restricted to the class of equivalent local martingale measures $\mathcal{Q}_{elmm}(\mathbb{Q})$,

$$v_{\mathbb{Q}}(y) = \inf_{\tilde{\mathbb{Q}} \in \mathcal{Q}_{elmm}(\mathbb{Q})} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y d\tilde{\mathbb{Q}}/d\mathbb{Q} \right) \right] \right\}.$$

The previous assertions (i) - (iii) hold when the classical problem (1.2) is finite for at least some $x > 0$, and the non-arbitrage condition $\mathcal{Q}_{elmm}(\mathbb{Q}) \neq \emptyset$ together with the Inada conditions for U are satisfied. Clearly, the asymptotic elasticity hypothesis involves only the utility function U and hence such condition is independent of the financial market.

In a more recent contribution, Kramkov and Schachermayer [17] proved that a necessary and sufficient condition for (i) - (iii) to hold is that the *dual function is finite*. Moreover, the authors showed that the following assertions are equivalent:

$$\begin{aligned} v_{\mathbb{Q}}(y) &< \infty, \text{ for all } y > 0, \\ \lim_{x \rightarrow \infty} \frac{u_{\mathbb{Q}}(x)}{x} &= 0, \\ \inf_{\tilde{\mathbb{Q}} \in \mathcal{Q}_{elmm}(\mathbb{Q})} \mathbb{E}_{\mathbb{Q}} \left[V \left(y d\tilde{\mathbb{Q}}/d\mathbb{Q} \right) \right] &< \infty, \text{ for all } y > 0. \end{aligned} \quad (1.6)$$

When any of these conditions is satisfied, it can be concluded that:

(iv) $u_{\mathbb{Q}}(x) < \infty$, for all $x > 0$.

(v) The primal and dual problems have optimal solutions, $\widehat{X} \in \mathcal{X}(x)$ and $\widehat{Y} \in \mathcal{Y}_{\mathbb{Q}}(y)$ respectively, and are unique. Moreover, for $y = u'_{\mathbb{Q}}(x)$ it follows that

$$U'(\widehat{X}_T(x)) = \widehat{Y}_T(y).$$

(vi) The primal and dual value functions, $u_{\mathbb{Q}}(x)$ and $v_{\mathbb{Q}}(y)$ respectively, are conjugate

$$\begin{aligned} u_{\mathbb{Q}}(x) &= \inf_{y>0} \{v_{\mathbb{Q}}(y) + xy\}, \\ v_{\mathbb{Q}}(y) &= \sup_{x>0} \{u_{\mathbb{Q}}(x) - xy\}. \end{aligned}$$

Let \mathcal{Q} be the family of probability measures on the measurable space (Ω, \mathcal{F}) . The choice of the market measure $\mathbb{Q} \in \mathcal{Q}$ (model uncertainty or ambiguity) has risen many empirical studies, and has also motivated (beside some incongruous paradox) a reexamination of the axiomatic foundations of the theory of choice under uncertainty. Gilboa and Schmeidler [6] gave a significant step in this direction, introducing the “certainty-independence” axiom, what led to robust utility functionals

$$X \longrightarrow \inf_{\mathbb{Q} \in \mathcal{Q}'} \{\mathbb{E}_{\mathbb{Q}}[U(X)]\},$$

where the set of “prior” models $\mathcal{Q}' \subset \mathcal{Q}$ is assumed to be a convex set of probability measures on the measurable space (Ω, \mathcal{F}) . For an overview and details about preference orders and its (robust) representation see Föllmer and Schied [2]. The corresponding robust utility maximization problem

$$\inf_{\mathbb{Q} \in \mathcal{Q}'} \{\mathbb{E}_{\mathbb{Q}}[U(X)]\} \rightarrow \max, \tag{1.7}$$

has being studied by several authors. See [28], [7], [31], [4] and [8] and references therein.

A natural observation is that the worst case approach in (1.7) does not discriminate among all possible models in \mathcal{Q}' , what again is reflected in inconsistencies in the axiomatic system proposed in [6]. Maccheroni, Marinacci and Rustichini [20] proposed a relaxed axiomatic system, which led to utility functionals

$$X \longrightarrow \inf_{\mathbb{Q} \in \mathcal{Q}'} \{\mathbb{E}_{\mathbb{Q}}[U(X)] + \vartheta(\mathbb{Q})\}, \tag{1.8}$$

where the penalty function ϑ assigns a weight $\vartheta(\mathbb{Q})$ to each model $\mathbb{Q} \in \mathcal{Q}'$. Such preference representations take into account both the risk preferences and model uncertainty. Schied [30] developed the corresponding dual theory for utility functions defined in the positive halfline and utility functionals of the form (1.8). The goal of the economic agent, with an initial capital $x > 0$, will be now to maximize the penalized expected utility from a terminal

wealth in the worst case model. This means that the agent seeks to solve the associated robust expected utility problem with value function

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[U(X_T)] + \vartheta(\mathbb{Q})\}, \quad (1.9)$$

where $\mathcal{Q}_{\ll}^{\vartheta} := \{\mathbb{Q} \ll \mathbb{P} : \vartheta(\mathbb{Q}) < \infty\}$ for a fixed reference measure \mathbb{P} . To guarantee that the \mathbb{Q} -expectation is well defined, we extend the operator $\mathbb{E}_{\mathbb{Q}}[U(\cdot)]$ to \mathcal{L}^0 , as in Schied [30, p. 111], by

$$\mathbb{E}_{\mathbb{Q}}[X] := \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[X \wedge n] = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[X \wedge n] \quad X \in \mathcal{L}^0(\Omega, \mathcal{F}). \quad (1.10)$$

The corresponding dual value function is defined by

$$v(y) := \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \{v_{\mathbb{Q}}(y) + \vartheta(\mathbb{Q})\}. \quad (1.11)$$

In this robust setting the necessary and sufficient condition (1.6) is transformed into

$$v_{\mathbb{Q}}(y) < \infty \quad \text{for all } \mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta} \quad \text{and } y > 0, \quad (1.12)$$

where $\mathcal{Q}_{\approx}^{\vartheta} := \{\mathbb{Q} \approx \mathbb{P} : \vartheta(\mathbb{Q}) < \infty\}$.

Remark 1.1 *When the conjugate convex function V is bounded from above it follows immediately that the penalized robust utility maximization problem (1.9) has a solution for any proper penalty function ϑ . This is the case, for instance, of the power utility function $U(x) := \frac{1}{q}x^q$, for $q \in (-\infty, 0)$, where the convex conjugate function $V(x) = \frac{1}{p}x^{-p} \leq 0$, with $p := \frac{q}{1-q}$. Moreover, condition (1.12) points out also that the existence of solution to the problem (1.9) relies on the positive part V^+ of the convex conjugate function.*

Let ϑ be a penalty function bounded from below, which corresponds to the minimal penalty function of a normalized and sensitive convex risk measure, see Section 2.3 for details and further references. Assuming condition (1.12), the following assertions hold for the robust problem (1.9).

- (vii) The robust value function $u(x)$ is strictly concave and takes only finite values.
- (viii) The “minimax property” is satisfied

$$\sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \{\mathbb{E}_{\mathbb{Q}}[U(X_T)] + \vartheta(\mathbb{Q})\} = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \sup_{X \in \mathcal{X}(x)} \{\mathbb{E}_{\mathbb{Q}}[U(X_T)] + \vartheta(\mathbb{Q})\};$$

in other words,

$$u(x) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \{u_{\mathbb{Q}}(x) + \vartheta(\mathbb{Q})\}.$$

- (ix) u and v are conjugate

$$u(x) = \inf_{y > 0} (v(y) + xy) \quad \text{and} \quad v(y) = \sup_{x > 0} (u(x) - xy).$$

- (x) v is convex, continuously differentiable, and take only finite values.
- (xi) The dual problem (1.11) has an optimal solution. That is, there exist $\mathbb{Q}^* \in \mathcal{Q}_{\ll}^\vartheta$ and $Y^* \in \mathcal{Y}_{\mathbb{Q}^*}(y)$ such that

$$\mathbb{E}_{\mathbb{Q}^*} [V(Y_T^*)] + \vartheta(\mathbb{Q}^*) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^\vartheta} \left\{ \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \{ \mathbb{E}_{\mathbb{Q}} [V(Y_T)] \} + \vartheta(\mathbb{Q}) \right\},$$

which is maximal in the sense that any other solution (\mathbb{Q}, Y) satisfies $\mathbb{Q} \ll \mathbb{Q}^*$ and $Y_T = Y_T^*$ \mathbb{Q} -a.s. .

- (xii) For each $x > 0$ there exists an optimal solution $X^* \in \mathcal{X}(x)$ to the robust problem (1.9). Furthermore, let $y > 0$, such that $v'(y) = -x$, and (\mathbb{Q}^*, Y^*) be a solution to the dual problem (1.11). Then (\mathbb{Q}^*, X^*) , with

$$X_T^* := -V'(Y_T^*),$$

is a saddlepoint for the robust problem

$$u(x) = \mathbb{E}_{\mathbb{Q}^*} [U(X_T^*)] + \vartheta(\mathbb{Q}^*) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^\vartheta} \sup_{X \in \mathcal{X}(x)} \{ \mathbb{E}_{\mathbb{Q}} [U(X_T)] + \vartheta(\mathbb{Q}) \}.$$

The outline and description of the main contributions of the paper are as follows: In Section 2 we propose the probability space on which we shall develop our work, and describe the class of absolutely continuous probabilities with respect to a reference probability measure \mathbb{P} . We also recall some fundamental facts about static convex measures of risk needed to establish the main results.

Samuelson [29] seems to be the first to propose a geometric Brownian motion as a model for the prices of the underlying assets in a market; it is often referred (wrongly) as the Black & Scholes model. This idea led to the, almost ubiquitous, exponential semimartingales models. We use one of them to introduce the market model in Section 3, which need not to have independent increments but include certain Lévy exponential models, and has been used to study some problems close to ours; see for instance [23] and [26]. We also give in this section a characterization of the equivalent local martingale measures for the proposed model. This contribution extends to our setting a result of Kunita [19] for Lévy exponential models. We finish this section introducing a family of penalties, which are minimal for the convex measures of risk generated by duality.

Once we have introduced necessary conditions for the penalization and the corresponding convex measure of risk ρ , which are relevant to develop the duality theory for the maximization of a penalized robust expected utility problem as in Schied [30], we address in Section 4 the relationship between the choice of a penalty function and the existence of a solution to the dual problem. For the power and the logarithmic utility functions we provide, in each case, thresholds for the family of penalty functions, which guarantee the existence of solutions to the optimal allocation problem. These results are the main contributions of this

work and their proof are based on Theorem 3.2 in Section 3.2. For stochastic volatility models, the robust utility maximization problem was addressed in [13] and [14] using stochastic control technics. We finish this section with a representation of the dual problem, given in Theorem 4.5, in terms of certain coefficients for an arbitrary utility function.

2 Preliminaries

Within a probability space which supports a semimartingale with the weak predictable representation property, there is a representation of the density processes of the absolutely continuous probability measures by means of two coefficients. Roughly speaking, the weak predictable representation property means that the “dimension” of the linear space of local martingales is two. Throughout these coefficients we can represent every local martingale as a combination of two components, namely an stochastic integral with respect to the continuous part of the semimartingale and an integral with respect to its compensated jump measure. This is of course the case for local martingales, and with more reason this observation about the dimensionality holds for the martingales associated with the corresponding densities processes. In this section we also review some concepts of stochastic calculus needed to understand these representation properties.

2.1 Fundamentals of Lévy and semimartingales processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that $L := \{L_t\}_{t \in \mathbb{R}_+}$ is a Lévy process for this probability space if it is an adapted càdlàg process with independent stationary increments starting at zero. The filtration considered is $\mathbb{F} := \{\mathcal{F}_t^{\mathbb{P}}(L)\}_{t \in \mathbb{R}_+}$, the completion of its natural filtration, i.e. $\mathcal{F}_t^{\mathbb{P}}(L) := \sigma\{L_s : s \leq t\} \vee \mathcal{N}$ where \mathcal{N} is the σ -algebra generated by all \mathbb{P} -null sets. The jump measure of L is denoted by $\mu : \Omega \times (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0)) \rightarrow \mathbb{N}$ where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. The dual predictable projection of this measure, also known as its Lévy system, satisfies the relation $\mu^{\mathbb{P}}(dt, dx) = dt \times \nu(dx)$, where $\nu(\cdot) := \mathbb{E}[\mu([0, 1] \times \cdot)]$ is the, so called, intensity or Lévy measure of L .

The Lévy-Itô decomposition of L is given by

$$L_t = bt + W_t + \int_{[0, t] \times \{0 < |x| \leq 1\}} x \{\mu(ds, dx) - \nu(dx) ds\} + \int_{[0, t] \times \{|x| > 1\}} x \mu(ds, dx). \quad (2.1)$$

It implies that $L^c = W$ is the Wiener process, and hence $[L^c]_t = t$, where $(\cdot)^c$ and $[\cdot]$ denote the continuous martingale part and the process of quadratic variation of any semimartingale, respectively. For the predictable quadratic variation we use the notation $\langle \cdot \rangle$.

Even though most of the paper deals with Lévy processes, we need to introduce some notation from the theory of semimartingales, and present some results needed in the next sections. Denote by \mathcal{V} the set of càdlàg, adapted processes with finite variation, and let $\mathcal{V}^+ \subset \mathcal{V}$ be the subset of non-decreasing processes in \mathcal{V} starting at zero.

Let $\mathcal{A} \subset \mathcal{V}$ be the class of processes with integrable variation, i.e. $A \in \mathcal{A}$ if and only if $\bigvee_0^\infty A \in L^1(\mathbb{P})$, where $\bigvee_0^t A$ denotes the variation of A over the finite interval $[0, t]$. The subset $\mathcal{A}^+ \subset \mathcal{A}$ represents those processes which are also increasing i.e. with non-negative right-continuous increasing trajectories. Furthermore, \mathcal{A}_{loc} (resp. \mathcal{A}_{loc}^+) is the collection of adapted processes with locally integrable variation (resp. adapted locally integrable increasing processes). For a càdlàg process X we denote by $X_- := (X_{t-})$ the left hand limit process, with $X_{0-} := X_0$ by convention, and by $\Delta X = (\Delta X_t)$ the jump process $\Delta X_t := X_t - X_{t-}$.

Given an adapted càdlàg semimartingale U , the jump measure and its dual predictable projection (or compensator) are denoted by $\mu_U([0, t] \times A) := \sum_{s \leq t} \mathbf{1}_A(\Delta U_s)$ and $\mu_U^{\mathcal{P}}$, respectively. Further, we denote by $\mathcal{P} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ the predictable σ -algebra and by $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0)$. With some abuse of notation, we write $\theta_1 \in \tilde{\mathcal{P}}$ when the function $\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \rightarrow \mathbb{R}$ is $\tilde{\mathcal{P}}$ -measurable and $\theta \in \mathcal{P}$ for predictable processes.

Let

$$\mathcal{L}(U^c) := \left\{ \theta \in \mathcal{P} : \exists \{\tau_n\}_{n \in \mathbb{N}} \text{ sequence of stopping times with } \tau_n \uparrow \infty \text{ and } \mathbb{E} \left[\int_0^{\tau_n} \theta^2 d[U^c] \right] < \infty \forall n \in \mathbb{N} \right\} \quad (2.2)$$

be the class of predictable processes $\theta \in \mathcal{P}$ integrable with respect to U^c in the sense of local martingale, and by

$$\Lambda(U^c) := \left\{ \int \theta_0 dU^c : \theta_0 \in \mathcal{L}(U^c) \right\}$$

the linear space of processes which admit a representation as the stochastic integral with respect to U^c . For an integer valued random measure $\tilde{\mu}$ we denote by $\mathcal{G}(\tilde{\mu})$ the class of $\tilde{\mathcal{P}}$ -measurable processes $\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (i) $\theta_1 \in \tilde{\mathcal{P}}$,
- (ii) $\int_{\mathbb{R}_0} |\theta_1(t, x)| \tilde{\mu}^{\mathcal{P}}(\{t\}, dx) < \infty \forall t > 0$,
- (iii) The process
$$\left\{ \sqrt{\sum_{s \leq t} \left\{ \int_{\mathbb{R}_0} \theta_1(s, x) \tilde{\mu}(\{s\}, dx) - \int_{\mathbb{R}_0} \theta_1(s, x) \tilde{\mu}^{\mathcal{P}}(\{s\}, dx) \right\}^2} \right\}_{t \in \mathbb{R}_+} \in \mathcal{A}_{loc}^+.$$

The set $\mathcal{G}(\tilde{\mu})$ represents the domain of the functional $\theta_1 \rightarrow \int \theta_1 d(\tilde{\mu} - \tilde{\mu}^{\mathcal{P}})$. We use the notation $\int \theta_1 d(\tilde{\mu} - \tilde{\mu}^{\mathcal{P}})$ to write the value of this functional in θ_1 . It is important to point out that this integral functional is not, in general, the integral with respect to the difference of two measures. But for $\theta_1 \in \tilde{\mathcal{P}}$ with $\int_{[0, t] \times \mathbb{R}_0} \theta_1 d\mu \in \mathcal{A}_{loc}$ we have $\theta_1 \in \mathcal{G}(\tilde{\mu})$ and

$$\int \theta_1(t, x) d\{\tilde{\mu} - \tilde{\mu}^{\mathcal{P}}\} = \int \theta_1(t, x) \tilde{\mu}(dt, dx) - \int \theta_1(t, x) \tilde{\mu}^{\mathcal{P}}(dt, dx).$$

For a detailed exposition on these topics see He, Wang and Yan [11] or Jacod and Shiryaev [15], which are our basic references.

In particular, for the Lévy process L with jump measure μ ,

$$\mathcal{G}(\mu) \equiv \left\{ \theta_1 \in \tilde{\mathcal{P}} : \left\{ \sqrt{\sum_{s \leq t} \{\theta_1(s, \Delta L_s)\}^2 \mathbf{1}_{\mathbb{R}_0}(\Delta L_s)} \right\}_{t \in \mathbb{R}_+} \in \mathcal{A}_{loc}^+ \right\}, \quad (2.3)$$

since $\mu^{\mathcal{P}}(\{t\} \times A) = 0$, for any Borel set A of \mathbb{R}_0 . Recall also that for an adapted process of finite variation $A \in \mathcal{V}$ we have

$$A \in \mathcal{A}_{loc} \iff \sqrt{\sum_{s \leq \cdot} (\Delta A_s)^2} \in \mathcal{A}_{loc}^+. \quad (2.4)$$

Therefore for $\theta_1 \in \mathcal{G}(\mu)$ with $\int_{[0,t] \times \mathbb{R}_0} |\theta_1| d\mu < \infty \forall t$ \mathbb{P} -a.s. it follows that $\int_{[0,t] \times \mathbb{R}_0} \theta_1 d\mu \in \mathcal{A}_{loc}$. Furthermore using a localizing argument we have for $\theta_1, \theta'_1 \in \mathcal{G}(\mu)$ with $\{\theta'_1(t, \Delta L_t)\}_t$ a locally bounded process that $\int_{[0,t] \times \mathbb{R}_0} |\theta_1 \theta'_1| d\mu \in \mathcal{A}_{loc}^+$.

We say that the semimartingale U has the *weak property of predictable representation* when

$$\mathcal{M}_{loc,0} = \Lambda(U^c) + \left\{ \int \theta_1 d(\mu_U - \mu_U^{\mathcal{P}}) : \theta_1 \in \mathcal{G}(\mu_U) \right\}, \quad (2.5)$$

where the previous sum is the linear sum of the vector spaces, and $\mathcal{M}_{loc,0}$ is the linear space of local martingales starting at zero.

The integral representation of a semimartingale U asserts that

$$U_t = U_0 + \alpha_t^U + U_t^c + \int_{[0,t] \times \{0 < |x| \leq 1\}} x \{ \mu_U(ds, dx) - \mu_U^{\mathcal{P}}(dx, ds) \} + \int_{[0,t] \times \{|x| > 1\}} x \mu_U(ds, dx), \quad (2.6)$$

where α_t^U is a predictable process with finite variation and $\alpha_0^U = 0$. Taking $\beta_t^U := [U^c]_t$ we define $(\alpha^U, \beta^U, \mu_U^{\mathcal{P}})$ as the *predictable characteristics (predictable triplet, local characteristics)* of the semimartingale U .

2.2 Density processes

Given an absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$ in a filtered probability space, where a semimartingale with the weak predictable representation property is defined, the structure of the density process has been studied extensively by several authors; see Theorem 14.41 in He, Wang and Yan [11] or Theorem III.5.19 in Jacod and Shiryaev [15].

It is well known that the Lévy-processes satisfy the weak property of predictable representation when the completed natural filtration is considered. In the following lemma we present the characterization of the density processes for the case of these processes. For Lévy processes the proof can be found in [12].

Lemma 2.1 *Given an absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$, there exist coefficients $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ such that*

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \mathcal{E}(Z^\theta)(t),$$

where

$$Z_t^\theta := \int_{]0,t]} \theta_0 dW + \int_{]0,t] \times \mathbb{R}_0} \theta_1(s, x) (\mu(ds, dx) - ds \nu(dx)), \quad (2.7)$$

and \mathcal{E} represents the Doleans-Dade exponential of a semimartingale. The coefficients θ_0 and θ_1 are unique, \mathbb{P} -a.s. and $\mu_{\mathbb{P}}^{\mathcal{P}}(ds, dx)$ -a.s., respectively.

For $\mathbb{Q} \ll \mathbb{P}$ the function $\theta_1(\omega, t, x)$ described in Lemma 2.1 determines the density of the predictable projection $\mu_{\mathbb{Q}}^{\mathbb{P}}(dt, dx)$ with respect to $\mu_{\mathbb{P}}^{\mathbb{P}}(dt, dx)$ (see He, Wang and Yan [11] or Jacod and Shiryaev [15]). More precisely, for $B \in (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0))$ we have

$$\mu_{\mathbb{Q}}^{\mathbb{P}}(\omega, B) = \int_B (1 + \theta_1(\omega, t, x)) \mu_{\mathbb{P}}^{\mathbb{P}}(dt, dx). \quad (2.8)$$

In what follows we restrict ourself to the time interval $[0, T]$, for some $T > 0$ fixed, and take $\mathcal{F} = \mathcal{F}_T$. We denote by $\mathcal{Q}_{\ll}(\mathbb{P})$ the subclass of absolutely continuous probability measure with respect to \mathbb{P} and by $\mathcal{Q}_{\approx}(\mathbb{P})$ the subclass of equivalent probability measures. The corresponding classes of density processes associated to $\mathcal{Q}_{\ll}(\mathbb{P})$ and $\mathcal{Q}_{\approx}(\mathbb{P})$ are denoted by $\mathcal{D}_{\ll}(\mathbb{P})$ and $\mathcal{D}_{\approx}(\mathbb{P})$, respectively. For instance, in the former case

$$\mathcal{D}_{\ll}(\mathbb{P}) := \left\{ D = \{D_t\}_{t \in [0, T]} : \exists \mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P}) \text{ with } D_t = \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} \right\}, \quad (2.9)$$

and the processes in this set are of the form

$$\begin{aligned} D_t = & \exp \left\{ \int_{]0, t]} \theta_0 dW + \int_{]0, t] \times \mathbb{R}_0} \theta_1(s, x) (\mu(ds, dx) - \nu(dx) ds) - \frac{1}{2} \int_{]0, t]} (\theta_0)^2 ds \right\} \times \\ & \times \exp \left\{ \int_{]0, t] \times \mathbb{R}_0} \{\ln(1 + \theta_1(s, x)) - \theta_1(s, x)\} \mu(ds, dx) \right\}, \end{aligned} \quad (2.10)$$

for $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$.

If $\int \theta_1(s, x) \mu(ds, dx) \in \mathcal{A}_{loc}(\mathbb{P})$ the previous formula can be written as

$$\begin{aligned} D_t = & \exp \left\{ \int_{]0, t]} \theta_0 dW - \frac{1}{2} \int_{]0, t]} (\theta_0(s))^2 ds + \int_{]0, t] \times \mathbb{R}_0} \ln(1 + \theta_1(s, x)) \mu(ds, dx) \right. \\ & \left. - \int_{]0, t] \times \mathbb{R}_0} \theta_1(s, x) \nu(dx) ds \right\}. \end{aligned} \quad (2.11)$$

2.3 Static measures of risk

Let $X : \Omega \rightarrow \mathbb{R}$ be a mapping from a set Ω of possible market scenarios, representing the discounted net worth of the position. Uncertainty is represented by the measurable space (Ω, \mathcal{F}) , and we denote by \mathcal{X} the linear space of bounded financial positions, including constant functions.

Definition 2.2 *The function $\rho : \mathcal{X} \rightarrow \mathbb{R}$, quantifying the risk of X , is a monetary risk measure if it satisfies the following properties:*

$$\text{Monotonicity: } \text{If } X \leq Y \text{ then } \rho(X) \geq \rho(Y) \quad \forall X, Y \in \mathcal{X}. \quad (2.12)$$

$$\text{Translation Invariance: } \rho(X + a) = \rho(X) - a \quad \forall a \in \mathbb{R} \quad \forall X \in \mathcal{X}. \quad (2.13)$$

When this function satisfies also the convexity property

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y) \quad \forall \lambda \in [0, 1] \quad \forall X, Y \in \mathcal{X}, \quad (2.14)$$

it is said that ρ is a convex risk measure.

We say that a set function $\mathbb{Q} : \mathcal{F} \rightarrow [0, 1]$ is a *probability content* if it is finite additive and $\mathbb{Q}(\Omega) = 1$. The set of *probability contents* on this measurable space is denoted by \mathcal{Q}_{cont} . From the general theory of static convex risk measures, we know that any map $\psi : \mathcal{Q}_{cont} \rightarrow \mathbb{R} \cup \{+\infty\}$, with $\inf_{\mathbb{Q} \in \mathcal{Q}_{cont}} \psi(\mathbb{Q}) \in \mathbb{R}$, induces a static convex measure of risk as a mapping $\rho : \mathfrak{M}_b \rightarrow \mathbb{R}$ given by

$$\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{cont}} \{ \mathbb{E}_{\mathbb{Q}}[-X] - \psi(\mathbb{Q}) \}. \quad (2.15)$$

Here \mathfrak{M} denotes the class of measurable functions and \mathfrak{M}_b the subclass of bounded measurable functions. Föllmer and Schied [3, Theorem 3.2] and Frittelli and Rosazza Gianin [5, Corollary 7] proved that any convex risk measure is essentially of this form.

More precisely, a convex measure of risk ρ on the space of bounded functions $\mathfrak{M}_b(\Omega, \mathcal{F})$ has the representation

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_{cont}} \{ \mathbb{E}_{\mathbb{Q}}[-X] - \psi_{\rho}^*(\mathbb{Q}) \}, \quad (2.16)$$

where

$$\psi_{\rho}^*(\mathbb{Q}) := \sup_{X \in \mathcal{A}_{\rho}} \mathbb{E}_{\mathbb{Q}}[-X], \quad (2.17)$$

and $\mathcal{A}_{\rho} := \{X \in \mathfrak{M}_b : \rho(X) \leq 0\}$ is the *acceptance set* of ρ .

The penalty ψ_{ρ}^* is called the *minimal penalty function* associated to ρ because, for any other penalty function ψ fulfilling (2.16), $\psi(\mathbb{Q}) \geq \psi_{\rho}^*(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{Q}_{cont}$. Furthermore, for the minimal penalty function, the next biduality relation is satisfied

$$\psi_{\rho}^*(\mathbb{Q}) = \sup_{X \in \mathfrak{M}_b(\Omega, \mathcal{F})} \{ \mathbb{E}_{\mathbb{Q}}[-X] - \rho(X) \}, \quad \forall \mathbb{Q} \in \mathcal{Q}_{cont}. \quad (2.18)$$

Remark 2.3 Among the measures of risk, the class of them that are concentrated on the set of probability measures $\mathcal{Q} \subset \mathcal{Q}_{cont}$ are of special interest. Recall that a function $I : E \subset \mathbb{R}^{\Omega} \rightarrow \mathbb{R}$ is sequentially continuous from below (above) when $\{X_n\}_{n \in \mathbb{N}} \uparrow X \Rightarrow \lim_{n \rightarrow \infty} I(X_n) = I(X)$ (respectively $\{X_n\}_{n \in \mathbb{N}} \downarrow X \Rightarrow \lim_{n \rightarrow \infty} I(X_n) = I(X)$). Föllmer and Schied [2] proved that any sequentially continuous from below convex measure of risk is concentrated on the set \mathcal{Q} . Later, Krätschmer [18, Prop. 3 p. 601] established that the sequential continuity from below is not only a sufficient but also a necessary condition in order to have a representation, by means of the minimal penalty function in terms of probability measures.

3 The market model

In this section, we introduce the market model considered in this paper. It is based on the generalization of the classical geometric Brownian setting, but in this case the coefficients are not constant and jumps are included in the model through an exogenous stochastic process. One of the most debatable feature about an stochastic process used for modelling stock market prices is the issue about the independent increments. A remarkable property of the proposed model is the fact that it need not to have independent increments. Also, it includes a certain subclass of exponential Lévy models. This section is concluded with a characterization of the set of equivalent local martingale measures.

3.1 General description and martingale measures

First, consider the stochastic process Y_t with dynamics given by

$$Y_t := \int_{]0,t]} \alpha_s ds + \int_{]0,t]} \beta_s dW_s + \int_{]0,t] \times \mathbb{R}_0} \gamma(s, x) (\mu(ds, dx) - \nu(dx) ds), \quad (3.1)$$

where the processes α, β are càdlàg, with $\beta \in \mathcal{L}(W)$ and $\gamma \in \mathcal{G}(\mu)$. Throughout we assume that the coefficients α, β and γ fulfill the following conditions:

$$(A 1) \quad \int_{]0,t]} (\alpha_s)^2 ds < \infty \quad \forall t \in \mathbb{R}_+ \quad \mathbb{P}\text{-a.s. .}$$

$$(A 2) \quad 0 < c \leq |\beta_t| \quad \forall t \in \mathbb{R}_+ \quad \mathbb{P}\text{-a.s. .}$$

$$(A 3) \quad \int_0^T \left(\frac{\alpha_u}{\beta_u} \right)^2 du \in \mathcal{L}^\infty(\mathbb{P}).$$

$$(A 4) \quad \gamma(t, \Delta L_t) \times \mathbf{1}_{\mathbb{R}_0}(\Delta L_t) \geq -1 \quad \forall t \in \mathbb{R}_+ \quad \mathbb{P}\text{-a.s. .}$$

$$(A 5) \quad \{\gamma(t, \Delta L_t) \mathbf{1}_{\mathbb{R}_0}(\Delta L_t)\}_{t \in \mathbb{R}_+} \text{ is a locally bounded process.}$$

The market model consists of two assets, one of them is the numéraire, having a strictly positive price. The dynamics of the other risky asset will be modeled as a function of the process Y_t defined above. More specifically, since we are interested in the problem of robust utility maximization, the discounted capital process can be written in terms of the wealth invested in this asset, and hence the problem can be written using only the dynamics of the discounted price of this asset. For this reason, throughout we will be concentrated in the dynamics of this price.

The dynamics of the discounted price process S are determined by the process Y as its Doleans-Dade exponential

$$S_t = S_0 \mathcal{E}(Y_t). \quad (3.2)$$

Condition (A 4) ensures that the price process is non-negative. This process is an exponential semimartingale, as it would be the case of an arbitrary semimartingale Y , if and only if the following two conditions are fulfilled:

$$\begin{aligned} (i) \quad & S = S \mathbf{1}_{[0, \tau]}, \text{ for } \tau := \inf \{t > 0 : S_t = 0 \text{ or } S_{t-} = 0\}. \\ (ii) \quad & \frac{1}{S_{t-}} \mathbf{1}_{[S_{t-} \neq 0]} \text{ is integrable w.r.t. } S. \end{aligned} \tag{3.3}$$

The first property is conceptually very appropriate when we are interested in modelling the dynamics of a price process. Recall that a stochastically continuous semimartingale has independent increments if and only if its predictable triplet is non-random. Therefore, in general, the price process S is not a Lévy exponential model, because $[Y^c]_t = \int_0^t (\beta_u)^2 du$ need not to be deterministic. However, observe that the price dynamics (3.2) includes Lévy exponential models, for Lévy processes with $\Delta L_t \geq -1$.

For the model (3.2) the price process can be written explicitly as

$$\begin{aligned} S_t = & S_0 \exp \left\{ \int_{]0, t]} \alpha_s ds + \int_{]0, t]} \beta_s dW_s + \int_{]0, t] \times \mathbb{R}_0} \gamma(s, x) (\mu(ds, dx) - \nu(dx) ds) - \frac{1}{2} \int_{]0, t]} (\beta_s)^2 ds \right\} \\ & \times \exp \left\{ \int_{]0, t] \times \mathbb{R}_0} \{\ln(1 + \gamma(s, x)) - \gamma(s, x)\} \mu(ds, dx) \right\}. \end{aligned} \tag{3.4}$$

Observe that (A 5) is a necessary and sufficient condition for S to be a locally bounded process.

The predictable càdlàg process $\{\pi_t\}_{t \in \mathbb{R}_+}$, satisfying the integrability condition $\int_0^t (\pi_s)^2 ds < \infty$ \mathbb{P} -a.s. for all $t \in \mathbb{R}_+$, shall denote the proportion of wealth at time t invested in the risky asset S . For an initial capital x , the discounted wealth $X_t^{x, \pi}$ associated with a self-financing investment strategy (x, π) fulfills the equation

$$X_t^{x, \pi} = x + \int_0^t \frac{X_{u-}^{x, \pi} \pi_u}{S_{u-}} \mathbf{1}_{[S_{u-} \neq 0]} dS_u. \tag{3.5}$$

We say that a self-financing strategy (x, π) is *admissible* if the wealth process satisfies $X_t^{x, \pi} > 0$ for all $t > 0$. The class of admissible wealth processes with initial wealth less than or equal to x is denoted by $\mathcal{X}(x)$.

Next result characterizes the class of *equivalent local martingale measures* defined as

$$\mathcal{Q}_{elmm} := \{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : \mathcal{X}(1) \subset \mathcal{M}_{loc}(\mathbb{Q})\} = \{\mathbb{Q} \in \mathcal{Q}_{\approx}(\mathbb{P}) : S \in \mathcal{M}_{loc}(\mathbb{Q})\}. \tag{3.6}$$

The class of density processes associated with \mathcal{Q}_{elmm} is denoted by $\mathcal{D}_{elmm}(\mathbb{P})$. Kunita [19] gave conditions on the parameters (θ_0, θ_1) of a measure $\mathbb{Q} \in \mathcal{Q}_{\approx}$ in order that it is a local martingale measure for a Lévy exponential model i.e. when $S = \mathcal{E}(L)$. Observe that in this case $\mathcal{Q}_{elmm}(S) = \mathcal{Q}_{elmm}(L)$. Next proposition extends those results, giving conditions on the parameters (θ_0, θ_1) under which an equivalent measure is a local martingale measure for the price model (3.2).

Proposition 3.1 *Given $\mathbb{Q} \in \mathcal{Q}_{\approx}$, let $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ be the corresponding processes describing the density processes found in Lemma 2.1. Then, the following equivalence holds:*

$$\mathbb{Q} \in \mathcal{Q}_{elmm} \iff \alpha_t + \beta_t \theta_0(t) + \int_{\mathbb{R}_0} \gamma(t, x) \theta_1(t, x) \nu(dx) = 0 \quad \forall t \geq 0 \quad \mathbb{P}\text{-a.s.} \quad (3.7)$$

Proof. Let $\mathbb{Q} \in \mathcal{Q}_{\approx}$ be an equivalent probability measure with density process given by $D_t := \mathbb{E}[d\mathbb{Q}/d\mathbb{P} | \mathcal{F}_t] = \mathcal{E}(Z^\theta)_t$, where the last equality follows from Lemma 2.1. Then, we have that

$$S \in \mathcal{M}_{loc}^1(\mathbb{Q}) \iff SD \in \mathcal{M}_{loc}^1(\mathbb{P}).$$

Since $\theta_1, \gamma \in \mathcal{G}(\mu)$, from (A 5) the process $\{\gamma(t, \Delta L_t) \mathbf{1}_{\mathbb{R}_0}(\Delta L_t)\}_{t \in \mathbb{R}_+}$ is a locally bounded process, we have that $\int \gamma \theta_1 d\mu \in \mathcal{A}_{loc}$, which yields that $\gamma \theta_1 \in \mathcal{G}(\mu)$ and

$$\int \gamma \theta_1 d\{\mu - \mu^{\mathcal{P}}\} = \int \gamma \theta_1 d\mu - \int \gamma \theta_1 d\mu^{\mathcal{P}}.$$

Therefore,

$$[Y, Z^\theta]_t = \int_0^t \beta_s \theta_0 ds + \int_{]0, t] \times \mathbb{R}_0} \gamma \theta_1 d\{\mu - \mu^{\mathcal{P}}\} + \int_{]0, t] \times \mathbb{R}_0} \gamma \theta_1 d\mu^{\mathcal{P}}.$$

Now, we write

$$S_t D_t = S_0 \mathcal{E}(Y)_t \mathcal{E}(Z^\theta)_t = S_0 \mathcal{E}(Y + Z^\theta + [Y, Z^\theta])_t,$$

and making some rearrangements we have that

$$\begin{aligned} & S_t D_t \\ &= S_0 + \int S_{u-} D_{u-} d\{Y + Z^\theta + [Y, Z^\theta]\}_u \\ &= S_0 + \int S_{u-} D_{u-} d\left\{ \int (\beta + \theta_0) dW + \int (\gamma + \theta_1 + \gamma \theta_1) d\{\mu - \mu^{\mathcal{P}}\} \right\}_u \\ &+ \int S_{u-} D_{u-} d\left\{ \int \left(\alpha_s + \beta_s \theta_0(s) + \int \gamma \theta_1 \nu(dx) \right) ds \right\}_u. \end{aligned}$$

On the other hand, observe that

$$\int S_{u-} D_{u-} d\left\{ \int (\beta + \theta_0) dW + \int (\gamma + \theta_1 + \gamma \theta_1) d\{\mu - \mu^{\mathcal{P}}\} \right\}_u$$

belongs to the set of local martingales \mathcal{M}_{loc} , and

$$\int S_{u-} D_{u-} d\left\{ \int \left(\alpha_s + \beta_s \theta_0(s) + \int \gamma \theta_1 \nu(dx) \right) ds \right\}_u$$

is a finite variation continuous process in \mathcal{V}^c . To verify this claim, observe first that (A 1) implies that $\int_0^t \alpha_s ds \in \mathcal{V}$. Further, for $\beta_s, \theta_0 \in \mathcal{L}(W)$ we know that $\int_{[0,t]} \{\beta_s\}^2 ds < \infty$ \mathbb{P} -a.s., and $\int_{[0,t]} \{\theta_0(s)\}^2 ds < \infty$ \mathbb{P} -a.s., and from the Rogers-Hölder inequality

$$\int_0^t |\beta_s| |\theta_0(s)| ds \leq \left(\int_0^t (\beta_s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^t (\theta_0(s))^2 ds \right)^{\frac{1}{2}} < \infty.$$

Then, $\int_0^t \beta_s \theta_0 ds$ is of finite variation due to the absolutely integrability of the integrand, i.e. $\int_0^t \beta_s \theta_0 ds \in \mathcal{V}$. Since $\int \gamma(s, x) \theta_1(s, x) \mu(ds, dx) \in \mathcal{A}_{loc}$, it follows that

$$\int_{[0,t] \times \mathbb{R}_0} \gamma(s, x) \theta_1(s, x) \nu(dx) ds \in \mathcal{V} \quad \mathbb{P} - a.s. \forall t \in \mathbb{R}_+.$$

Summarizing,

$$\int_0^t \alpha_s ds + \int_0^t \beta_s \theta_0 ds + \int_{]0,t] \times \mathbb{R}_0} \gamma \theta_1 \nu(dx) ds \in \mathcal{V}.$$

The equivalence (3.7) follows now observing that a predictable local martingale with locally integrable variation is constant. ■

3.2 Minimal penalties

Now, we shall introduce a family of penalty functions for the density processes described in Section 2.2, for the absolutely continuous measures $\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})$.

Let h_0 and h_1 be \mathbb{R}_+ -valued convex functions defined in \mathbb{R} with $h_0(0) = 0 = h_1(0)$, and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be increasing convex function continuous at zero with $h(0) = 0$. Define the penalty function

$$\vartheta(\mathbb{Q}) := \mathbb{E}_{\mathbb{Q}} \left[\int_0^T h \left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} \delta(t, x) h_1(\theta_1(t, x)) \nu(dx) \right) dt \right] \mathbf{1}_{\mathcal{Q}_{\ll}}(\mathbb{Q}) + \infty \times \mathbf{1}_{\mathcal{Q}_{cont} \setminus \mathcal{Q}_{\ll}}(\mathbb{Q}), \quad (3.8)$$

where θ_0, θ_1 are the processes associated to \mathbb{Q} from Lemma 2.1 and $\delta(t, x) : \mathbb{R}_+ \times \mathbb{R}_0 \rightarrow \mathbb{R}_+$ is an arbitrary but fix nonnegative function $\delta(t, x) \in \mathcal{G}(\mu)$. Further, define the convex measure of risk

$$\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})} \{ \mathbb{E}_{\mathbb{Q}}[-X] - \vartheta(\mathbb{Q}) \}. \quad (3.9)$$

Notice that ρ is a normalized and sensitive measure of risk. Next theorem establishes the minimality of the penalty function introduced above for the risk measure ρ . The proof can be found in [12].

Theorem 3.2 *The penalty function ϑ defined in (3.8) is the minimal penalty function of the convex risk measure ρ given by (3.9).*

4 Robust utility maximization

In this section the connection between penalty functions and the existence of solutions to the penalized robust expected utility problem is established. We also formulate the dual problem in terms of control processes for an arbitrary utility function.

4.1 Penalties and solvability

Let us now introduce the class

$$\mathcal{C} := \left\{ \mathcal{E}(Z^\xi) : \begin{array}{l} \xi := (\xi^{(0)}, \xi^{(1)}), \xi^{(0)} \in \mathcal{L}(W), \xi^{(1)} \in \mathcal{G}(\mu), \text{ with} \\ \alpha_t + \beta_t \xi_t^{(0)} + \int_{\mathbb{R}_0} \gamma(t, x) \xi^{(1)}(t, x) \nu(dx) = 0 \text{ Lebesgue } \forall t \end{array} \right\}, \quad (4.1)$$

with Z^ξ as in (2.7). Observe that $\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1)$; see (1.5) for the definition of $\mathcal{Y}_{\mathbb{P}}(1)$. It should be pointed out that this relation between these three sets plays a crucial role in the formulation of the dual problem, even in the non-robust case.

Theorem 4.1 *For $q \in (-\infty, 1) \setminus \{0\}$, let $U(x) := \frac{1}{q}x^q$ be the power utility function, and consider the functions h, h_0 and h_1 as in Subsection 3.2, satisfying the following conditions:*

$$\begin{aligned} h(x) &\geq \exp(\kappa_1 x^2) - 1 \text{ where } \kappa_1 := 1 \vee 2(2p^2 + p)T \text{ and } p := \frac{q}{1-q}, \\ h_0(x) &\geq |x|, \\ h_1(x) &\geq \frac{|x|}{c}, \text{ for } c \text{ as in assumption (A 2)}. \end{aligned}$$

Then, for the penalty function

$$\vartheta_{x^q}(\mathbb{Q}) := \mathbb{E}_{\mathbb{Q}} \left[\int_0^T h \left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} |\gamma(t, x)| h_1(\theta_1(t, x)) \nu(dx) \right) dt \right],$$

the penalized robust utility maximization problem (1.9) has a solution.

Proof. The penalty function ϑ_{x^q} is bounded from below, and by Theorem 3.2 it is the minimal penalty function of the normalized and sensitive convex measure of risk defined in (2.15). Therefore, we only need to prove that condition (1.12) holds. In order to prove that, fix an arbitrary probability measure $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{x^q}} = \{\mathbb{Q} \approx \mathbb{P} : \vartheta_{x^q}(\mathbb{Q}) < \infty\}$ and let $\theta = (\theta_0, \theta_1)$ be the corresponding coefficients obtained in Lemma 2.1.

(1) In Lemma 4.2, Schied [30] establishes that even for $\mathbb{Q} \in \mathcal{Q}_{\ll}$, with density process D , the next equivalence holds

$$Y \in \mathcal{Y}_{\mathbb{Q}}(y) \Leftrightarrow YD \in \mathcal{Y}_{\mathbb{P}}(y).$$

Therefore, for $\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta_{x^q}}$, with coefficient $\theta = (\theta_0, \theta_1)$, it follows that

$$\begin{aligned} v_{\mathbb{Q}}(y) &\equiv \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \{ \mathbb{E}_{\mathbb{Q}} [V(Y_T)] \} \\ &= \inf_{Y \in \mathcal{Y}_{\mathbb{P}}(y)} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{Y_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} \leq \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^\xi)_T}{\mathcal{E}(Z^\theta)_T} \right) \right] \right\}. \end{aligned}$$

(2) Define

$$\varepsilon_t := \alpha_t + \beta_t \theta_0(t) + \int_{\mathbb{R}_0} \gamma(t, x) \theta_1(t, x) \nu(dx),$$

the process involved in the definition of the class \mathcal{C} in (4.1).

When ε_t is identically zero for all $t > 0$, Proposition 3.1 implies that $\mathbb{Q} \in \mathcal{Q}_{elmm}$. However, for $\mathbb{Q} \in \mathcal{Q}_{elmm}$ the constant process $Y \equiv y$ belongs to $\mathcal{Y}_{\mathbb{Q}}(y)$, and it follows that $v_{\mathbb{Q}}(y) < \infty$, for all $y > 0$. In this case the proof is concluded.

If ε is not identically zero, consider $\xi_t^{(0)} := \theta_0(t) - \frac{\varepsilon_t}{\beta_t}$ and $\xi^{(1)} := \theta_1$. Since

$$\infty > \vartheta_{x^q}(\mathbb{Q}) \geq \mathbb{E}_{\mathbb{Q}} \left[\int_0^T \left(\frac{1}{\beta_t} \int_{\mathbb{R}_0} \gamma(t, x) \theta_1(t, x) \nu(dx) \right)^2 dt \right] - T,$$

it follows that $\left\{ \frac{1}{\beta_t} \int_{\mathbb{R}_0} \gamma(t, x) \theta_1(t, x) \nu(dx) \right\}_{t \in [0, T]} \in \mathcal{L}(W')$ for W' a \mathbb{Q} -Wiener process and thus also $\xi^{(0)} \in \mathcal{L}(W')$. Moreover, for $\xi = (\xi^{(0)}, \xi^{(1)})$ we have that $\mathcal{E}(Z^\xi) \in \mathcal{C}$.

Using Girsanov's theorem, we obtain $\frac{\mathcal{E}(Z^\xi)_t}{\mathcal{E}(Z^\theta)_t} = \exp \left\{ \int_{[0, t]} \left(-\frac{\varepsilon_u}{\beta_u} \right) dW'_u - \frac{1}{2} \int_{[0, t]} \left(\frac{\varepsilon_u}{\beta_u} \right)^2 du \right\}$.

(3) The Cauchy-Bunyakovsky-Schwarz inequality yields

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^\xi)_T}{\mathcal{E}(Z^\theta)_T} \right) \right] &= \frac{1}{p} y^{-p} \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ p \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right) dW' + \frac{p}{2} \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right)^2 dt \right\} \right] \\ &\leq \frac{1}{p} y^{-p} \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ 2p \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right) dW' - \frac{4p^2}{2} \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right)^2 dt \right\} \right]^{\frac{1}{2}} \\ &\quad \times \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ \left(\frac{4p^2}{2} + p \right) \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right)^2 dt \right\} \right]^{\frac{1}{2}}. \end{aligned} \quad (4.2)$$

On the other hand, the process

$$\exp \left\{ 2p \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right) dW' - \frac{4p^2}{2} \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right)^2 dt \right\} \in \mathcal{M}_{loc}(\mathbb{Q})$$

is a local \mathbb{Q} -martingale and, since it is positive, is a supermartingale. Hence,

$$\mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ 2p \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right) dW' - \frac{4p^2}{2} \int_{[0, T]} \left(\frac{\varepsilon_t}{\beta_t} \right)^2 dt \right\} \right] \leq 1.$$

Finally, observe that for $\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta_{x^q}}$, using that it has finite penalization $\vartheta_{x^q}(\mathbb{Q}) < \infty$ and Jensen's inequality, we have

$$\begin{aligned} \infty &> \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ \frac{\kappa_1}{T} \int_0^T \left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} |\gamma(t, x)| h_1(\theta_1(t, x)) \nu(dx) \right)^2 dt \right\} \right] \\ &\geq \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ 2(2p^2 + p) \int_0^T \left(|\theta_0(t)| + \frac{1}{|\beta_t|} \left| \int_{\mathbb{R}_0} \gamma(t, x) \theta_1(t, x) \nu(dx) \right| \right)^2 dt \right\} \right]. \end{aligned}$$

From the last two displays it follows that the r.h.s. of (4.2) is finite and the theorem follows. ■

Next theorem establishes a sufficient condition for the existence of a solution to the robust utility maximization problem (1.9) for an arbitrary utility function.

Theorem 4.2 *Suppose that the utility function \tilde{U} is bounded above by a power utility U , with penalty function ϑ_{x^q} associated to U as in Theorem 4.1. Then, the robust utility maximization problem (1.9) for \tilde{U} with penalty ϑ_{x^q} has an optimal solution.*

Proof. Since $U(x) := \frac{1}{q}x^{-q} \geq \tilde{U}(x)$ for all $x > 0$, for some $q \in (-\infty, 1) \setminus \{0\}$ the corresponding convex conjugate functions satisfy $V(y) \geq \tilde{V}(y)$ for each $y > 0$. As it was pointed out in Remark 1.1, we can restrict ourself to the positive part $\tilde{V}^+(y)$. From Proposition 4.1, we can fix some $Y \in \mathcal{Y}_{\mathbb{Q}}(y)$ such that $\mathbb{E}_{\mathbb{Q}}[V(Y_T)] < \infty$ for any $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{x^q}}$ and $y > 0$, arbitrary, but fixed. Furthermore, the inequality $V(y) \geq \tilde{V}(y)$ implies that their inverse functions satisfy $(V^+)^{(-1)}(n) \geq (\tilde{V}^+)^{(-1)}(n)$ for all $n \in \mathbb{N}$, and hence

$$\sum_{n=1}^{\infty} \mathbb{Q} \left[Y_T \leq (\tilde{V}^+)^{(-1)}(n) \right] \leq \sum_{n=1}^{\infty} \mathbb{Q} \left[Y_T \leq (V^+)^{(-1)}(n) \right] < \infty.$$

The moments Lemma ($\mathbb{E}_{\mathbb{Q}}[X] < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mathbb{Q}[|X| \geq n] < \infty$) yields $\mathbb{E}_{\mathbb{Q}}[\tilde{V}^+(Y_T)] < \infty$, and the assertion follows. ■

Example 4.3 *The logarithm utility function satisfies conditions of Theorem 4.2. However, this case will be studied more deeply in Section 4.2, since the techniques involve interesting arguments related to the relative entropy.*

From the proof of Theorem 4.2 it is clear that the behavior of the convex conjugate function in a neighborhood of zero is fundamental. From this observation we conclude the following.

Corollary 4.4 *Let U be a utility function with convex conjugate V , and ϑ a penalization function such that the robust utility maximization problem (1.9) has a solution. For a utility function \tilde{U} such that their convex conjugate function \tilde{V} is majorized in an ε -neighborhood of zero by V , the corresponding utility maximization problem (1.9) has a solution.*

Next we give an alternative representation of the robust dual value function, introduced in (1.11), in terms of the family \mathcal{C} of stochastic processes.

Theorem 4.5 *For a utility function U satisfying condition (1.12), the dual value function can be written as*

$$\begin{aligned} v(y) &= \inf_{\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta}} \left\{ \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\} \\ &= \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \left\{ \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\}. \end{aligned} \quad (4.3)$$

Proof. Condition (1.12), together with Lemma 4.4 in [30] and Theorem 2 in [17], imply the following identity

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta}} \left\{ \inf_{\tilde{\mathbb{Q}} \in \mathcal{D}_{elmm}(\mathbb{Q})} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y d\tilde{\mathbb{Q}}/d\mathbb{Q} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\}.$$

Since $\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C}$, we get

$$\begin{aligned} v(y) &\geq \inf_{\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta}} \left\{ \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\} \\ &\geq \inf_{\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta}} \left\{ \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\} + \vartheta(\mathbb{Q}) \right\}. \end{aligned} \quad (4.4)$$

Finally, from Lemma 4.2 in Schied [30] and $\mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1)$ follows

$$v_{\mathbb{Q}}(y) \leq \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{\mathcal{E}(Z^{\xi})_T}{D_T^{\mathbb{Q}}} \right) \right] \right\},$$

and we have the inequalities (4.4) in the other direction, and the result follows. ■

4.2 The logarithmic utility case

As it was pointed out above in Example 4.3, the existence of a solution to the dual problem for the logarithmic utility function $U(x) = \log(x)$ can be read from the results presented in the previous subsection. However, the nature of the optimization problem arising in the case of a logarithmic utility deserves a deeper study. Let h, h_0 and h_1 be as in Subsection 3.2, satisfying also the following growth conditions:

$$\begin{aligned} h(x) &\geq x, \\ h_0(x) &\geq \frac{1}{2}x^2, \\ h_1(x) &\geq \{|x| \vee x \ln(1+x)\} \mathbf{1}_{(-1,0)}(x) + x(1+x) \mathbf{1}_{\mathbb{R}_+}(x). \end{aligned}$$

Now, define the penalization function

$$\begin{aligned} \vartheta_{\log}(\mathbb{Q}) &:= \mathbb{E}_{\mathbb{Q}} \left[\int_0^T h \left(h_0(\theta_0(t)) + \int_{\mathbb{R}_0} h_1(\theta_1(t,x)) \nu(dx) \right) dt \right] \mathbf{1}_{\mathcal{Q}_{\ll}}(\mathbb{Q}) \\ &\quad + \infty \times \mathbf{1}_{\mathcal{Q}_{cont} \setminus \mathcal{Q}_{\ll}}(\mathbb{Q}). \end{aligned} \quad (4.5)$$

Remark 4.6 Notice that when $\mathbb{Q} \in \mathcal{Q}_{\ll}^{\vartheta_{\log}}(\mathbb{P})$ with coefficient $\theta = (\theta_0, \theta_1)$ has a finite penalization, the following \mathbb{Q} -integrability properties hold:

$$\begin{aligned}
(4.6.i) \quad & \int_{[0,T] \times \mathbb{R}_0} \theta_1(t, x) \mu_{\mathbb{P}}^{\mathcal{P}}(dt, dx) \in \mathcal{L}^1(\mathbb{Q}) \\
(4.6.ii) \quad & \int_{[0,T] \times \mathbb{R}_0} \{1 + \theta_1(t, x)\} \ln(1 + \theta_1(t, x)) \mu_{\mathbb{P}}^{\mathcal{P}}(dt, dx) \in \mathcal{L}^1(\mathbb{Q}) \\
(4.6.iii) \quad & \int_{[0,T] \times \mathbb{R}_0} \ln(1 + \theta_1(s, x)) \mu(ds, dx) \in \mathcal{L}^1(\mathbb{Q}) \\
(4.6.iv) \quad & \mathbb{E}_{\mathbb{Q}} \left[\int_{]0,T] \times \mathbb{R}_0} \ln(1 + \theta_1) d\mu \right] = \mathbb{E}_{\mathbb{Q}} \left[\int_{]0,T] \times \mathbb{R}_0} \{\ln(1 + \theta_1)\} (1 + \theta_1) d\mu_{\mathbb{P}}^{\mathcal{P}} \right]
\end{aligned}$$

In addition, for $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{\log}}(\mathbb{P})$ we have

$$(4.6.v) \quad \int_{[0,T] \times \mathbb{R}_0} \theta_1(s, x) \mu(ds, dx) < \infty \mathbb{P} - a.s.$$

For $\mathbb{Q} \in \mathcal{Q}_{\ll}(\mathbb{P})$, the relative entropy function is defined as

$$H(\mathbb{Q}|\mathbb{P}) = \mathbb{E} [D_T^{\mathbb{Q}} \log(D_T^{\mathbb{Q}})].$$

Lemma 4.7 Given $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{\log}}(\mathbb{P})$, it follows that

$$H(\mathbb{Q}|\mathbb{P}) \leq \vartheta_{\log}(\mathbb{Q}).$$

Proof. For $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\vartheta_{\log}}(\mathbb{P})$, Remark 4.6 implies that

$$\begin{aligned}
H(\mathbb{Q}|\mathbb{P}) &= \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{2} \int_0^T (\theta_0)^2 ds + \int_{]0,T] \times \mathbb{R}_0} \ln(1 + \theta_1(s, x)) \mu(ds, dx) - \int_0^T \int_{\mathbb{R}_0} \theta_1(s, x) \nu(dx) ds \right] \\
&\leq \mathbb{E}_{\mathbb{Q}} \left[\int_0^T \left\{ \frac{1}{2} (\theta_0)^2 ds + \int_{\mathbb{R}_0} \{\ln(1 + \theta_1(s, x))\} \theta_1(s, x) \nu(dx) \right\} ds \right] \\
&\leq \vartheta_{\log}(\mathbb{Q}).
\end{aligned}$$

■

Lemma 4.8 Let $U(x) = \log(x)$ and ϑ_{\log} be as in (4.5). Then the robust utility maximization problem (1.9) has an optimal solution.

Proof. Again, we only need to verify that condition (1.12) holds. Observe that for $\mathbb{Q} \in \mathcal{Q}_{\ll}$ fix we have that

$$v_{\mathbb{Q}}(y) \leq \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E} \left[D_T^{\mathbb{Q}} \log \left(\frac{D_T^{\mathbb{Q}}}{\mathcal{E}(Z^{\xi})_T} \right) - \log(y) - 1 \right] \right\}.$$

Also, Proposition 3.1 and the Novikov condition yield for $\tilde{\xi} \in \mathcal{C}$, with $\tilde{\xi}^{(0)} := -\frac{\alpha_s}{\beta_s}$ and $\tilde{\xi}^{(1)} := 0$, that $\tilde{\mathbb{Q}} \in \mathcal{Q}_{elmm}$, where $d\tilde{\mathbb{Q}} \backslash d\mathbb{P} = D_T^{\tilde{\xi}} := \mathcal{E}(Z^{\tilde{\xi}})_T$. Further, from Lemma 4.7 we conclude for $\mathbb{Q} \in \mathcal{Q}_{\approx}^{\psi_{\log}}(\mathbb{P})$ that

$$\mathbb{E} \left[D_T^{\mathbb{Q}} \log \left(\frac{D_T^{\mathbb{Q}}}{D_T^{\tilde{\xi}}} \right) \right] = H(\mathbb{Q} | \mathbb{P}) + \mathbb{E}_{\mathbb{Q}} \left[\int_0^T \frac{\alpha_s}{\beta_s} \theta_s^{(0)} ds + \frac{1}{2} \int_0^T \left(\frac{\alpha_s}{\beta_s} \right)^2 ds \right] < \infty$$

and the claim follows. ■

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