Generalized wordlength patterns and strength

Jay H. Beder
Department of Mathematical Sciences
University of Wisconsin-Milwaukee
P.O. Box 413
Milwaukee, WI 53201-0413
beder@uwm.edu

Jesse S. Beder
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street
Urbana, IL 61801
beder@math.uiuc.edu

Abstract

Xu and Wu (2001) defined the *generalized wordlength pattern* $(A_1, \ldots, A_k)$ of an arbitrary fractional factorial design (or orthogonal array) on k factors. They gave a coding-theoretic proof of the property that the design has strength t if and only if $A_1 = \cdots = A_t = 0$. The quantities A_i are defined in terms of characters of cyclic groups, and so one might seek a direct character-theoretic proof of this result. We give such a proof, in which the specific group structure (such as cyclicity) plays essentially no role. Nonabelian groups can be used if the counting function of the design satisfies one assumption, as illustrated by a couple of examples.

1 Introduction

A *fractional factorial design* is a multisubset D of a finite Cartesian product $G = G_1 \times \cdots \times G_k$, that is, a set of elements of G, the element $\mathbf{x}$ possibly repeated with some multiplicity $O(\mathbf{x})$. We will say that D is *based on* G, and refer to O as the *counting* or *multiplicity function*[1] of D. In statistical terminology, the set G_i indexes the levels of the ith factor in an experiment, and G is the set of treatment combinations. The treatment combinations used in the design are referred to as *runs*, and the number of runs in the design, counting multiplicities, is

$$|D| = \sum_{\mathbf{x} \in G} O(\mathbf{x}). \tag{1}$$

---

[1]It is called the *indicator function* of D by a number of authors – for example, in [3].

1

Xu and Wu [9] associated to a design D a k-tuple $(A_1(D), \ldots, A_k(D))$, called its *generalized wordlength pattern*, defined as follows. If G_i has s_i elements, we take $G_i = \mathbb{Z}_{s_i}$, the additive cyclic group of integers modulo s_i. This makes G an abelian group. To each $u \in \mathbb{Z}_s$ we associate a complex-valued function χ_u on $\mathbb{Z}_s$ such that

$$\chi_u(x) = \xi^{ux}, \tag{2}$$

where ξ is a primitive sth root of unity (say $\xi = e^{2\pi i/s}$). For $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{x} = (x_1, \ldots, x_k) \in G$, we let

$$\chi_{\mathbf{u}}(\mathbf{x}) = \prod_i \chi_{u_i}(x_i), \tag{3}$$

and define the *J-characteristics*[2] of the design to be the quantities

$$\chi_{\mathbf{u}}(D) = \sum_{\mathbf{x} \in G} O(\mathbf{x})\overline{\chi_{\mathbf{u}}(\mathbf{x})}, \tag{4}$$

the bar denoting the complex conjugate. This formula departs superficially from that given in [9]. The introduction of the conjugate does not change the value of $\chi_{\mathbf{u}}(D)$ since the choice of ξ in (2) is arbitrary and may be replaced by $\xi = e^{-2\pi i/s}$. The factor $O(\mathbf{x})$ makes it explicit that each summand is repeated according to its multiplicity.

Finally, the "generalized wordlengths" are given by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(\mathbf{u})=j} |\chi_{\mathbf{u}}(D)|^2 \qquad \text{for } j = 1, \ldots, k, \tag{5}$$

where $N = |D|$ is defined as in (1) and $\mathrm{wt}(\mathbf{u})$ is the *Hamming weight* of $\mathbf{u}$, that is, the number of non-zero components of $\mathbf{u}$. For the statistical meaning of the generalized wordlength pattern, the reader is referred to [9].

The design D may also be viewed as an orthogonal array, particularly if its runs are displayed in matrix form, say as columns of a $k \times N$ matrix. Xu and Wu [9, Theorem 4(ii)] use a coding-theoretic result to show that $A_1(D) = \cdots = A_t(D) = 0$ iff D has strength t. They note in passing that the functions (2) and (3) are group characters, which might lead us to expect a character-theoretic proof of this result. Providing such a proof is the purpose of this paper.

Using a suggestive idea from [2], we first reexpress the numbers $A_j(D)$ in terms of certain Fourier coefficients.

The functions χ_u in (2) are the *irreducible characters* of the group $\mathbb{Z}_s$, and so the functions $\chi_{\mathbf{u}}$ are the irreducible characters of G. They form an orthonormal basis of the set of all functions from G to $\mathbb{C}$ under the inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{\mathbf{x} \in G} \phi(\mathbf{x})\overline{\psi(\mathbf{x})}. \tag{6}$$

If we express O in this basis as

$$O = \sum_{\mathbf{u} \in G} \mu_{\mathbf{u}} \chi_{\mathbf{u}},$$

---

[2]When $s_1 = \cdots = s_k = 2$, the quantities $\chi_g(D)$ reduce to the *J-characteristics* of Deng and Tang [4]. We are following Ai and Zhang [1] in using the same term for these quantities in the general case.

then its Fourier coefficients $\mu_{\mathbf{u}}$ satisfy

$$\mu_{\mathbf{u}} = \langle O, \chi_{\mathbf{u}} \rangle = \frac{1}{|G|} \sum_{\mathbf{x} \in G} O(\mathbf{x}) \overline{\chi_{\mathbf{u}}(\mathbf{x})} = \frac{1}{|G|} \chi_{\mathbf{u}}(D),$$

so that the generalized wordlengths (5) are given by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(\mathbf{u})=j} |\chi_{\mathbf{u}}(D)|^2 = \frac{|G|^2}{N^2} \sum_{\mathrm{wt}(u)=j} |\mu_{\mathbf{u}}|^2.$$

To establish our claim, we need to show that D has strength t iff $\mu_{\mathbf{u}} = 0$ for all $\mathbf{u}$ such that $1 \leq \mathrm{wt}(\mathbf{u}) \leq t$.

It turns out that this result does not depend on the fact that the groups G_i are cyclic, or even abelian, although in the nonabelian case we will need to recast the concept of weight and to impose one restriction on O. We recast the main result in Section 4, and give the proof in Section 5. Background on character theory and on strength is given in Sections 2 and 3. We conclude with two examples illustrating the restriction on O in the nonabelian case.

There are many excellent expositions of character theory, and we will sometimes mention known results without citation. We will often refer to [5]; other texts include [6] and [8].

Notation and terminology. As already indicated, the complex conjugate of z will be denoted by $\bar{z}$. We denote the complex numbers by $\mathbb{C}$, the integers modulo s by $\mathbb{Z}_s$, the cardinality of a set E by $|E|$, and vectors (k-tuples) by boldface. The set of complex-valued functions on G will be written $\mathbb{C}^G$.

All groups are finite. The identity element of a group will generally be denoted by e.

When $G = G_1 \times \cdots \times G_k$ is a direct product of groups, the *Hamming weight* $\mathrm{wt}(\mathbf{u})$ of an element $\mathbf{u} \in G$ will be defined as the number of nonidentity components of $\mathbf{u}$. Here we have modified the usual definition of Hamming weight as G_i may have no zero symbol. Each G_i may be identified with a subgroup of G, namely the subgroup $e_1 \times \cdots e_{i-1} \times G_i \times e_{i+1} \times \cdots \times e_k$ where e_j is the identity of G_j. A similar identification holds for $G_{i_1} \times \cdots \times G_{i_m}$ where $1 \leq i_1 < \cdots < i_m \leq k$. For such subgroups it will be useful to introduce the following terminology.

Definition 1.1. If $H = G_{i_1} \times \cdots \times G_{i_m}$, we call H a *factorial subgroup* of G. The number m will be called the *rank* of H. The *factorial complement* of H in G is $\prod_{i \notin I} G_i$, where $I = \{i_1, \ldots, i_m\}$.

2   Characters

We will deal only with complex-valued characters. We refer the reader to a treatment of character theory for more detail, and simply quote the results that we will need.

The set of characters on the group G is closed under pointwise addition, and contains a finite set $\mathrm{Irr}(G)$ that generates it in the sense that every character on G is a unique linear combination of characters in $\mathrm{Irr}(G)$ with nonnegative integer coefficients. The characters in $\mathrm{Irr}(G)$ are called *irreducible*. Among them is the *principal character* $\chi \equiv 1$. The irreducible characters of the cyclic group $\mathbb{Z}_s$ are given by (2), while for an abelian group G they are the homomorphisms from G to the multiplicative group $\mathbb{C}^*$ [5, Corollary 2.6].

If $G = G_1 \times \cdots \times G_k$ and χ_i is a character on G_i, then

$$\chi(\mathbf{x}) = \prod_i \chi_i(x_i) \qquad (\mathbf{x} = (x_1, \ldots, x_m) \in G)$$

defines a character on G, and $\chi \in \mathrm{Irr}(G)$ iff $\chi_i \in \mathrm{Irr}(G_i)$ for all i [5, Theorem 4.21].

Definition 2.1. For χ a character of G, $\ker(\chi) = \{g \in G : \chi(g) = \chi(e)\}$, where e is the identity of G.

One can show[3] that $\ker(\chi)$ is a normal subgroup of G. The number $\chi(e)$ is a positive integer, called the *degree* of χ.

A character on G is a *class function*, that is, a function that is constant on the conjugacy classes of G. Let

$$\mathrm{Cf}(G) = \text{the set of class functions from } G \text{ to } \mathbb{C}.$$

This is clearly a vector space over $\mathbb{C}$, in which the irreducible characters play a special role (see, e.g., [5, Theorem 2.8 and Corollary 2.14]):

Theorem 2.2. *Under the inner product (6), $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathrm{Cf}(G)$. In particular, if $f \in \mathrm{Cf}(G)$ then f has a unique orthonormal expansion*

$$f = \sum_{\chi \in \mathrm{Irr}(G)} \mu_\chi \chi,$$

where the Fourier coefficients are given by

$$\mu_\chi = \langle f, \chi \rangle.$$

Remark 2.3. Two points should be noted when G is abelian. First, the conjugacy classes of G are singletons, and so *all* functions are class functions. In this case $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathbb{C}^G$, the set of all complex-valued functions on G. We made use of this in Section 1.

Second, $\mathrm{Irr}(G)$ is also a group under pointwise multiplication, and is isomorphic to G itself. In particular, we note the following:

- The irreducible characters may be indexed one-to-one by group elements. This indexing is given explicitly in equation (2) for the cyclic group $\mathbb{Z}_s$, and by (3) when G is a direct product of cyclic groups. The same holds when G is abelian.[4]

- χ_u is principal iff u is the identity of G.

We will use these facts in Lemma 4.2.

---

[3] A representation of G is a homomorphism, and $\ker(\chi)$ is the kernel of the representation affording χ. When G is abelian and χ is irreducible, $\ker(\chi)$ is the kernel of the homomorphism χ.

[4] Because of the Fundamental Theorem of Abelian Groups.

3   Strength

If a design D on the set $G_1 \times \cdots \times G_k$ is displayed as columns of a $k \times N$ matrix, the *projection* of D on factors $i_1 < \cdots < i_m$ is the sub-matrix consisting of rows $i_1, \ldots, i_m$. The resulting design D' is a multisubset of $H = G_{i_1} \times \cdots \times G_{i_m}$, with counting function

$$O'(\mathbf{y}) = \sum_{p(\mathbf{X})=\mathbf{y}} O(\mathbf{x}), \tag{7}$$

where p is the projection of G on H (namely, $p(x_1, \ldots, x_k) = (x_{i_1}, \ldots, x_{i_m})$).

Definition 3.1. D has *strength $t \geq 1$* if the projection of D onto any t factors has constant counting function.

In other words, for every $I = \{i_1, \ldots, i_t\} \subset \{1, \ldots, k\}$, the projection D'_I of D on the factors $i_1, \ldots, i_t$ consists of λ_I copies of the full factorial $G_{i_1} \times \cdots \times G_{i_t}$, so that the counting function of D'_I is the constant function $O'_I \equiv \lambda_I$.

We note that if D has strength t then it also has strength t' for all $t' < t$.

When G is a *group*, the map p projecting G onto $H = G_{i_1} \times \cdots \times G_{i_m}$ is a group homomorphism. Any such group H has its own set of irreducible characters, of course. Rephrasing Definition 3.1, we see that *D has strength t iff whenever we project G onto a factorial subgroup H with at most t factors, O' is simply a multiple of the principal character of H.*

4   Restatement of the theorem

For a design D based on the group $G = G_1 \times \cdots \times G_k$ with counting function O, the J-characteristics (4) of D are now given by

$$\chi(D) = \sum_{\mathbf{X} \in G} O(\mathbf{x}) \overline{\chi(\mathbf{x})} \tag{8}$$

for each $\chi \in \mathrm{Irr}(G)$.

As we have noted, when the group G_i is abelian (and cyclic in particular), its irreducible characters may be indexed by G_i. Without this, the concept of the weight of an element $\mathbf{u} = (u_1, \ldots, u_k) \in G$ is no longer relevant, and so we must transfer this concept to the irreducible characters of G.

Definition 4.1. For $\chi \in \mathrm{Irr}(G)$, let K be the largest factorial subgroup contained in $\ker(\chi)$. We define the *base* of χ to be the factorial complement of K in G, and the *weight* of χ by

$$\mathrm{wt}(\chi) = \mathrm{rank}(\mathrm{base}(\chi)).$$

Note that $\mathrm{wt}(\chi) = 0$ iff χ is the principal character of G.

If K is as in Definition 4.1, then $\chi \equiv \chi(e)$ on K. Now $e \in \mathrm{base}(\chi)$, so if $\chi \equiv 1$ on its base, then $\chi(e) = 1$, and so $\chi \equiv 1$. In other words, *if χ restricted to its base is principal, then χ itself is principal*, and conversely. (The converse is trivial.)

The following lemma relates Definition 4.1 to the Hamming weight of elements $\mathbf{u} \in G$ in the abelian case. It makes use of the facts mentioned in Remark 2.3.

Lemma 4.2. *Let $G = G_1 \times \cdots \times G_k$ where G_i is abelian for every i. Fix an isomorphism indexing the irreducible characters by the elements of G. Then $\mathrm{wt}(\chi_{\mathbf{u}}) = \mathrm{wt}(\mathbf{u})$.*

Proof. Given $\mathbf{u} = (u_1, \ldots, u_k)$, let $I = \{i : u_i \neq e_i\}$, where e_i is the identity of G_i. For $i \notin I$, $\chi_{u_i} = \chi_{e_i} \equiv 1$, so

$$\chi_{\mathbf{u}} = \prod_{i=1}^{k} \chi_{u_i} = \prod_{i \in I} \chi_{u_i}.$$

Let $K = \prod_{i \notin I} G_i$. We claim that K is the largest factorial subgroup of G contained in $\ker(\chi_{\mathbf{u}})$. If so, then $\mathrm{base}(\chi_{\mathbf{u}}) = \prod_{i \in I} G_i$, and so

$$\mathrm{wt}(\chi_{\mathbf{u}}) = |I| = \mathrm{wt}(\mathbf{u}).$$

(This still holds if $I = \emptyset$.)

To prove our claim, note that if $\mathbf{x} \in K$ then $x_i = e_i$ for all $i \in I$, from which we have

$$\chi_{\mathbf{u}}(\mathbf{x}) = \prod_{i \in I} \chi_{u_i}(e_i) = 1 = \chi_{\mathbf{u}}(\mathbf{e}),$$

so that $\mathbf{x} \in \ker(\chi_{\mathbf{u}})$. Thus $K \subset \ker(\chi_{\mathbf{u}})$. To show that K is the largest such factorial subgroup, consider $K' = K \times G_j$ for some $j \in I$. Since $\chi_{u_j} \neq \chi_{e_j}$, we may choose $x_j \in G_j$ such that $\chi_{u_j}(x_j) \neq 1$. Let $\mathbf{x} = (x_1, \ldots, x_k)$ where

$$x_i = \begin{cases} x_j, & i = j \\ e_i, & i \neq j \end{cases}.$$

Then $\chi_{\mathbf{u}}(\mathbf{x}) = \chi_{u_j}(x_j) \neq 1 = \chi_{\mathbf{e}}(\mathbf{x})$, so $\mathbf{x} \notin \ker(\chi_{\mathbf{u}})$. Thus K' is not contained in $\ker(\chi_{\mathbf{u}})$, which proves our claim. $\square$

We now replace the definition of generalized wordlengths given in (5) by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(\chi)=j} |\chi(D)|^2 \qquad \text{for } j = 1, \ldots, k, \tag{9}$$

where $N = |D|$, defined as in (1). With this, we restate our theorem as follows:

Theorem 4.3. *Let D be a fractional factorial design on $G = G_1 \times \cdots \times G_k$ with counting function O, and assume O is a class function on G. For each $\chi \in \mathrm{Irr}(G)$ define $\chi(D)$ by (8), and let $\mu_\chi = \langle O, \chi \rangle$. Define $A_j(D)$ by (9), and assume $t \geq 1$. Then the following are equivalent:*

a. *D has strength t.*

b. *$A_1(D) = \cdots = A_t(D) = 0$.*

c. *$\mu_\chi = 0$ for all $\chi \in \mathrm{Irr}(G)$ with $1 \leq \mathrm{wt}(\chi) \leq t$.*

In Section 6 we give two nonabelian examples with counting functions that are class functions.

5   Proof of the theorem

As in the abelian case, we have

$$\mu_\chi = \langle O, \chi \rangle = \frac{1}{|G|} \sum_{\mathbf{x} \in G} O(\mathbf{x}) \overline{\chi(\mathbf{x})} = \frac{1}{|G|} \chi(D)$$

for each $\chi \in \mathrm{Irr}(G)$, so that the generalized wordlengths (9) are given by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(\chi)=j} |\chi(D)|^2 = \frac{|G|^2}{N^2} \sum_{\mathrm{wt}(\chi)=j} |\mu_\chi|^2.$$

Thus we immediately have the equivalence of (b) and (c) in Theorem 4.3. Our goal is to prove the equivalence of (a) and (c).

We noted in Section 3 that D has strength t iff whenever we project G onto a factorial subgroup H of rank at most t, the counting function O' of the projected design is a simply a multiple of the principal character of H. Assuming that O' is a class function on H, we have the orthonormal expansion

$$O' = \sum_{\hat{\chi} \in \mathrm{Irr}(H)} \mu_{\hat{\chi}} \hat{\chi} \tag{10}$$

from which we see that D has strength t iff, for the projection on any H with $\mathrm{rank}(H) \le t$, the Fourier coefficients $\mu_{\hat{\chi}} = \langle O', \hat{\chi} \rangle$ vanish for all non-principal irreducible characters $\hat{\chi}$ of H.

On the other hand, when O is a class function on G we have

$$O = \sum_{\chi \in \mathrm{Irr}(G)} \mu_\chi \chi. \tag{11}$$

Thus the proof requires a comparison of equations (10) and (11). It rests on the following two lemmas. In both, G is an arbitrary finite group, and we denote the coset Kg by $\bar{g}$.[5]

Lemma 5.1. *[5, Lemma 2.22] Let K be a normal subgroup of G.*

  a. *If χ is a character of G and $K \subseteq \ker(\chi)$, then χ is constant on cosets of K in G and the function $\hat{\chi}$ on G/K defined by $\hat{\chi}(\bar{g}) = \chi(g)$ is a character of G/K.*

  b. *If $\hat{\chi}$ is a character of G/K, then the function χ defined by $\chi(g) = \hat{\chi}(\bar{g})$ is a character of G and $K \subseteq \ker(\chi)$.*

  c. *In both (a) and (b), $\chi \in \mathrm{Irr}(G)$ iff $\hat{\chi} \in \mathrm{Irr}(G/K)$.*

Lemma 5.2. *Let K be normal in G and let $H = G/K$. Let $f \in \mathbb{C}^G$ and $\chi \in \mathrm{Irr}(G)$. Define $f' \in \mathbb{C}^H$ by*

$$f'(\bar{y}) = \sum_{x \in \bar{y}} f(x), \tag{12}$$

and define $\hat{\chi}$ as in Lemma 5.1(a). If $f \in \mathrm{Cf}(G)$ then $f' \in \mathrm{Cf}(H)$, and

$$\langle f', \hat{\chi} \rangle = |K| \langle f, \chi \rangle.$$

---

[5]We also use the "bar" notation to indicate complex conjugates; context will determine which is meant.

Note that when G is a group, the counting function O' of a projected design, defined in (7), is of the form (12).

Proof of Lemma 5.2. First, suppose that $\bar{y}_1$ and $\bar{y}_2$ are conjugate in H. Then

$$\bar{y}_2 = (Kh^{-1})(Ky_1)(Kh) = h^{-1}Ky_1h$$

for some $h \in G$, so the elements of the cosets $\bar{y}_1$ and $\bar{y}_2$ may be paired in such a way that each $x_2 \in \bar{y}_2$ is the conjugate of a unique $x_1 \in \bar{y}_1$. Since f is a class function, $f(x_1) = f(x_2)$, so $\sum_{x \in \bar{y}_1} f(x) = \sum_{x \in \bar{y}_2} f(x)$. This shows that f' is a class function on H.

By Lemma 5.1(a), χ is constant on each coset $\bar{y}$, and $\hat{\chi}(\bar{y}) = \chi(y)$. We then have

$$
\begin{aligned}
\langle f, \chi \rangle &= \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\chi(x)} \\
&= \frac{1}{|G|} \sum_{\bar{y} \in H} \sum_{x \in \bar{y}} f(x)\overline{\chi(x)} \\
&= \frac{1}{|G|} \sum_{\bar{y} \in H} \overline{\hat{\chi}(\bar{y})} \sum_{x \in \bar{y}} f(x) \\
&= \frac{1}{|G|} \sum_{\bar{y} \in H} f'(y)\overline{\hat{\chi}(\bar{y})} = \frac{\langle f', \hat{\chi} \rangle}{|K|}.
\end{aligned}
$$

$\square$

We are now ready to complete the proof of Theorem 4.3. We begin by noting two things. First, according to Lemma 5.2, the assumption that O is a class function guarantees that the counting function O' of *every* projected design is also a class function. In particular, O' has an orthonormal expansion (10).

Second, when $G = H \times K$ is a direct product, H is isomorphic to G/K, and the character $\hat{\chi}$ in Lemma 5.1 is the restriction of χ to H. Recall that χ is nonprincipal iff its restriction to its base is nonprincipal.

(a) $\Rightarrow$ (c): Assume that D has strength t, and let $\chi \in \mathrm{Irr}(G)$ with $1 \leq \mathrm{wt}(\chi) \leq t$. We need to show that the Fourier coefficient μ_χ of O vanishes. Let $H = \mathrm{base}(\chi)$, and let $\hat{\chi}$ be defined by χ as in Lemma 5.1(a) where K is the complement of H in G. Now χ is nonprincipal, as $\mathrm{wt}(\chi) \geq 1$, so $\hat{\chi}$ is as well. On the other hand, since H has at most t factors and D has strength t, O' is a multiple of the principal character of H. But then $\mu_{\hat{\chi}} = \langle O', \hat{\chi} \rangle = 0$, and so by Lemma 5.2 $\mu_\chi = \langle O, \chi \rangle = 0$.

(c) $\Rightarrow$ (a): Assuming the condition on the coefficients μ_χ given by (c), we must show that D has strength t. To this end, consider any factorial subgroup H of G having at most t factors, let K be its factorial complement, and let O' be the counting function of the design projected on H. Let $\hat{\chi} \in \mathrm{Irr}(H)$ be a nonprincipal character on $H = G/K$, and let $\chi \in \mathrm{Irr}(G)$ correspond to it via Lemma 5.1(b). In particular, χ is nonprincipal and $K \subseteq \ker(\chi)$. Let K_1 be the largest factorial subgroup contained in $\ker(\chi)$, so that $K_1 \supseteq K$. Taking complements, we have $\mathrm{base}(\chi) \subseteq H$, so that

$$\mathrm{wt}(\chi) = \mathrm{rank}(\mathrm{base}(\chi)) \leq \mathrm{rank}(H) \leq t.$$

Therefore, $\langle O, \chi \rangle = \mu_\chi = 0$ by assumption. But then $\langle O', \hat{\chi} \rangle = 0$ as well, by Lemma 5.2, so O' must be a multiple of the principal character of H. Since this holds for all such H, D has strength t.

6 Two examples

We conclude by giving two examples of designs whose treatment combinations are indexed by nonabelian groups and whose counting functions are class functions of those groups. Both examples make use of S_3, the symmetric group on 3 letters. We write

$$S_3 = \{e, a, b, c, x, y\}$$

where e is the identity, a, b and c are transpositions, and x and y are 3-cycles. As is well known, the conjugacy classes of S_3 are $\{e\}, \{a, b, c\}$ and $\{x, y\}$.

We also make use of the facts that the conjugacy classes of an abelian group are the singleton subsets, and that in a direct product, $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are conjugate iff x_i and y_i are conjugate for each i.

Example 6.1. A 1/2-fraction of $6 \times 2 \times 2$ experiment of strength 2.

We index the treatment combinations by $G = S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The following array displays the runs as columns, the vertical lines separating conjugacy classes.

$$D = \left[\begin{array}{ccc|ccc|ccc|ccc} e & x & y & e & x & y & a & b & c & a & b & c \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

This makes use of 6 of the 12 conjugacy classes of G. The other 6 classes would furnish another example. Since S_3 is the smallest nonabelian group, this is the smallest non-trivial fractional factorial design of strength 2 that can be indexed by a nonabelian group.

Example 6.2. A 1/2-fraction of a $6 \times 2 \times 2 \times 2$ experiment of strength 3.

$$D = \left[\begin{array}{cccc|cc|cc|cc|cc|ccc|ccc|ccc|ccc} e & e & e & e & x & y & x & y & x & y & x & y & a & b & c & a & b & c & a & b & c & a & b & c \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

We have indexed the treatment combinations by $G = S_3 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Note that the last three rows consist of three copies of the full 2^3 factorial design, split into its two regular 2^{3-1} fractions given by the solutions (X, Y, Z) of $X + Y + Z = 0$ and $= 1$ modulo 2. We have attached the first fraction to e, x, and y, and the second fraction to a, b and c.

Further examples are given in [7]

References

[1] Ming-Yao Ai and Run-Chu Zhang. Projection justification of generalized minimum aberration for asymmetrical fractional factorial designs. *Metrika*, 60:279–285, 2004.

[2] Jürgen Bierbrauer. Bounds on orthogonal arrays and resilient functions. *Journal of Combinatorial Designs*, 3:179–183, 1995.

[3] Shao-Wei Cheng and Kenny Q. Ye. Geometric isomorphism and minimum aberration for factorial designs with quantitative factors. *The Annals of Statistics*, 32:2168–2185, 2004.

[4] Lih-Yuan Deng and Boxin Tang. Generalized resolution and minimum aberration criteria for Plackett–Burman and other nonregular factorial designs. *Statistica Sinica*, 9:1071–1082, 1999.

[5] I. Martin Isaacs. *Character Theory of Finite Groups.* Academic Press, New York, 1976.

[6] Walter Ledermann. *Introduction to Group Characters.* Cambridge University Press, Cambridge, 2nd edition, 1987.

[7] Margaret Ann McComack. Constructing orthogonal arrays on non-abelian groups. Master's thesis, University of Wisconsin – Milwaukee, 2012.

[8] Jean-Pierre Serre. *Linear Representations of finite Groups.* Springer-Verlag, New York, 1977. Leonard L. Scott, transl.

[9] Hongquan Xu and C. F. J. Wu. Generalized minimum aberration for asymmetrical fractional factorial designs. *The Annals of Statistics*, 29:1066–1077, 2001.