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Nonexistence of (3,2,1)-conjugate *r*-orthogonal Latin Squares of Order *v* for $r\in\{v+2,v+3,v+5\}$

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Abstract: Two latin squares of order v, $L=(l_{ij})$ and $M=(m_{ij})$ are called to be r-orthogonal if their superposition produces exactly r distinct ordered pairs, that is $\left|\{(l_{ij},m_{ij}):1\leqslant i,j\leqslant v\}\right|=r$, which is denoted by r-MOLS(v). It has been proved that there does not exist an r-MOLS(v) for $r\in\{v+1,v^2-1\}$. If M is the (3,2,1)-conjugate of L, then L is called to be (3,2,1)-conjugate r-orthogonal, as denoted by (3,2,1)-r-COLS(v). In this paper, the nonexistence of (3,2,1)-r-COLS(v) for $r\in\{v+2,v+3,v+5\}$ is proved.

Key words: latin square; r-orthogonal; (3,2,1)-conjugate

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0 Introduction

A quasigroup is an ordered pair $(Q,\otimes)$, where Q is a set and $\otimes$ is a binary operation on Q, such that the equations $a\otimes x=b$ and $y\otimes a=b$ are uniquely solvable for every pair of elements a,b in Q. It's fairly well know that the multiplication table of a quasigroup defines a latin square, that is, a latin square can be viewed as the multiplication table of a quasigroup with the headline and sideline removed.

If $(Q,\otimes)$ is a quasigroup, we may define six binary operations $\otimes_{(1,2,3)}$, $\otimes_{(1,3,2)}$, $\otimes_{(2,1,3)}$, $\otimes_{(2,3,1)}$, $\otimes_{(3,1,2)}$, $\otimes_{(3,2,1)}$ on the set Q as follows: $a\otimes b=c$ if and only if

$$a\otimes_{(1,2,3)} b=c,\quad a\otimes_{(1,3,2)} c=b,\quad b\otimes_{(2,1,3)} a=c,$$
$$b\otimes_{(2,3,1)} c=a,\quad c\otimes_{(3,1,2)} a=b,\quad c\otimes_{(3,2,1)} b=a.$$

These six (not necessarily distinict) quasigroups $(Q,\otimes_{(i,j,k)})$, where $\{i,j,k\}=\{1,2,3\}$ are called the conjugates of $(Q,\otimes)$. As the multiplication table of a quasigroup $(Q,\otimes)$ defines a latin square which is L, then these six latin squares defined by the multiplication tables of its conjugates $(Q,\otimes_{(i,j,k)})$ are called the conjugates of L.

Two latin squares of order v, $L=(l_{ij})$ and $M=(m_{ij})$ are said to be orthogonal if their superposition produces exactly v^2 distinct ordered pairs, that is

$$|\{(l_{ij},m_{ij}):1\leqslant i,j\leqslant v\}|=v^2.$$

If the superposition produces r distinct ordered pairs, that is

$$|\{(l_{ij},m_{ij}):1\leqslant i,j\leqslant v\}|=r,$$

then L and M are said to be r-orthogonal. Belyavs-

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kaya[1-3] first discussed the practical utilizetion of *r*-orthogonal latin squares in coding theory and some problems raised thereby, and systematically treated the following question. For which integers *v* and *r* does a pair of *r*-orthogonal latin squares of order *v* exist? Evidently, $v\leqslant r\leqslant v^2$. In papers by Colbourn and Zhu[4], Zhu and Zhang[5-6], this question has been completely answered. And for the existence of (v+1)-MOLS(v) and (v^2-1)-MOLS(v), the answer is negative. From [6, Theorem 2.1], we have the following result.

Theorem 1 There exists no *r*-MOLS(*v*) with *v* and *r* shown in Table 1.

Table 1 The genuine exception of *r* -MOLS(*v*)

order *v*	genuine exceptions of *r*
2	4
3	5, 6, 7
4	7, 10, 11, 13, 14
5	8, 9, 20, 22, 23
6	33, 36

If *M* is the transpose ((2,1,3)-conjugate) of *L*, then *L* is said to be *r*-self-orthogonal. The spectrum of *r*-self-orthogonal latin squares (*r*-SOLS for short) have almost completely determined by Xu and Chang[7-8]. The following result is from [8, Theorem 6.2].

Theorem 2 There exists no *r*-SOLS(*v*) with *v* and *r* shown in Table 2.

Table 2 The genuine exception of *r*-SOLS(*v*)

order *v*	genuine exceptions of *r*
2	4
3	5, 6, 7, 9
4	6, 7, 8, 10, 11, 12, 13, 14
5	8, 9, 12, 16, 18, 20, 22, 23
6	32, 33, 34, 36
7	46

If *M* is the (3,2,1)-conjugate of *L*, then *L* is said to be (3,2,1)-conjugate *r*-orthogonal and denoted by (3,2,1)-*r*-COLS(*v*). It is much more difficult to determine the spectrum of (3,2,1)-*r*-COLS than that of *r*-MOLS and *r*-SOLS. By exhaustive computer search, we have the following nonexistence result.

Theorem 3 There exists no (3,2,1)-*r*-COLS(*v*) with *v* and *r* shown in Table 3.

Table 3 The genuine exception of (3,2,1)-*r*-COLS(*v*)

order *v*	genuine exceptions of *r*
2	4
3	5, 6, 7
4	6, 7, 9, 10, 11, 13, 14
5	7, 8, 9, 10, 12, 14, 18, 20, 21, 22, 23
6	8, 9, 11, 13, 31, 32, 33, 34, 36
7	9, 10, 12, 14, 16, 45, 46
8	10, 11, 13, 15, 17, 61

For the existence of (3,2,1)-*r*-COLS(*v*) with $r\in\{v+1,v^2-1\}$, the answer is negative according to the spectrum of *r*-MOLS(*v*). In this paper, we shall show the nonexistence of (3,2,1)-*r*-COLS(*v*) for $r\in\{v+2,v+3,v+5\}$.

1 The Nonexistence of (3,2,1)-*r*-COLS(*v*) for $r\in\{v+2,v+3,v+5\}$

Suppose $L=(l_{ij})_{v\times v}$ is a (3,2,1)-*r*-COLS(*v*), $M=(m_{ij})_{v\times v}$ is the (3,2,1)-conjugate of *L*. Let $P=\{(l_{ij},m_{ij}):1\leqslant i<j\leqslant v\}$. It is obvious that $|P|=r$. We call *P* the (3,2,1)-DOP set (distinct ordered pairs set) of *L*. In this section, we always suppose that every latin square of order *v* is based on set $\{1,2,\cdots,v\}$.

Lemma 1 For any positive integer *v*, if $L=(l_{ij})_{v\times v}$ is a (3,2,1)-*r*-COLS(*v*) with (3,2,1)-DOP set *P*, then *P* contains $\{(i,i):1\leqslant i\leqslant v\}$.

Proof Let $L=(l_{ij})_{v\times v}$ be a latin square and $M=(m_{ij})_{v\times v}$ be the (3,2,1)-conjugate of *L*. For any $i\in\{1,2,\cdots,v\}$, there exists $j\in\{1,2,\cdots,v\}$ such that

$l_{ij} = i$ since L is a latin square.

Furthermore, since M is the (3,2,1)-conjugate of L, we have $m_{ij} = i$ and $(i,i) \in P = \{(l_{ij}, m_{ij}) : 1 \leqslant i, j \leqslant v\}$.

Lemma 2 Let $L = (l_{ij})_{v\times v}$ be a latin square and $M = (m_{ij})_{v\times v}$ be the (3,2,1)-conjugate of L. Let σ_p and τ_p be permutations associated with the pth columns of L and M, respectively:

$$\sigma_p = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ l_{1p} & l_{2p} & l_{3p} & \cdots & l_{vp} \end{pmatrix},$$

$$\tau_p = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ m_{1p} & m_{2p} & m_{3p} & \cdots & m_{vp} \end{pmatrix}.$$

Then $\tau_p = \sigma_p^{-1}$.

Proof It is easy to see from the definition of (3,2,1)-conjugate.

Definition 1 Write σ_p in Lemma 2 into disjoint cycles:

$$\sigma_p = (x_1^{(p)}) \cdots (x_{r_{1p}}^{(p)})(y_{11}^{(p)} y_{12}^{(p)}) \cdots (y_{r_{2p}1}^{(p)} y_{r_{2p}2}^{(p)}) \cdots (z_{11}^{(p)} z_{12}^{(p)} \cdots z_{1v}^{(p)}),$$

then,

$$\tau_p = (x_1^{(p)}) \cdots (x_{r_{1p}}^{(p)})(y_{11}^{(p)} y_{12}^{(p)}) \cdots (y_{r_{2p}1}^{(p)} y_{r_{2p}2}^{(p)}) \cdots (z_{1v}^{(p)} \cdots z_{12}^{(p)} z_{11}^{(p)}).$$

The type of the two permutations is defined as $1^{r_{1p}} 2^{r_{2p}} \cdots v^{r_{vp}}$, where $r_{1p} + 2r_{2p} + \cdots + vr_{vp} = v$.

Let $P_p = \{(l_{ij}, m_{ij}) : 1 \leqslant i, j \leqslant v\} \setminus \{(i,i) : 1 \leqslant i \leqslant v\}$. It is easy to see that

$$|P_p| = \sum_{l=3}^{v} (l \cdot r_{lp}), \quad p = 1, 2, \cdots, v.$$

Combined with Lemma 1, we have the following theorem.

Theorem 4 For any positive integer v, there exists no (3,2,1)-(v+2)-COLS(v).

Theorem 5 For any positive integer v, there exists no (3,2,1)-(v+3)-COLS(v).

Proof It is obviously true for $1 \leqslant v \leqslant 2$. We suppose that $v \geqslant 3$ in the following of this proof. Let L be a (3,2,1)-(v+3)-COLS(v) and M be the (3,2,1)-conjugate of L. Besides the pairs in $\{(i,i) : 1 \leqslant i \leqslant v\}$, there are only three distinct ordered pairs in the (3,2,1)-DOP set of L. Then there exists some $p \in \{1, 2, \cdots, v\}$ such that there is only one cycle of length 3 in σ_p as defined in Definition 1. Let (ijk) be the cycle of length 3. From the definition of (3,2,1)-conjugate, (ikj) must be a cycle in the permutation associated with the pth column of M. That is $l_{ip} = j$, $l_{jp} = k$, $l_{kp} = i$, $m_{ip} = k$, $m_{jp} = i$, $m_{kp} = j$. They produce three distinct ordered pairs (j,k), (k,i) and (i,j) as shown in Figure 1, where $\otimes L$ and $\otimes M$ are the multiplication tables of quasigroups corresponding to L and M, respectively.

	p	q
i	j	k
j	k	
k	i	

$\otimes L$

	p	q
i	k	i/k
j	i	
h	j	i

$\otimes M$

Fig.1 The multiplication tables of quasigroups corresponding to L and M

The sidelines of $\otimes L$ and $\otimes M$ are the row indexes of L and M, respectively. The headlines of $\otimes L$ and $\otimes M$ are the column indexes of L and M. For the ith row of L, there exists some $q \in \{1, 2, \cdots, v\}$. such that $l_{iq} = k$. From the definition of (3,2,1)-conjugate we have $m_{kq} = i$. As three distinct ordered pairs (j,k), (k,i) and (i,j) have already occurred, m_{iq} must be i or k. If $m_{iq} = i$, it is in contradiction to $m_{kq} = i$. If $m_{iq} = k$, it is in contradiction to $m_{ip} = k$. This completes the proof.

Lemma 3 Let L be a latin square of order v, σ_1 and σ_2 be two cycles in permutations associated with columns of L as defined in Definition 1. Denote the (3,2,1)-DOP sets associated with σ_1 and σ_2 by P_1 and P_2, respectively.

(1) If σ_1 and σ_2 have the same length 3 and $|P_1\cap P_2|=1$, then $\sigma_1=\sigma_2$.

(2) If σ_1 and σ_2 have the same length 4 and $|P_1\cap P_2|=3$, then $\sigma_1=\sigma_2$.

(3) If the length of σ_1 is 4 and the length of σ_2 is 3, then $|P_1\cap P_2|\neq 2$.

Proof (1) Let $\sigma_1=(ijk)$. From the definition of (3,2,1)-conjugate, σ_1 produces three distinct ordered pairs (j,k), (k,i) and (i,j). Suppose $P_1\cap P_2=\{(i,j)\}$ and $\sigma_2=(ijm)$. From the definition of latin square, we get $\sigma_1=\sigma_2$.

(2) For any three distinct ordered pairs in P_1, as they are not in $\{(i,i):1\leqslant i\leqslant v\}$, they must be produced by four different elements in $\{1,2,\cdots,v\}$, and each element occurs in two pairs. It's easy to see that $P_1=P_2$ and $\sigma_1=\sigma_2$.

(3) Any two distinct ordered pairs in P_1 are formed by two or four different elements, and any two distinct ordered pairs in P_2 are formed by three different elements.

Theorem 6 For any positive integer *v*, there exists no (3,2,1)-(*v*+5)-COLS(*v*).

Proof It is obviously true for $1\leqslant v\leqslant 2$. The nonexistence of (3,2,1)-8-COLS(3) and (3,2,1)-9-COLS(4) are from the nonexistence of (v^2-1)-MOLS(*v*) and Theorem 3, respectively. We suppose that $v\geqslant 5$ in the following of this proof. Suppose $L=(l_{ij})$ is a (3,2,1)-(*v*+5)-COLS(*v*), and $L'=(l'_{ij})$ is the (3,2,1)-conjugate of *L*. Then $|\{(l_{ij},l'_{ij}):l_{ij}\neq l'_{ij},1\leqslant i,\ j\leqslant v\}|=5$. From Lemma 3, we know that the five distinct ordered pairs must occur in the same column of the superposition of L and L', and be produced by a cycle of length 5 in σ_p as defined in Definition 1 for some $p\in\{1,2,\cdots,v\}$. Let $(ijkmn)$ be the cycle of length 5. From the definition of (3,2,1)-conjugate, $(inmkj)$ must be a cycle in the permutation associated with the *p*th column of L'. That is $l_{ip}=j$, $l_{jp}=k$, $l_{kp}=m$, $l_{mp}=n$, $l_{np}=i$, $l'_{ip}=n$, $l'_{np}=m$, $l'_{mp}=k$, $l'_{kp}=j$, $l'_{jp}=i$. They produce five distinct ordered pairs (j,n), (k,i), (m,j), (n,k) and (i,m) as shown in Figure 2, where $\otimes L$ and $\otimes L'$ are the multiplication tables of quasigroups corresponding to *L* and *L'*, respectively.

$\otimes L$	*p*	*q*
i	*j*	*n*
j	*k*	
k	*m*	*i*
m	*n*	*k*
n	*i*	

$\otimes L'$	*p*	*q*
i	*n*	*n*/*k*
j	*i*	
k	*j*	*m*
m	*k*	*i*/*k*
n	*m*	*i*

Fig.2 The multiplication tables of quasigroups corresponding to *L* and *L'*

The sidelines of $\otimes L$ and $\otimes L'$ are the row indexes of L and L', respectively. The headlines of $\otimes L$ and $\otimes L'$ are the column indexes of *L* and L'. For the *i*th row of L, there exists some $q\in\{1,2,\cdots,v\}$ such that $l'_{iq}=n$. From the definition of (3,2,1)-conjugate we have $l'_{nq}=i$. Since the five distinct ordered pairs are (j,n), (k,i), (m,j), (n,k) and (i,m), l'_{iq} must be *n* or k. If $l'_{iq}=n$, it is in contradiction to $l'_{ip}=n$. If $l'_{iq}=k$, from the definition of (3,2,1)-conjugate, $l_{kq}=i$ and then we have $l'_{kq}=m$, $l_{mq}=k$. Then l'_{mq} must be i or k. If $l'_{mq}=i$, it is in contradiction to $l'_{nq}=i$. If $l'_{mq}=k$, it is in contradiction to $l'_{iq}=k$. This completes the proof.

2 Remarks

From Table 3 in Theorem 3, it's easy to see that there exists no (3,2,1)-(*v*+7)-COLS(*v*) for $v\in\{4,5,6,7,8\}$. For the existence of (3,2,1)-(*v*+7)-COLS(*v*), the answer may be negative also.

Suppose that L and M are r-orthogonal latin squares of order v. If M is the (1,3,2)- conjugate of L, then L is said to be (1,3,2)- conjugate r-orthogonal and denoted by (1,3,2)-r-COLS(v). It is obvious that if a latin square L is (3,2,1)-conjugate r-orthogonal, then its transpose L^{T} is (1,3,2)-conjugate r-orthogonal. Combined with Theorems 4, 5 and 6, we have the following theorem.

Theorem 7 For any positive integer v, there exist no (3,2,1)-r-COLS(v) and (1,3,2)-r-COLS(v) for $r \in \{v+2, v+3, v+5\}$.

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v 阶(3,2,1)-共轭 r-正交拉丁方在集合 $r \in \{v+2, v+3, v+5\}$上的不存在性

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摘要： 2 个 v 阶拉丁方，$L=(l_{ij})$和 $M=(m_{ij})$被称为是 r-正交的，如果把它们重叠起来可以得到恰好 r 个不同的有序元素偶，即$\left|\{(l_{ij}, m_{ij}): 1 \leqslant i, j \leqslant v\}\right| = r$，记为 r-MOLS(v)．r-MOLS(v) 在 $r \in \{v+1, v^2-1\}$上的不存在性已经得到证明．如果 M 是 L 的(3,2,1)-共轭，可认为 L 是(3,2,1)-共轭 r-正交的，可记为(3,2,1)-r-COLS(v)．并且证明了(3,2,1)-r-COLS(v)在 $r \in \{v+2, v+3, v+5\}$上的不存在性．

关键词： 拉丁方; r-正交; (3,2,1)-共轭

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