Stability of Impulsive Stochastic Partial Delay Differential Equations with Markovian Jumps

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Abstract: Based on the fixed point theory, the asymptotical stability of mild solution to impulsive stochastic partial differential equations with infinite delays and Markovian jumps is studied. In addition, some conditions are derived to ensure the ensuing result. In particular, since Markovian jumps are considered in this work, the result derived from this paper generalizes the result obtained in Sakthivel et al's publication.

Key words: impulse; mild solution; asymptotical stability; Markovian jumps

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Recently, stochastic partial differential equations in a separable Hilbert space have been studied by many authors and there are many valuable results on stability. For example, Caraballo et al[1] investigated the exponential stability of mild solutions of stochastic partial differential equations with delays by stochastic integral techniques. By the fixed point theory, Luo[2] studied the asymptotical mean square stability of mild solution of neutral stochastic delay differential equations. Based on the method[2], Sakthivel et al[3] proved mild solution of nonlinear impulsive stochastic differential equation is asymptotically stable. However, so far, there are few results about the stability of impulsive stochastic partial differential equations with infinite delays and Markovian jumps.

Motivated by the above discussion, in this paper, we study the asymptotical stability of mild solution for impulsive stochastic partial differential equations with infinite delays and Markovian jumps of the form

$$\mathrm{d}x(t)=[Ax(t)+a(t,x(t-\alpha(t)),r(t))]\mathrm{d}t+ b(t,x(t-\beta(t)),r(t))\,\mathrm{d}W(t), t\geqslant 0, t\neq t_k, \tag{1}$$

$$\triangle x(t_k)=x(t_k^+)-x(t_k^-)= I_k(x(t_k)), t=t_k, k=1,2,\cdots,m, \tag{2}$$

$$x_0=\phi\in C_{F_0}^b([\hat{n},0];X). \tag{3}$$

1 Preliminaries

Let X, Y be two real separable Hilbert spaces and we denote by $\|\cdot\|_X, \|\cdot\|_Y$ their vector norms. We denote the notation $\|\cdot\|$ for the norm of $L(Y,X)$, where $L(Y,X)$ denotes the set of all bounded linear operators from Y into X. Besides, let (Ω,F,P) be a complete probability space equipped with some filtration $\{F_t\}_{t\geqslant 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and F_0 contains all P-null sets. Moreover, We use $C_{F_0}^b([\hat{n},0];X)$ to denote the family of all

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almost surely bounded, F_0-measurable and continuous random variables from $[\hat{n},0]$ to X, equipped with the norm $\|\phi\|_C = \sup_{\theta\in[\hat{n},0]}\|\phi(\theta)\|_X$.

Let $W(t)$ be a Wiener process defined on (Ω,F,P) and taking values in the separable Hilbert space Y, with covariance operator Q[4].

Furthermore, let $\{r(t),\ t\in R_+\}$ be a right-continuous Markov chain on (Ω,F,P) which take values in a finite state space $S=\{1,2,\cdots,N\}$.

Now, we make the system (1)~(3) precise: A is a infinitesimal generator of a semigroup of bounded linear operators $T(t), t\geqslant 0$ defined on X. Let $R_+=[0,\infty)$ and the mappings $a: R_+\times X\to X$, $b: R_+\times X\to L(Y,X)$ are all Borel measurable, meanwhile, $\alpha: R_+\to R_+$, $\beta: R_+\to R_+$ are all continuous, $I_k: X\to X$. Besides, $t-\alpha(t)\to\infty$, $t-\beta(t)\to\infty$ as $t\to\infty$, and $\hat{n}=\max\{\inf(s-\alpha(s), s\geqslant 0),\ \inf(s-\beta(s), s\geqslant 0)\}$. Furthermore, $x(t_k^+)$ and $x(t_k^-)$ denote the right-hand and left-hand limits of $x(t)$ at time $t=t_k$; and the fixed moments t_k satisfy $0<t_1<\cdots<t_m<\lim_{k\to\infty}t_k=\infty$; $\vartriangle x(t_k)=x(t_k^+)-x(t_k^-)$ denote the jumps in the state at time t_k with I_k which determine the size of the jumps, $k=1,2,\cdots,m$.

Definition 1 A stochastic process $\{x(t), t\in[0,T]\}$, $0\leqslant T<+\infty$, is called mild solution of (1)~(3) if

(1) $x(t)$ is adapted to $F_t, t\geqslant 0$.

(2) $x(t)\in X$ is continuous on $t\in[0,T]$ almost surely, and for arbitrary $0\leqslant t\leqslant T$,

$$x(t)=T(t)\phi(0)+\int_0^t T(t-s)a\cdot (s,x(s-\alpha(s)),r(s))\,\mathrm{d}s+\int_0^t T(t-s)b(s,x(s-\beta(s)),r(s))\,\mathrm{d}W(s)+\sum_{0<t_k<t}T(t-t_k)I_k(x(t_k)),\tag{4}$$

and $x_0=\phi\in C_{F_0}^b([\hat{n},0];X)$.

Definition 2 Let $p\geqslant 2$ be an integer. Eq. (4) is said to be stable in p-th moment, if for any given $\varepsilon>0$, there exist a positive constant δ such that $\|\phi\|_C<\delta$ and

$$E\{\sup_{t\geqslant 0}\|x(t)\|_X^p\}<\varepsilon.\tag{5}$$

Definition 3 Let $p\geqslant 2$ be an integer. Eq. (4) is said to be asymptotically stable in p-th moment if it is stable in p-th moment and for any $\phi\in C_{F_0}^b([\hat{n},0];X)$, the following holds

$$\lim_{\tau\to\infty}E\{\sup_{t\geqslant\tau}\|x(t)\|_X^p\}=0.\tag{6}$$

In order to study the stability of system (1)~(3), we impose the following assumptions hold:

(B1) $\|T(t)\|_X\leqslant Me^{-\lambda t}$, $\forall t\geqslant 0$, where $M>0$ and $\lambda>0$.

(B2) $\|a(t,x,i)-a(t,y,i)\|_X\leqslant M_1\|x-y\|_X$, $\forall t\geqslant 0$, $x,y\in X$ and $i\in S$, where $M_1>0$.

(B3) $\|b(t,x,i)-b(t,y,i)\|\leqslant M_2\|x-y\|_X$, $\forall t\geqslant 0$, $x,y\in X$ and $i\in S$, where $M_2>0$.

(B4) $\|I_k(x)-I_k(y)\|_X\leqslant q_k\|x-y\|_X$, $\forall x,y\in X$, where $0<q_k<+\infty$, $k=1,2,\cdots,m$.

2 Stability analysis

Let S be the Banach space of all F_t-adapted processes $\varphi(t,\omega):[\hat{n},\infty)\times\Omega\to R$, meanwhile, for any fixed $\omega\in\Omega$, $\varphi(t,\omega)$ is almost surely continuous in t; when $s\in[\hat{n},0]$, $\varphi(s,\omega)=\phi(s)$; and $E\|\varphi(t,\omega)\|_X^p\to 0$ as $t\to\infty$.

Moreover, we shall assume that

$$f(t,0,i)\equiv 0,\quad g(t,0,i)\equiv 0,$$

and

$$I_k(0)\equiv 0\ (k=1,2,\cdots,m),$$

for any

$$t\geqslant 0\ \text{ and }\ i\in S.$$

Then, Eqs. (1)~(3) has a trivial solution when $\phi\equiv 0$.

Theorem 1 Let $p\geqslant 2$ be an integer. Suppose Assumptions (B1)~(B4) hold, then Eq. (4) is asymptotic stability in p-th moment if

$$4^{p-1}M^p(M_1^pN^p\lambda^{-p}+M_2^pN^{p/2}C_pK_p\lambda^{-1}+m^{p-1}\sum_{k=1}^m q_k^p)<1,\tag{7}$$

where

$$C_p=(p^{p+1}/2(p-1)^{p-1})^{p/2},$$
$$K_p=(2\lambda(p-1)/(p-2))^{1-p/2}.$$

Proof Define an operator $\upsilon: S\to S$ by $(\upsilon x)(t)=\phi(t)$, for $t\in[\hat{n},0]$, and for $t\geqslant 0$,

$$(\upsilon x)(t)=T(t)\phi(0)+\int_0^t T(t-s)a\cdot$$

$$(s,x(s-\alpha(s)),r(s))\,\mathrm{d}s+$$
$$\int_0^t T(t-s)b(s,x(s-\beta(s)),r(s))\,\mathrm{d}W(s)+$$
$$\sum_{0<t_k<t} T(t-t_k)I_k(x(t_k)):=$$
$$I_1(t)+I_2(t)+I_3(t)+I_4(t). \tag{8}$$

Next, we divide the proof into three steps.

Firstly, we show the p-th moment continuity of υ, for $t\in[0,\infty)$. Let every $x\in S$ and $t_0\geqslant 0$, we have

$$E\|(\upsilon x)(t_0+r)-(\upsilon x)(t_0)\|_X^p\leqslant$$
$$4^{p-1}\sum_{i=1}^4 E\|I_i(t_0+r)-I_i(t_0)\|_X^p. \tag{9}$$

It is easy to know that

$$\lim_{r\to 0}E\|I_i(t_0+r)-I_i(t_0)\|_X^p=0, i=1,2. \tag{10}$$

Moreover, by using the Bukhölder-Davis-Gundy inequality, we have

$$E\|I_3(t_0+r)-I_3(t_0)\|_X^p\leqslant$$
$$2^{p-1}C_pE(\sum_{i=1}^N\int_0^{t_0}\|T(t_0+r-s)-T(t_0-s)\|_X^2\cdot$$
$$\|b(s,x(s-\beta(s)),i)\|^2\,\mathrm{d}s)^{p/2}+$$
$$2^{p-1}C_pE(\sum_{i=1}^N\int_{t_0}^{t_0+r}\|T(t_0+r-s)\|_X^2\cdot$$
$$\|b(s,x(s-\beta(s)),i)\|^2\,\mathrm{d}s)^{p/2}, \tag{11}$$

and it is easy to see

$$E\|I_4(t_0+r)-I_4(t_0)\|_X^p\leqslant$$
$$2^{p-1}M^{2p}(e^{-\lambda r}-1)^p(\sum_{0<t_k<t_0}q_k)^{p-1}\cdot$$
$$\sum_{0<t_k<t_0}q_ke^{-\lambda p(t_0-t_k)}E\|x(t_k)\|_X^p+2^{p-1}M^p\cdot$$
$$E(\sum_{t_0<t_k<t_0+r}q_ke^{-\lambda p(t_0-t_k+r)}\|x(t_k)\|_X)^p. \tag{12}$$

So from (11) and (12) we have

$$\lim_{r\to 0}E\|I_j(t_0+r)-I_j(t_0)\|_X^p=0,\quad j=3,4. \tag{13}$$

Therefore, we know that υ is continuous in p-th moment on $t\in[0,\infty)$.

Secondly, we prove that $\upsilon(S)\subset S$. From (2) we have

$$E\|(\upsilon x)(t)\|_X^p\leqslant 4^{p-1}E\|T(t)\phi(0)\|_X^p+4^{p-1}E\|\cdot$$
$$\int_0^t T(t-s)a(s,x(s-\alpha(s)),r(s))\mathrm{d}s\|_X^p+$$
$$4^{p-1}E\|\int_0^t T(t-s)b(s,x(s-\beta(s)),r(s))\cdot$$
$$\mathrm{d}W(s)\|_X^p+4^{p-1}E\|\sum_{0<t_k<t}T(t-t_k)I_k(x(t_k))\|_X^p:=$$
$$J_1(t)+J_2(t)+J_3(t)+J_4(t), \tag{14}$$

obviously, we get

$$J_1(t)\leqslant 4^{p-1}M^pe^{-\lambda pt}\|\phi(0)\|_X^p\leqslant$$
$$4^{p-1}M^pe^{-\lambda pt}\|\phi\|_C^p\to 0,\ \text{ as }\ t\to\infty. \tag{15}$$

Besides, it follows from Assumptions (B1), (B2) and Hölder’s inequality that

$$J_2(t)\leqslant 4^{p-1}M^pM_1^pE(\sum_{i=1}^N\int_0^t e^{-\lambda(t-s)}\cdot$$
$$\|x(s-\alpha(s))\|_X\,\mathrm{d}s)^p\leqslant$$
$$4^{p-1}M^pM_1^pN^p(1/\lambda)^{(p-1)}\cdot$$
$$\int_0^t e^{-\lambda(t-s)}E\|x(s-\alpha(s))\|_X^p\mathrm{d}s. \tag{16}$$

Based on the definition of S, then for $x(t)\in S$ and any $\varepsilon>0$, there exists some $t_1>0$ such that $E\|x(t-\alpha(t))\|_X^p<\varepsilon$ when $t\geqslant t_1$. Hence, we obtain

$$J_2(t)\leqslant 4^{p-1}M^pM_1^pN^p(1/\lambda)^{(p-1)}\cdot$$
$$\int_0^{t_1}e^{-\lambda(t-s)}E\|x(s-\alpha(s))\|_X^p\,\mathrm{d}s+$$
$$4^{p-1}M^pM_1^pN^p(1/\lambda)^p\varepsilon. \tag{17}$$

It is easy to know that there exists some $t_2\geqslant t_1$, such that for any $t\geqslant t_2$,

$$4^{p-1}M^pM_1^pN^p(1/\lambda)^{(p-1)}e^{-\lambda t}\cdot$$
$$\int_0^{t_1}e^{\lambda s}E\|x(s-\alpha(s))\|_X^p\,\mathrm{d}s\leqslant$$
$$\varepsilon-4^{p-1}M^pM_1^pN^p(1/\lambda)^p\varepsilon, \tag{18}$$

which together with (17) yields that

$$J_2(t)=4^{p-1}E\|\int_0^t T(t-s)a(s,x(s-\alpha(s)),$$
$$r(s))\mathrm{d}s\|_X^p\to 0,\ \text{ as }\ t\to\infty. \tag{19}$$

Furthermore, by using the Bukhölder-Davis-Gundy inequality, the following holds

$$J_3(t)\leqslant 4^{p-1}M^pM_2^pC_pN^{p/2}\cdot$$
$$E(\int_0^t e^{-2\lambda(t-s)}\|x(s-\beta(s))\|_X^2\,\mathrm{d}s)^{p/2}\leqslant$$
$$4^{p-1}M^pM_2^pC_pN^{p/2}K_p\cdot$$
$$\int_0^t e^{-\lambda(t-s)}E\|x(s-\beta(s))\|_X^p\,\mathrm{d}s. \tag{20}$$

It is necessary to note when $p=2$, inequality (20) also holds with $0^0:=1$. Similar to the proof of (19), from (20), it is easy to derive that

$$J_3(t)=4^{p-1}E\|\int_0^t T(t-s)b(s,x(s-\beta(s)),$$
$$r(s))\mathrm{d}W(s)\|_X^p\to 0,\ \text{ as }\ t\to\infty. \tag{21}$$

Using Assumption (B4) and Hölder’s inequality, we get

$$J_4(t) \leqslant 4^{p-1} m^{p-1} M^p \sum_{k=1}^{m} e^{-\lambda p(t-t_k)} .$$

$$q_k^p E\| x(t_k)\|_X^p \to 0, \text{ as } \ t \to \infty. \tag{22}$$

Therefore, from the above discussion, we know that $E\|(\upsilon x)(t)\|_X^p \to 0$ as $t\to\infty$. That is to say, υ maps S into itself.

Next, we verify that υ is contractive. Similar to the above discussion, we obtain for any $x, y \in S$, $\tau \in (0,\infty)$, the following holds

$$E \sup_{t\in[0,\tau]} \|(\upsilon x)(t) - (\upsilon y)(t)\|_X^p \leqslant Q_p E \sup_{t\in[0,\tau]} \| x(t) - y(t)\|_X^p, \tag{23}$$

where

$$Q_p = 3^{p-1} M^p (M_1^p N^p \lambda^{-p} + M_2^p N^{p/2} C_p K_p \lambda^{-1} + m^{p-1} \sum_{k=1}^{m} q_k^p).$$

Form (7) and (23), we know υ is a contraction mapping. Hence, by the contraction mapping principle, we obtain that υ has a unique fixed point $x(t)$ in S, and $x(t)$ is a solution of Eq. (4), meanwhile, when $t\in[\hat{n},0]$, $x(s)=\phi(s)$; and $E\|x(t)\|_X^p \to 0$, as $t\to\infty$.

For the purpose of asymptotic stability, we need to show that the mild solution of Eqs. (1)~(3) is stable in p-th moment. For any fixed $\varepsilon > 0$, we choose

$$0 < \delta < \varepsilon$$

which satisfies the condition

$$4^{p-1} M^p \delta + 4^{p-1} M^p (M_1^p N^p \lambda^{-p} + M_2^p N^{p/2} C_p K_p \lambda^{-1} + m^{p-1} \sum_{k=1}^{m} q_k^p)\varepsilon < \varepsilon .$$

If $x(t) = x(t,0,\phi)$ is a mild solution of system (1)~(3) with $\|\phi\|_C^p < \delta$, then $x(t)$ is defined in (4) and $(\upsilon x)(t)= x(t)$. Next, we show that for any $t \geqslant 0$, $E\|x(t)\|_X^p < \varepsilon$. It is easy to know that $E\|x(t)\|_X^p < \varepsilon$ on $t\in[\hat{n},0]$. Suppose there exists some $\tilde{t}$ such that

$$E\|x(\tilde{t})\|_X^p = \varepsilon,$$

and

$$E\|x(t)\|_X^p < \varepsilon,$$

where

$$t\in[\hat{n},\tilde{t}).$$

Then, from (14), we obtain that

$$E\|x(\tilde{t})\|_X^p \leqslant 4^{p-1} M^p \delta + 4^{p-1} M^p (M_1^p N^p \lambda^{-p} + M_2^p N^{p/2} C_p K_p \lambda^{-1} + m^{p-1} \sum_{k=1}^{m} q_k^p)\varepsilon < \varepsilon,$$

which contradicts the definition of $\tilde{t}$. Therefore, the mild solution of Eqs. (1)~(3) is asymptotically stable in p-th moment. This completes the proof of Theorem 1.

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带马尔科夫跳的脉冲随机时滞偏微分方程的稳定性

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摘要：应用不动点定理讨论了带马尔科夫跳的脉冲随机时滞偏微分方程的适定性解的渐近稳定性，得到一些条件确保了所证结论. 由于考虑了马尔科夫跳，因此文中所得的结论推广了 Sakthivel 等作者所得到的结论.

关键词：脉冲；适定性解；渐近稳定性；马尔科夫跳

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