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Notes on Neumann Problem for Schrödinger Operators in Weighted Lipschitz Domains

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Abstract: Let Ω be a bounded Lipschitz domain in $R^n, n\geq 3$. Let $\omega_\alpha(Q)=|Q-Q_0|^\alpha$, where Q_0 is a fixed point on $\partial\Omega$. For Schrödinger equation $-\Delta u+Vu=0$ in Ω, with singular non-negative potentials V belonging to the reverse Hölder class B_∞, we study the Neumann problem with boundary data in the weighted space $L^2(\partial\Omega,\omega_\alpha \mathrm{d}\sigma)$, where $\mathrm{d}\sigma$ denotes the surface measure on $\partial\Omega$. We show that a unique solution u can be found for the Neumann problem provided $0<\alpha<n-1$. Also proven is that the non-tangential maximal function of ∇u exists in $L^2(\partial\Omega,\omega_\alpha \mathrm{d}\sigma)$.

Key words: Schrödinger equation; Neumann problem; weighted Lipschitz domains

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1 Introduction and main results

In recent years, there has been considerable interest in boundary value problems in non-smooth domains. Let Ω be a bounded Lipschitz domain in $R^n, n\geq 3$. The Dirichlet problem and the Neumann problem with boundary date in $L^p(\partial\Omega)$ for the Laplace's equations in Ω are well understood due to the work of Dahlberg[1,2], Jerison-Kenig[3], Verchota[4] and Dahlberg-Kenig[5]. It was Shen[6] in 1995 that firstly studied the L^p-Neumann problem for the Schrödinger equation $-\Delta u+Vu=0$ in Ω, where $V\in B_\infty$ and $\Omega\subset R^n, n\geq 3$ is the region above a Lipschitz graph. He shows that the Neumann problem exists a unique solution u such that the non-tangential maximal function of ∇u is in $L^p(1<p\leq 2)$. In [7], Tao extends to the Schrödinger equation $-\Delta u+Vu=0$ in Ω, with singular non-negative potentials V belonging to the reverse Hölder class B_n.

Recall that Ω is a bounded Lipschitz domain in $R^n, n\geq 3$. Let $\omega_\alpha=\omega_\alpha(Q)=|Q-Q_0|^\alpha$, where $\alpha>1-n$ and Q_0 is a fixed a point on $\partial\Omega$. In this paper, we initiate the study of solvability of the boundary value problems for Neumann problem for Schrödinger equation $-\Delta u+Vu=0$ in Ω with boundary date in $L^2(\partial\Omega,\omega_\alpha \mathrm{d}\sigma)$, where $\mathrm{d}\sigma$ denotes the surface measure on $\partial\Omega$. We obtain certain ranges of α for which the Neumann problem is uniquely solvable.

More precisely, we consider the Neumann problem

$$\begin{cases} -\Delta u+Vu=0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{v}}=g\in L^2(\partial\Omega,\omega_\alpha), & \text{on } \partial\Omega, \\ \left\|(\nabla u)^*\right\|_{L^2(\partial\Omega,\omega_\alpha)}<\infty. \end{cases} \tag{1}$$

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where $\vec{v}$ denotes the outward unit normal to $\partial\Omega$ and $(\nabla u)^*$ denotes the non-tangential maximal function of ∇u, which is defined by

$$(\nabla u)^*(Q)=\sup\{|\nabla u(X)|: X\in\Omega: |X-Q|<2\delta(X)\}, \quad (2)$$

for $Q\in\partial\Omega$. We remark that in (1), the boundary values are taken in the sense of non-tangential convergence, i.e. $\lim_{X\to Q, X\in\Gamma(Q)}\nabla u(X)\vec{v}(Q)=g(Q)$ for almost every $Q\in\partial\Omega$.

And in (1), $V(X)$ denotes a non-negative locally L^∞ integrable function on R^n, which belongs to B_∞. As it is known, a nonnegative locally L^∞ integrable function $V(X)$ on R^n is said to belong to B_∞ if there exists a positive constant C_q such that the reverse Hölder inequality

$$\|V(X)\|_{L^\infty(B)}\leq\frac{C_q}{|B|}\int_B V(X)\mathrm{d}X, \quad (3)$$

holds for every ball B in R^n [8].

The following is the main results of the paper.

Theorem 1 Let Ω be a bounded Lipschitz domain in $R^n, n\geq 3$. Then given any $g\in L^2(\partial\Omega,\omega_\alpha)$ with $0<\alpha<n-1$, the Neumann problem (1) has a unique solution. Moreover, the solution u satisfies

$$\int_{\partial\Omega}|(\nabla u)^*|^2\,\omega_\alpha\mathrm{d}\sigma\leq C\int_{\partial\Omega}|g|^2\,\omega_\alpha\mathrm{d}\sigma, \quad (4)$$

where C depends only on n, α and the Lipschitz character of Ω.

We remark that the Dirichlet and Neumann problems for Laplace equation $\Delta u=0$ in Lipschitz domain with boundary date in $L^2(\partial\Omega,\omega_\alpha\mathrm{d}\sigma)$ were studied recently in [9].

Throughout this paper we will use C to denote positive constant, which may be different from line to line, and depend at most on n, the Lipschitz constant and the constant in (3). We will use $\|.\|_p$ to denote the norm in $L^p(\partial\Omega)$. For $P\in\partial\Omega$ and $r>0$, we say $B(P,r)\cap\partial\Omega$ is a coordinate patch for $\partial\Omega$, if there exists a Lipschitz function $\varphi: R^{n-1}\to R$ such that, after a rotation of the coordinate system, we have

$$\Omega\cap B(P,r)=\{(X',x_n)\in R: x_n>\varphi(X')\}\cap B(P,r).$$

In this new coordinate system, we let

$$\Delta(P,r)=\{(X',\varphi(X'))\in R^n:|X'-P'|<\rho\},$$
$$D(P,r)=\{(X',x_n)\in R^n:|X'-P'|<\rho, \varphi(X')<x_n<\varphi(X'+\rho)\}.$$

Recall that Ω is a Lipschitsz domain if there exists $r_0=r_0(\Omega)>0$ such that $B(P,r_0)\cap\partial\Omega$ is a coordinate patch for any $P\in\partial\Omega$. Clearly, if $0<\rho<cr_0$, we have $\Delta(P,r)\subset\partial\Omega$ and $D(P,r)\subset\Omega$.

The main ingredients of the proofs of the theorem stated above are (1) the un-weighted L^2 estimates originated in [10], (2) certain localization techniques originated in [5], and (3) the representation formulas in terms of the Green's and Neumann's functions.

The paper is organized as follows. In section 2, we establish the estimates for fundamental solution of the Schrödinger operator $-\Delta+V$ in R^n, and our proofs of the Theorem 1 will be given in section 3.

2 Estimates of fundamental solutions

This section is devoted to the estimates of fundamental solutions for the operator $-\Delta+V$. We will assume that $V\in B_\infty$.

Suppose $-\Delta u+Vu=0$ in $B(x_0,2R)$ for some $x_0\in R^n$, $R>0$. Then let $\Gamma(X,Y)$ denotes the fundamental solution for the Schrödinger operator $-\Delta+V$. Clearly, $\Gamma(X,Y)=\Gamma(Y,X)$. Since $V\geq 0$, it is well know that

$$0\leq\Gamma(X,Y)\leq\Gamma_0(X,Y)=1/(\omega_n(n-2)|X-Y|^{n-2}). \quad (5)$$

Lemma 1[11] Assume $X,Y\in R^n$ and $|X-Y|\leq 2/m(V,X)$. Then

$$|\nabla_X\Gamma(X,Y)-\nabla_X\Gamma_0(X,Y)|\leq\frac{Cm(V,X)^2}{|X-Y|^{n-3}},$$

with the constant C independent of X and Y.

Given $f\in L^p(\partial\Omega)$, $1<p<\infty$, we define the single potential as follow

$$S(f)(X)=\int_{\partial\Omega}\Gamma(X,Q)f(Q)\mathrm{d}\sigma \text{ for } X\in R^n. \quad (6)$$

By using well-known techniques from the theorem of Coifman, et al[12], one can show the following

Lemma.

Lemma 2[6] Let $f\in L^p(\partial\Omega)$, $1<p<\infty$, and $u=S(f)$. Then $\|(\nabla u)^*\|_{L^p(\partial\Omega)}\le C\|f\|_{L^p(\partial\Omega)}$, and for $P\in\partial\Omega$,

$$\frac{\partial u}{\partial X_i}(P)=\frac{1}{2}f(P)v_i(P)+ pv\int_{\partial\Omega}\nabla_p\Gamma(P,Q)f(Q)\mathrm{d}\sigma.$$

The rest of this section is devoted to establishing the estimate for the Neumann function on Ω. Let

$$S(f)(X)=\int_{\partial\Omega}\Gamma(X,Q)f(Q)\mathrm{d}\sigma,$$

$$S_0(f)(X)=\int_{\partial\Omega}\Gamma_0(X,Q)f(Q)\mathrm{d}\sigma,$$

Then

$$\frac{\partial}{\partial v}S(f)=(\frac{1}{2}I+K)f,$$

$$\frac{\partial}{\partial v}S_0(f)=(\frac{1}{2}I+K_0)f.$$

It is known that

$$I/2+K_0:L^2(\partial\Omega)\to L^2(\partial\Omega),$$

is a Fredholm operator with index zero. One can show that $K-K_0$ is a compact operator on $L^2(\partial\Omega)$. It then follows that

$$I/2+K:L^2(\partial\Omega)\to L^2(\partial\Omega),$$

is also a Fredholm operator with index zero. It is easy to see that the operator is one-to-one on $L^2(\partial\Omega)$. So $I/2+K$ is invertible on $L^2(\partial\Omega)$. Hence, the Neumann problem

$$\begin{cases}-\Delta u+Vu=0, & \text{in } \Omega,\\ \partial u/\partial\vec{v}=g, & \text{on } \partial\Omega,\\ \|(\nabla u)^*\|_{L^2(\partial\Omega)}<\infty,\end{cases}$$

is uniquely solvable. For $Y\in\Omega$, we let $v^Y(X)$ be the solution with Neumann date $\frac{\partial}{\partial v}\Gamma(Q,Y)$ and

$$N(X,Y)=\Gamma(X,Y)-v^Y(X). \tag{7}$$

Then

$$\begin{cases}\{-\Delta_X+V(X)N(X,Y)=\delta_Y(X), & \text{in } \Omega,\\ \partial N(Q,Y)/\partial\vec{v}=0, & \text{on } \partial\Omega.\end{cases}$$

The following Lemmas is very important to us.

Lemma 3[11] Let Ω be a bounded Lipschitz domain. Assume $k>0$ be any integer, then

$$|N(X,Y)|\le\frac{C_k}{\{1+m(V,X)|X-Y|\}^k|X-Y|^{n-2}},$$

with the constant C_k independent of X, Y and the diameter of domains Ω.

Lemma 4[10] Let $V\in B_\infty$, and $k>0$ be any integer. Then there exist $0<\delta<1$, and a positive constant C_k such that, for $X,Y,Z\in\overline{\Omega}$ with $|Z-X|\le|X-Y|/10$,

$$|N(X,Y)-N(Z,Y)|\le \frac{C_k}{\{1+m(V,x)\}^k}\frac{|Z-X|^\delta}{|X-Y|^{n-2+\delta}}. \tag{8}$$

We end this section with following inner estimates.

$$|\nabla N(X,Y)|\le \frac{C_k}{\{1+m(V,x)|X-Y|\}^k}\frac{1}{|X-Y|^{n-1}}. \tag{9}$$

3 The proof of the theorem

This section is devoted to the proof of the theorem, using the same technique originated in [10], and Lemma 3, decay estimates of Neumann function (9) we can easily get following Lemma.

Lemma 5 Let u be a solution of Schrödinger equation $-\Delta u+Vu=0$ in Ω such that $(\nabla u)^*\in L^2(\partial\Omega)$ and $\partial u/\partial\vec{v}=f\in L^2(\partial\Omega)$ on $\partial\Omega$. Then for any $Q_0\in\partial\Omega$ and $r>0$,

$$\int_{Q\in\partial\Omega,|Q-Q_0|\geqslant r}|(\nabla u)^*|^2\mathrm{d}\sigma\le C\int_{Q\in\partial\Omega,|Q-Q_0|\geqslant r}|f|^2\mathrm{d}\sigma+ (C/r^\lambda)\int_{Q\in\partial\Omega,|Q-Q_0|<r}|f|^2|Q-Q_0|^\lambda\,\mathrm{d}\sigma, \tag{10}$$

where $0\le\lambda<n-1$.

Proof Fix $Q_0\in\partial\Omega$. There exists $r_0>0$ depending only on the Lipschitz character of Ω such that, after a possible rotation of the coordinate system,

$$\Omega\cap B(Q_0,r_0)=\{(X',x_n)\in R^n:x_n> \psi(X')\}\cap B(Q_0,r_0), \tag{11}$$

where $\psi:R^{n-1}\to R$ is Lipschitz continuous. We may assume that $\psi(0)=0$, $Q_0=(0,0)$.

Let $\Delta_r=\{(X',\psi(X'):|X'|\le r\}$. We now write $f=g+h$, where $g=f\chi_{\Delta_{8r}}, h=f\chi^c_{\Delta_{8r}}$. Let

$$u_1(X)=\int_{\partial\Omega}N(X,Q)g(Q)\mathrm{d}\sigma, \tag{12}$$

and

$$u_2(X)=\int_{\partial\Omega}N(X,Q)h(Q)\mathrm{d}\sigma, \tag{13}$$

where u_1 and u_2 are the solutions of the L^2 Neumann problem with boundary date g and h respectively. Then $u=u_1+u_2$. By the L^2 estimate[11],

$$\int_{\partial\Omega\setminus\Delta_{8r}}|(\nabla u_2)^*|^2\,\mathrm{d}\sigma\le C\int_{\partial\Omega}|h|^2\,\mathrm{d}\sigma\le C\int_{\partial\Omega\setminus\Delta_{8r}}|f|^2\,\mathrm{d}\sigma. \qquad (14)$$

Next, we will show that, for some λ,

$$\int_{Q\in\partial\Omega,|Q|\geqslant 8r}|(\nabla u_1)^*|^2\,\mathrm{d}\sigma\le \frac{C}{r^\lambda}\int_{Q\in\partial\Omega,|Q|<8r}|f|^2|Q|^\lambda\,\mathrm{d}\sigma, \qquad (15)$$

Clearly, estimate (10) follows from (14) and (15).

To estimate $(\nabla u)^*$ on $\partial\Omega\setminus\Delta_{8r}$, first we note that $\partial u_1/\partial\vec{v}=0$ on $\partial\Omega\setminus\Delta_{8r}$. Also for $X\in\Omega$ and $\mathrm{dist}(X,\Delta_r)\ge cr$, in view of estimate (12), and Lemma 3, we have

$$|u_1(X)|\le\int_{\partial\Omega}|N(Q,X)||g(Q)|\,\mathrm{d}\sigma\le \int_{\partial\Omega}\frac{C_k|g(Q)|}{\{1+m(V,X)|X-Q|\}^k|X-Q|^{n-2}}\mathrm{d}\sigma\le \int_{\Delta_{8r}}\frac{C_k}{|X-Q|^{n-2}}|f|\,\mathrm{d}\sigma\le \frac{C_kC}{|X|^{n-2}}\int_{\Delta_{8r}}|f|\,\mathrm{d}\sigma. \qquad (16)$$

Here $C=(c-1)/c$, the last inequality followed because $|X|>cr>c|Q|$, and $|X-Q|>|X|-|Q|>(c-1)/c|X|$. Let $E_j=\Delta_{2^jr}\setminus\Delta_{2^{j-1}r}$, where $4\le j\le J$ and $2^Jr\sim r_0$. For $Q\in E_j$, let

$$M_1(F)(Q)=\sup\{|F(X)|: X\in\gamma(Q)\ \text{and}\ |X-Q|\le\theta 2^jr\}, \qquad (17)$$

$$M_2(F)(Q)=\sup\{|F(X)|: X\in\gamma(Q)\ \text{and}\ |X-Q|\ge\theta 2^jr\}, \qquad (18)$$

where $\gamma(Q)=\{X\in\Omega: |X-Q|<2\mathrm{dist}(X,\partial\Omega)\}$ and θ is chosen so that for $Q\in E_j$, $M_1(F)(Q)$ is less than the non-tangential maximal function of F. Clearly $(\nabla u_1)^*\le M_1(\nabla u_1)+M_2(\nabla u_1)$.

Note that if $X\in\gamma(Q)$ and $|X-Q|\ge\theta 2^jr$, we now use the interior estimate and (9) to obtain

$$|\nabla u_1(X)|\le\int_{\partial\Omega}|\nabla N(X,Q)||g(Q)|\,\mathrm{d}\sigma\le \int_{\partial\Omega}\frac{C_k}{\{1+m(V,x)|X-Q|\}^k}\frac{|g(Q)|}{|X-Q|^{n-1}}\mathrm{d}\sigma\le$$
$$C\int_{\partial\Omega}\frac{C_k}{|X-Q|^{n-1}}|g(Q)|\,\mathrm{d}\sigma\le \frac{C_kC}{(2^jr)^{n-1}}\int_{\Delta_{8r}}|f|\,\mathrm{d}\sigma\le \frac{C_kC}{(2^jr)^{n-1}}\int_{\Delta_{8r}}|f||Q|^{\frac{\lambda}{2}}\frac{1}{|Q|^{\frac{\lambda}{2}}}\mathrm{d}\sigma\le$$
$$\frac{C_kC}{(2^jr)^{n-1}}\{\int_{\Delta_{8r}}|f|^2|Q|^\lambda\,\mathrm{d}\sigma\}^{\frac12}\{\int_{\Delta_r}\frac{1}{|Q|^\lambda}\mathrm{d}\sigma\}^{\frac12}\le \frac{C_kC}{(2^jr)^{n-1}}\{\int_{\Delta_{8r}}|f|^2|Q|^\lambda\,\mathrm{d}\sigma\}^{\frac12}\{\int_0^r t^{n-2-\lambda}\mathrm{d}t\}^{\frac12}\le \frac{C_kCr^{\frac{n-\lambda-1}{2}}}{(2^jr)^{n-1}}\{\int_{\Delta_{8r}}|f|^2|Q|^\lambda\,\mathrm{d}\sigma\}^{\frac12},$$

where $0\le\lambda<n-1$. It follows that

$$\int_{E_j}|M_2(\nabla u_1)|^2\,\mathrm{d}\sigma\le \frac{C_kCr^{n-\lambda-1}}{(2^jr)^{2(n-1)}}|E_j|\int_{\Delta_{8r}}|f|^2|Q|^\lambda\,\mathrm{d}\sigma\le \frac{C_kC}{(2^j)^{n-1}r^\lambda}\int_{\Delta_{8r}}|f|^2|Q|^\lambda\,\mathrm{d}\sigma. \qquad (19)$$

For $M_1(\nabla u_1)$ on E_j, we use the L^2 estimate[11] on Lipschitz domain $D_\tau\setminus D_{\frac{\tau}{4}}$, where $D_\tau=\Omega\cap B_\tau$ and $\tau\in(2^jr,2^{j+1}r)$, to obtain

$$\int_{E_j}|M_1(\nabla u_1)|^2\,\mathrm{d}\sigma\le\int_{E_j}|\frac{\partial u_1}{\partial\vec{v}}|^2\mathrm{d}\sigma\le \int_{\Omega\cap\partial(D_\tau\setminus D_{\tau/4})}|\nabla u_1|^2\mathrm{d}\sigma. \qquad (20)$$

Integrating both sides of (20) in $\tau\in(2^jr,2^{j+1}r)$ then yields

$$\int_{E_j}|M_1(\nabla u_1)|^2\mathrm{d}\sigma\le\frac{C}{2^jr}\int_{D_{2^{j+1}r}\setminus D_{2^{j-2}r}}|\nabla u_1|^2\mathrm{d}X, \qquad (21)$$

$$\int_{E_j}|M_1(\nabla u_1)|^2\mathrm{d}\sigma\le\frac{C}{(2^jr)^3}\int_{D_{2^{j+2}r}\setminus D_{2^{j-3}r}}|u_1|^2\mathrm{d}X, \qquad (22)$$

where the second inequality follows from the Cacciopoli inequality. This, together with (16), gives

$$\int_{E_j}|M_1(\nabla u_1)|^2\mathrm{d}\sigma\le \frac{C}{(2^j)^{n-1}r^\lambda}\int_{\Delta_{8r}}|f(Q)|^2|Q|^\lambda\,\mathrm{d}\sigma. \qquad (23)$$

It follows from estimates (19) and (23) that

$$\int_{E_j}|(\nabla u_1)^*|^2\mathrm{d}\sigma\le$$

$$2\int_{E_j}\{|M_1(\nabla u_1)|^2+|M_2(\nabla u_1)|^2\}\mathrm{d}\sigma\le \frac{C}{(2^j)^{n-1}r^{\lambda}}\int_{\Delta_{8r}}|f(Q)|^2|Q|^{\lambda}\mathrm{d}\sigma, \tag{24}$$

by summation we can get

$$\int_{\Delta_{c_0r_0}\setminus\Delta_{8r}}|(\nabla u_1)^*|^2\mathrm{d}\sigma\le\frac{C}{r^{\lambda}}\int_{\Delta_{8r}}|f(Q)|^2|Q|^{\lambda}\mathrm{d}\sigma. \tag{25}$$

By covering technique we can also show that

$$\int_{\partial\Omega\setminus\Delta_{c_0r_0}}|(\nabla u_1)^*|^2\mathrm{d}\sigma\le\frac{C}{r^{\lambda}}\int_{\Delta_{8r}}|f(Q)|^2|Q|^{\lambda}\mathrm{d}\sigma. \tag{26}$$

By(25) and (26), we get

$$\int_{\partial\Omega\setminus\Delta_{8r}}|(\nabla u_1)^*|^2\mathrm{d}\sigma\le\frac{C}{r^{\lambda}}\int_{\Delta_{8r}}|f(Q)|^2|Q|^{\lambda}\mathrm{d}\sigma. \tag{27}$$

Thus we have

$$\int_{\partial\Omega\setminus\Delta_{8r}}|(\nabla u)^*|^2\mathrm{d}\sigma\le 2\int_{\partial\Omega\setminus\Delta_{8r}}|(\nabla u_1)^*|^2\mathrm{d}\sigma+ 2\int_{\partial\Omega\setminus\Delta_{8r}}|(\nabla u_2)^*|^2\mathrm{d}\sigma\le \int_{\partial\Omega\setminus\Delta_{8r}}|f|^2\mathrm{d}\sigma+\frac{C}{r^{\lambda}}\int_{\Delta_{8r}}|f(Q)|^2|Q|^{\lambda}\mathrm{d}\sigma.$$

By scaling, estimate (10) follows easily. The proof is complete.

Remark 1 It follows from (10) that, if $0<\lambda<n-1$,

$$\int_{Q\in\partial\Omega,|Q-Q_0|>r}|(\nabla u)^*|^2\mathrm{d}\sigma\le C_{\lambda}\int_{\partial\Omega}\{\frac{|Q-Q_0|}{|Q-Q_0|+r}\}^{\lambda}|f|^2\mathrm{d}\sigma. \tag{28}$$

Now, it is in the position to give the proof of theorem. It will follow from (28), as well as the solvability of the L^p Neumann problem for $1<p<2$ [5]. Indeed, by Hölder inequality, we have

$$\int_{\partial\Omega}|f|^2\,\omega_{\alpha}\mathrm{d}\sigma\le (\int_{\partial\Omega}|f|^p\mathrm{d}\sigma)^{\frac{p}{2}}(\int_{\partial\Omega}\omega_{\alpha}^{\frac{p}{2-p}}\mathrm{d}\sigma)^{1-\frac{p}{2}}\le C\|f\|_p^{\frac{p^2}{2}},$$

the last inequality holds because $\int_{\partial\Omega}\omega_{\alpha}^{\frac{p}{2-p}}\mathrm{d}\sigma<\infty$, if we choose some $p=p(\alpha)\in(1,2)$. So we have $L^2(\partial\Omega,\omega_{\alpha}\mathrm{d}\sigma)\subset L^p(\partial\Omega)$ for some $p=p(\alpha)\in(1,2)$, the uniqueness follows directly from the uniqueness in L^p. To prove the existence, we fix $g\in L^2(\partial\Omega,\omega_{\alpha}\mathrm{d}\sigma)$, where $\omega_{\alpha}(Q)=|Q-Q_0|^{\alpha}$. Let u be the solution of Schrödinger equation in Ω such that $(\nabla u)^*\subset L^p(\partial\Omega)$ and $\partial u/\partial\vec{v}=g$ on $\partial\Omega$. We need to show that

$$\|(\nabla u)^*\|_{L^2(\partial\Omega,\omega_{\alpha}\mathrm{d}\sigma)}\le C\|g\|_{L^2(\partial\Omega,\omega_{\alpha}\mathrm{d}\sigma)}. \tag{29}$$

To this end, we let

$$g_j(Q)=\begin{cases}g(Q),\text{for }Q\in\partial\Omega\setminus B(Q_0,1/j),\\ g_{\partial\Omega\cap B(Q_0,\frac{1}{j})},\text{for }Q\in\partial\Omega\cap B(Q_0,1/j).\end{cases}$$

It is easy to see that $g_j\in L^2(\partial\Omega)$, $g_j\to g$ in $L^2(\partial\Omega,\omega_{\alpha}\mathrm{d}\sigma)$ as $j\to\infty$.

Let u_j be a solution of Schrödinger equation in Ω such that $(\nabla u_j)^*\in L^2(\partial\Omega)$ and $\partial u_j/\partial\vec{v}=g_j$ on $\partial\Omega$. Choose $\lambda\in(\alpha,n-1)$. We multiply both sides of (28) by $r^{\alpha-1}$ and integrate the resulting inequality in $r\in(0,\infty)$. This gives

$$\int_{\partial\Omega}|(\nabla u_j)^*|^2|Q-Q_0|^{\alpha}\mathrm{d}\sigma\le C\int_0^{\infty}r^{\alpha-1}\{\int_{\partial\Omega\setminus B(Q_0,r)}|(\nabla u_j)^*|^2\mathrm{d}\sigma\}\mathrm{d}r\le C\int_0^{\infty}r^{\alpha-1}\{\int_{\partial\Omega}\left(\frac{|Q-Q_0|}{|Q-Q_0|+r}\right)^{\lambda}\cdot |g_j(Q)|^2\mathrm{d}\sigma\}\mathrm{d}r\le C\int_{\partial\Omega}|g_j(Q)|^2\cdot \{\int_0^{\infty}r^{\alpha-1}\left(\frac{|Q-Q_0|}{|Q-Q_0|+r}\right)^{\lambda}\mathrm{d}r\}\mathrm{d}\sigma\le C\int_{\partial\Omega}|g_j(Q)|^2|Q-Q_0|^{\alpha}\mathrm{d}\sigma. \tag{30}$$

Finally, we can proof that since $g_j\to g$ in $L^p(\partial\Omega)$ for some $p>1$, then we have $(\nabla u_j)^*\to(\nabla u)^*$ in $L^p(\partial\Omega)$. Thus there exists a subsequence $\{u_j\}$ such that $(\nabla u_{j_k})^*\to(\nabla u)^*$ a.e. on $\partial\Omega$. This, together with (30) and Fatou's Lemma, gives the desired estimate (29). The proof is completed.

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Lipschitz 区域上 Schrödinger 算子 Neumann 问题的讨论

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摘要： $\Omega \in R^n, n \geq 3$ 是一个有界 Lipschitz 区域. 令 $\omega_\alpha(Q) = |Q - Q_0|^\alpha$，其中 Q_0 是边界 $\partial\Omega$ 上的一个固定点. 对带有非负奇异位势的 Schrödinger 方程 $-\Delta u + Vu = 0$，$V \in B_\infty$，研究了边值在 $L^2(\partial\Omega, \omega_\alpha \mathrm{d}\sigma)$ 中的 Neumann 问题，证明了当 $0 < \alpha < n-1$ 时, Neumann 问题存在唯一解，并且 $(\nabla u)^* \in L^2(\partial\Omega, \omega_\alpha \mathrm{d}\sigma)$.

关键词: Schrödinger 算子; Neumann 问题; 加权 Lipschitz 区域

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