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On the Coupled of NBEM and FEM for an Anisotropic Quasilinear Problem in Elongated Domains*

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ABSTRACT

In this paper, based on the Kirchhoff transformation, the coupling of natural boundary element method and finite element method are discussed for solving exterior anisotropic quasilinear problems with elliptic artificial boundary. By the principle of the natural boundary reduction, we obtain natural integral equation on elliptic artificial boundaries, the coupled variational problem and its numerical method. Moreover, the convergence and error estimate of the approximate solutions are obtained. Finally, some numerical examples are presented to illuminate the feasibility of the method.

Keywords: Quasilinear Elliptic Equation; Elliptic Artificial Boundary; Natural Integral Equation

1. Introduction

Based on the Green's function and Green's formula, natural boundary element method (NBEM) reduces the boundary value problem of partial differential equation into a hypersingular integral equation on the boundary, and then solves the latter numerically [1,2]. It has advantages over the usual boundary reduction methods: such as the diminution of the number of space dimensions by 1, the conservation of energy functional, the preservation of self-adjointness and coerciveness. But it also has evident limitations, it's difficult to obtain Green's functions for solving problem in general domains. Therefore, the coupling of NBEM which is also called artificial boundary condition [3,4] or DtN method [5,6] and finite element method (FEM) [2] is useful and necessary for general cases.

The standard procedure of the coupling method can be described as follows. We introduce an artificial boundary to divide the original domain into two subregions, a bounded inner region and an unbounded one with a special boundary, such as circle, ellipse, and spherical surface, on which the boundary element method and finite element method are used respectively. This technique has been used to solve many linear problems [1,2,4-6] and it has also been successfully generalized to solve nonlinear boundary value problems [7-9] or quasilinear problems [3,10,11]. The problems were discussed in [3,10,11] take circle as artificial boundary, but for the problems with elongated domains, an elliptic boundary that leads to a smaller computational domain is obviously better than the circle one. The purpose of the paper is to study the coupling of NBEM and FEM to solve the anisotropic quasilinear problems with an elliptic artificial boundary.

Let Ω be a elongated, bounded and simple connected domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega = \Gamma_0$. $\Omega^c = \mathbb{R}^2/\bar{\Omega}$. We consider the numerical solution of the exterior anisotropic quasilinear problem

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(\mathrm{x},u)\dfrac{\partial u}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\beta a(\mathrm{x},u)\dfrac{\partial u}{\partial y}\right)\right)=f, & \text{in } \Omega^c, \\ u=0, & \text{on } \Gamma_0, \\ u(\mathbf{x})=O(1), & \text{as } |\mathbf{x}|\to\infty, \end{cases} \tag{1.1}$$

With $\beta > \alpha > 0$ or $\alpha = \beta = 1$, $\mathbf{x} = (x, y)$, $a(\cdot,\cdot)$ and f are given functions which will be ranked as below. Following [3,12], suppose that the given function $a(\cdot,\cdot)$ satisfies

$$0 < C_0 \le a(\mathbf{x},u) \le C_1, \tag{1.2}$$

$\forall u \in \mathbb{R}$, and for almost all $\mathbf{x} \in \Omega^c$, where two positive constants $C_0, C_1 \in \mathbb{R}$, and

$$|a(\mathbf{x},u) - a(\mathbf{x},v)| \le C_L |u - v|, \tag{1.3}$$

$\forall u, v \in \mathbb{R}$, and for almost all $\mathbf{x} \in \Omega^c$, with a constant $C_L > 0$. We also assume that $\partial a/\partial s$, $\partial^2 a/\partial s^2$ are con-

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tinuous. In the following, we suppose that the function $f \in L^2(\Omega^c)$ has compact support, *i.e.*, there exists a constant $\mu_0 > 0$, such that

$$\operatorname{supp} f \subset \Omega_{\mu_0} = \left\{ \mathbf{x} \in \mathbb{R}^2 \middle| |\mathbf{x}| \le \mu_0 \right\}. \tag{1.4}$$

We also assume that

$$a(\mathbf{x},u) = a_0(u), \text{ when } |\mathbf{x}| \ge \mu_0. \tag{1.5}$$

Now, we introduce an elliptic artificial boundary

$$\Gamma_{\mu_1} = \left\{ (\mu,\phi) \middle| \mu = \mu_1 > \mu_0, 0 \le \phi \le 2\pi \right\}.$$

Γ_{μ_1} divide Ω^c into two regions, a bounded domain Ω_i and an unbounded domain Ω_e with elliptic artificial boundary. Then the problem (1.1) can be rewritten in the coupled form:

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(\mathbf{x},u) \dfrac{\partial u}{\partial x} \right) + \dfrac{\partial}{\partial y}\left(\beta a(\mathbf{x},u) \dfrac{\partial u}{\partial y} \right) \right) = f, & \text{in } \Omega_i, \\ u = 0, & \text{on } \Gamma_0, \end{cases} \tag{1.6}$$

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(\mathbf{x},u) \dfrac{\partial u}{\partial x} \right) + \dfrac{\partial}{\partial y}\left(\beta a(\mathbf{x},u) \dfrac{\partial u}{\partial y} \right) \right) = 0, & \text{in } \Omega_e, \\ u(\mathbf{x}) = O(1), & \text{when } |\mathbf{x}| \to \infty, \end{cases} \tag{1.7}$$

$$u(\mathbf{x}) \text{ and } \alpha a_0(u) n_x \frac{\partial u}{\partial x} + \beta a_0(u) n_y \frac{\partial u}{\partial y} \text{ are continuous on } \Gamma_{\mu_1}. \tag{1.8}$$

where $\boldsymbol{n} = (n_x, n_y)$ is the unit exterior normal vector on Γ_{μ_1}. Particularly, when $a(\mathbf{x},u) = c$ which is independent of $\mathbf{x}$ and u, [13-15] have obtained the natural integral equation. We introduce the so-called Kirichhoff transformation [16]

$$w(\mathbf{x}) = \int_0^{u(\mathbf{x})} a_0(\zeta)\,\mathrm{d}\zeta, \text{ for } \mathbf{x} \in \Omega_e, \tag{1.9}$$

then we have

$$\nabla w = a_0(u) \nabla u. \tag{1.10}$$

and

$$\begin{aligned} &\left(\alpha \frac{\partial w}{\partial x}, \beta \frac{\partial w}{\partial y} \right) \\ &= \left(\alpha a_0(u) \frac{\partial u}{\partial x}, \beta a_0(u) \frac{\partial u}{\partial y} \right). \end{aligned} \tag{1.11}$$

From equation (1.7) we have that w satisfies the following problem

$$\begin{cases} -\left(\alpha \dfrac{\partial^2 w}{\partial x^2} + \beta \dfrac{\partial^2 w}{\partial y^2} \right) = 0, & \text{in } \Omega_e, \\ w(\mathbf{x}) = O(1), & \text{when } |\mathbf{x}| \to \infty. \end{cases} \tag{1.12}$$

The rest of the paper is organized as follows. In Section 2, we obtain the natural integral equation for elliptic unbounded domain cases. In Section 3, we give the equivalent variational problems and the finite element approximations. The reduced problem's well-posedness, the convergence results and error estimate are also discussed. At last, in Section 4, we present some numerical examples to illuminate the efficiency and feasibility of our method.

2. Natural Boundary Reduction

In this section, by virtue of the Poisson integral formula and natural integral equation for the linear problem, we shall obtain the corresponding results for the quasilinear problem in Ω^c. For this purpose, we need to discuss some properties between elliptic coordinates (μ,ϕ) and Cartesian coordinates (x,y) first. The relationship between the two coordinates can be expressed as below

$$\begin{cases} x = f_0 \cosh\mu \cos\phi, \\ y = f_0 \sinh\mu \sin\phi, \end{cases} \tag{2.1}$$

where $f_0 = \sqrt{a^2 - b^2}$, $a = f_0 \cosh\mu_1$, $b = f_0 \sinh\mu_1$. Following from [15], we have

Theorem 2.1 The transformation between elliptic coordinates and Cartesian coordinates (2.1) possesses the following property.

1) The Jacobi determinant of Equation (2.1) is

$$\begin{aligned} J &= f_0^2 \cosh^2\mu \sin^2\phi + f_0^2 \sinh^2\mu \cos^2\phi \\ &= f_0^2 \left(\cosh^2\mu - \cos^2\phi \right), \end{aligned} \tag{2.2}$$

$J = 0$ if and only if $(x,y) = (\pm f_0, 0)$;

2)

$$\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \phi^2} = J\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{2.3}$$

for $u \in C^2(\mathbb{R}^2)$;

3) For the exterior domain Ω_e

$$\frac{\partial u}{\partial \nu} = -\frac{1}{\sqrt{J}}\frac{\partial u}{\partial \mu}, \tag{2.4}$$

where ν refers to the unit exterior normal vector on Γ_{μ_1} (regarded as the inner boundary of Ω_e).

Proof The conclusions 1 and 2 can be obtained by direct computation. And 3 follows from the property

$$\boldsymbol{\upsilon} = -\frac{1}{\sqrt{J}}\left(f_0 \sinh\mu \cos\phi, f_0 \cosh\mu \sin\phi\right).$$

2.1. Natural Integral Equation for *α* = *β* = 1

Assume that $w(\mathbf{x})$ is the solution of the problem (1.12), and the value $w|_{|\mu|=\mu_1}$ is given, namely

$$w|_{|\mu|=\mu_1} = w_0(\phi).$$

Then based on the natural boundary reduction, there are the Poisson integral formulas

$$w(\mu,\phi) = \frac{e^{2\mu}-e^{2\mu_1}}{2\pi}\int_0^{2\pi}\frac{w_0(\phi')}{e^{2\mu}+e^{2\mu_1}-2e^{\mu+\mu_1}\cos(\phi-\phi')}d\phi', \quad \mu>\mu_1, \tag{2.5}$$

or

$$w(\mu,\phi) = \frac{1}{\pi}\sum_{n=1}^{\infty}e^{n(\mu_1-\mu)}\int_0^{2\pi}\cos n(\phi-\phi')w_0(\phi') + \frac{1}{2\pi}\int_0^{2\pi}w_0(\phi')d\phi', \quad \mu>\mu_1. \tag{2.6}$$

And the natural integral equation

$$\frac{\partial w}{\partial n} = \frac{1}{\sqrt{J_0}}\left[-\frac{1}{4\pi\sin^2\frac{\phi}{2}}\times w_0(\phi)\right], \quad \mu=\mu_1, \tag{2.7}$$

or

$$\frac{\partial w}{\partial n} = \frac{1}{\pi\sqrt{J_0}}\sum_{n=1}^{\infty}n\int_0^{2\pi}\cos n(\phi-\phi')w_0(\mu_1,\phi')d\phi, \quad \mu=\mu_1, \tag{2.8}$$

the definition of J_0 can be found in the following. The Poisson integral formulas (2.5) and (2.6) and the natural integral Equations (2.7) and (2.8) can also be expressed in the Fourier series forms

$$w(\mu,\phi) = \sum_{n=-\infty}^{+\infty}a_n e^{|n|(\mu_1-\mu)+in\phi}, \quad \mu>\mu_1, \tag{2.9}$$

$$\frac{\partial w}{\partial n} = \frac{1}{\sqrt{J_0}}\sum_{n=-\infty}^{+\infty}|n|a_n e^{in\phi}, \quad \mu=\mu_1, \tag{2.10}$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi}w_0(\phi')e^{-in\phi'}d\phi'$, $i=\sqrt{-1}$ and

$$J_0 = f_0^2\left(\cosh^2\mu_1\sin^2\phi + \sinh^2\mu_1\cos^2\phi\right) = f_0^2\left(\cosh^2\mu_1 - \cos^2\phi\right).$$

From (1.10), we obtain

$$\frac{\partial w}{\partial n} = a_0(u)\frac{\partial u}{\partial n}. \tag{2.11}$$

Combining (1.9), (2.10) and (2.11), we get the exact artificial boundary condition of u on Γ_{μ_1},

$$\begin{aligned}\left.\left(a_0(u)\frac{\partial u(\mu,\phi)}{\partial n}\right)\right|_{\mu=\mu_1} &= \frac{1}{\sqrt{J_0}}\sum_{n=-\infty}^{+\infty}|n|a_n e^{in\phi}\\ &= \frac{1}{\pi\sqrt{J_0}}\sum_{n=1}^{+\infty}n\int_0^{2\pi}\cos n(\phi-\phi')\left(\int_0^{u(\mu_1,\phi)}a_0(\mathbf{y})d\mathbf{y}\right)d\phi\\ &= K_1(u(\mu_1,\phi)),\end{aligned} \tag{2.12}$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{u(\mu_1,\phi')}a_0(\mathbf{y})d\mathbf{y}\right)e^{-in\phi'}d\phi'$, $i=\sqrt{-1}$, $J_0 = f_0^2\left(\cosh^2\mu_1 - \cos^2\phi\right)$. Then by (1.6)-(1.8) and (2.12), the original problem with $\alpha=\beta=1$ confines in Ω_i can be defined as follows

$$\begin{cases}-\nabla\cdot\left(a(\mathbf{x},u)\nabla u\right) = f, & \text{in } \Omega_i,\\ u=0, & \text{on } \Gamma_0,\\ a_0(u)\frac{\partial u}{\partial n} = K_1(u(\mu_1,\phi)), & \text{on } \Gamma_{\mu_1}.\end{cases} \tag{2.13}$$

Therefore, the solution of problem (2.13) is the solution of the problem (1.1) with $\alpha=\beta=1$ confining in the bounded domain Ω_i.

2.2. Natural Integral Equation for *β* > *α* > 0

Now we assume that Γ_{μ_1} can be expressed in the form: $\Gamma_{\mu_1} = \left\{(x,y)\,|\,px^2+qy^2=R^2\right\}$, with $\beta q > \alpha p > 0$. We also assume that $w(\mathbf{x})$ is the solution of the problem (1.12), and the value $w|_{|\mu|=\mu_1}$ is given, namely

$$w|_{|\mu|=\mu_1} = w_0(\phi).$$

Let $x=\sqrt{\alpha}\xi$, $y=\sqrt{\beta}\eta$, then the boundary Γ_{μ_1} is changed by the elliptic boundary

$$\tilde{\Gamma} = \left\{(\xi,\eta)\,\middle|\,\alpha p\xi^2+\beta q\eta^2=R^2\right\},$$

the unit exterior normal vector on $\tilde{\Gamma}$ is

$$\boldsymbol{\nu} = -\left(\sqrt{\alpha p}\cos\theta, \sqrt{\beta q}\sin\theta\right)\Big/\sqrt{\alpha p\cos^2\theta+\beta q\sin^2\theta}.$$

By the above transformation, the problem (1.12) changes into

$$\begin{cases} -\left(\dfrac{\partial^2 w}{\partial \xi^2}+\dfrac{\partial^2 w}{\partial \eta^2}\right)=0, & \text{in } \tilde{\Omega}_e, \\ w(\mathbf{x})=O(1), & \text{when } |\mathbf{x}|\to\infty. \end{cases} \tag{2.14}$$

This is the right problem we talked in Section 2.1. Similar with Equation (2.1), we let

$$\xi = f_0 \cosh\mu \cos\phi, \eta = f_0 \sinh\mu \sin\phi,$$

where

$$f_0 = \sqrt{\frac{\beta q-\alpha p}{\alpha p\beta q}}R,$$

$$\mu_1 = ln\left(\frac{\sqrt{\beta q}+\sqrt{\alpha p}}{\sqrt{\beta q-\alpha p}}\right).$$

Then just the same as the problem discussed in Section 2.1, we have the natural integral equation on Γ_{μ_1}

$$\begin{aligned} &\alpha n_x \frac{\partial w}{\partial x}+\beta n_y \frac{\partial w}{\partial y} \\ &= -\sqrt{\frac{\alpha p\beta q}{p\cos^2\phi+q\sin^2\phi}}\left[\frac{1}{4\pi R\sin^2\dfrac{\phi}{2}}\times w_0(\phi)\right], \end{aligned} \tag{2.15}$$

where $\boldsymbol{n}=(n_x,n_y)=(x/R,y/R)$ is the unit exterior normal vector on Γ_{μ_1}. From (1.11), we obtain

$$\begin{aligned} &\alpha n_x \frac{\partial w}{\partial x}+\beta n_y \frac{\partial w}{\partial y} \\ &= \alpha n_x a_0(u)\frac{\partial u}{\partial x}+\beta n_y a_0(u)\frac{\partial u}{\partial y}. \end{aligned} \tag{2.16}$$

Combining (1.9), (2.15) and (2.16), we obtain the exact artificial boundary condition of u on Γ_{μ_1},

$$\begin{aligned} &\left(\alpha n_x a_0(u)\frac{\partial u}{\partial x}+\beta n_y a_0(u)\frac{\partial u}{\partial y}\right)\bigg|_{\mu=\mu_1} \\ &= -\sqrt{\frac{\alpha p\beta q}{p\cos^2\phi+q\sin^2\phi}} \\ &\cdot\left[\frac{1}{4\pi R\sin^2\dfrac{\phi}{2}}*\left(\int_0^{u(\mu_1,\phi)} a_0(\mathbf{y})\mathrm{d}\mathbf{y}\right)\right] \\ &= \sqrt{\frac{\alpha p\beta q}{p\cos^2\phi+q\sin^2\phi}}\sum_{n=1}^{+\infty}\frac{n}{\pi R} \\ &\cdot\int_0^{2\pi}\cos n(\phi-\phi')\left(\int_0^{u(\mu_1,\phi)} a_0(\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\phi \\ &= K_1(u(\mu_1,\phi)). \end{aligned} \tag{2.17}$$

Then by (1.6)-(1.8) and (2.17), the original problem with $\beta>\alpha>0$ confines in Ω_i can be defined as follows

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(\mathbf{x},u)\dfrac{\partial u}{\partial x}\right)\dfrac{\partial}{\partial y}\left(\beta a(\mathbf{x},u)\dfrac{\partial u}{\partial y}\right)\right)=f, & \text{in } \Omega_i, \\ u=0, & \text{on } \Gamma_0, \\ \alpha n_x a_0(u)\dfrac{\partial u}{\partial x}+\beta n_y a_0(u)\dfrac{\partial u}{\partial y}=K_1(u(\mu_1,\phi)), & \text{on } \Gamma_{\mu_1}. \end{cases} \tag{2.18}$$

Therefore, the solution of problem (2.18) is the solution of the problem (1.1) with $\beta>\alpha>0$ confining in the bounded domain Ω_i.

3. Variational Problem and Finite Element Approximation

3.1. The Equivalent Variational Problems

Now we consider the problems (2.13) and (2.18). We shall use $W^{m,p}$ denoting the standard Sobolev spaces, $\|\cdot\|$ and $|\cdot|$ referring to the corresponding norms and semi-norms. Especially, we define $H^m(\Omega)=W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega}=\|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,\Omega}=|\cdot|_{m,2,\Omega}$. Let us introduce the space

$$V=\left\{v\in H^1(\Omega_i)\ v|_{\Gamma_0}=0\right\}, \tag{3.1}$$

and the corresponding norms

$$\|v\|_{0,\Omega_i}=\sqrt{\int_{\Omega_i}|v|^2\mathrm{d}\mathbf{x}}, \|v\|_{1,\Omega_i}=\sqrt{\int_{\Omega_i}\left(|v|^2+|\nabla v|^2\right)\mathrm{d}\mathbf{x}}.$$

The boundary value problems (2.13) and (2.18) are equivalent to the following variational problem

$$\begin{cases} \text{Find } u\in V, \text{ such that} \\ D(u;u,v)+\hat{D}(u;u,v)=F(v),\ \forall v\in V, \end{cases} \tag{3.2}$$

where

$$D(w;u,v)=\int_{\Omega_i} a(\mathbf{x},w)\left(\alpha\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\beta\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}\mathbf{x}, \tag{3.3}$$

$$\begin{aligned} \hat{D}(w;u,v) &= \sum_{n=1}^{+\infty}\frac{\sqrt{\alpha\beta}}{n\pi}\int_0^{2\pi}\int_0^{2\pi} a_0(w(\mu_1,\phi')) \\ &\cdot\frac{\partial u(\mu_1,\phi')}{\partial\phi'}\frac{\partial v(\mu_1,\phi)}{\partial\phi}\cos n(\phi'-\phi)\mathrm{d}\phi'\mathrm{d}\phi, \end{aligned} \tag{3.4}$$

where $\hat{D}(w;u,v)$ is gotten from Green's formula, (2.7) and (2.8) with $\mathrm{d}s=\sqrt{J_0}\mathrm{d}\phi$ and (2.17) with

$$\mathrm{d}s=\frac{R}{\sqrt{pq}}\sqrt{p\cos^2\phi+q\sin^2\phi}\mathrm{d}\phi.$$

$$F(v)=\int_{\Omega_i} f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{3.5}$$

For any real number $s > 0$, we let

$$H^s\left(\Gamma_{\mu_1}\right) = \left\{ f \in L^2\left(\Gamma_{\mu_1}\right) \middle| \|f\|_{s,\Gamma_{\mu_1}} < +\infty \right\} \tag{3.6}$$

with $\|f\|^2_{s,\Gamma_{\mu_1}} = \sum_{|m|=0}^{+\infty}\left(1+m^2\right)^s \left|F_m\right|^2$,

and $F_m = \frac{1}{2\pi}\int_0^{2\pi} f\left(\mu_1,\phi\right)e^{-in\phi}\mathrm{d}\phi$, $\bar{F}_m = F_{-m}$.

Lemma 3.1 There exists a constant $C > 0$ which has different meaning in different place and is related to α and β, such that

$$\left|\hat{D}\left(w;u,v\right)\right| \le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},\ \hat{D}\left(u;u,u\right) \ge C_0\left|u\right|^2_{1,\Omega_i},$$

$\forall u,v,w \in V$.

In practice, we need to truncate the series in (2.12) and (2.17) for some nonnegative integer N, that is

$$\begin{aligned}&\left\|\left(\alpha n_x a_0\left(u\right)\frac{\partial u}{\partial x} + \beta n_y a_0\left(u\right)\frac{\partial u}{\partial y}\right)_{\mu=\mu_1}\right. \\ &= K_1^N\left(u\left(\mu_1,\phi\right)\right),\end{aligned} \tag{3.7}$$

with

$$K_1^N\left(u\left(\mu_1,\phi\right)\right) = \frac{1}{\pi\sqrt{J_0}}\sum_{n=1}^{N} n\int_0^{2\pi}\cos n\left(\phi-\phi'\right)\left(\int_0^{u(\mu_1,\phi)} a_0\left(\mathbf{y}\right)\mathrm{d}\mathbf{y}\right)\mathrm{d}\phi, \tag{3.8}$$

when $\alpha = \beta = 1$, and

$$K_1^N\left(u\left(\mu_1,\phi\right)\right) = \sqrt{\frac{\alpha p\beta q}{p\cos^2\phi + q\sin^2\phi}}\sum_{n=1}^{N}\frac{n}{\pi R}\int_0^{2\pi}\cos n\left(\phi-\phi'\right)\left(\int_0^{u(\mu_1,\phi)} a_0\left(\mathbf{y}\right)\mathrm{d}\mathbf{y}\right)\mathrm{d}\phi, \tag{3.9}$$

when $\beta > \alpha > 0$. So we only use the summation of the first N terms in (2.13) and (2.18). We will consider the following approximate problem

$$\begin{cases} -\nabla\cdot\left(a\left(\mathbf{x},u^N\right)\nabla u^N\right) = f, & \text{in } \Omega_i, \\ u^N = 0, & \text{on } \Gamma_0, \\ a_0\left(u^N\right)\dfrac{\partial u^N}{\partial n} = K_1^N\left(u^N\left(\mu_1,\phi\right)\right), & \text{on } \Gamma_{\mu_1}. \end{cases} \tag{3.10}$$

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a\left(\mathbf{x},u^N\right)\dfrac{\partial u^N}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\beta a\left(\mathbf{x},u^N\right)\dfrac{\partial u^N}{\partial y}\right)\right) = f, & \text{in } \Omega_i, \\ u^N = 0, & \text{on } \Gamma_0, \\ \alpha n_x a_0\left(u^N\right)\dfrac{\partial u^N}{\partial x} + \beta n_y a_0\left(u^N\right)\dfrac{\partial u^N}{\partial y} = K_1^N\left(u^N\left(\mu_1,\phi\right)\right), & \text{on } \Gamma_{\mu_1}. \end{cases} \tag{3.11}$$

Both (3.10) and (3.11) are equivalent to the following variational problem

$$\begin{cases} \text{Find } u^N \in V, \text{ such that} \\ D\left(u^N;u^N,v\right) + \hat{D}_N\left(u^N;u^N,v\right) = F\left(v\right),\ \forall v \in V, \end{cases} \tag{3.12}$$

where

$$\begin{aligned}&\hat{D}_N\left(w;u,v\right) \\ &= \sum_{n=1}^{N}\frac{\sqrt{\alpha\beta}}{n\pi}\int_0^{2\pi}\int_0^{2\pi} a_0\left(w\left(\mu_1,\phi'\right)\right)\frac{\partial u\left(\mu_1,\phi'\right)}{\partial\phi'} \\ &\cdot\frac{\partial v\left(\mu_1,\phi\right)}{\partial\phi}\cos n\left(\phi'-\phi\right)\mathrm{d}\phi'\mathrm{d}\phi.\end{aligned} \tag{3.13}$$

Similar with Lemma 3.1, we have

Lemma 3.2 There exists a constant $C > 0$ which has different meaning in different place, such that

$$\left|\hat{D}_N\left(w;u,v\right)\right| \le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},$$

$$\hat{D}_N\left(u;u,u\right) \ge C_0\left|u\right|^2_{1,\Omega_i},\ \forall u,v,w \in V.$$

3.2. Finite Element Approximation

Divide the arc Γ_{μ_1} into M parts and take a finite element subdivision in Ω_i such that their nodes on Γ_{μ_1} are coincident. That is, we make a regular and quasi-uniform triangulation T_h on Ω_i, such that

$$\Omega_i = \bigcup_{K\in T_h} K, \tag{3.14}$$

with K is a (curved) triangle; h the maximum side of the triangles. Let

$$V_h = \left\{v_h \in V \left| v \right|_K \text{ is a linear polynomial}, \forall K \in T\right\}_h. \quad (3.15)$$

Then the approximate problem of (3.12) can be written as

$$\begin{cases} \text{Find } u_h^N \in V_h, \text{ such that} \\ D\left(u_h^N; u_h^N, v_h\right) + \widehat{D}_N\left(u_h^N; u_h^N, v_h\right) = F\left(v_h\right), \forall v_h \in V_h. \end{cases} \quad (3.16)$$

Some existence and uniqueness results for this type of problem are given in [12,17,18] under some conditions on the coefficients a, so by the constraint conditions (1.2) and (1.3) we have

Lemma 3.3 Problems (3.2), (3.12) and (3.16) have unique solvability.

3.2.1. Convergence Theorems

In this section, we obtain the convergence result of the problems discussed above. We let $u, u^N \in H^2\left(\Omega_i\right)$ and $u_h^N \in V_h$ be the solution of problems (3.2), (3.12), (3.16) respectively. We also assume that

$$V_h \subset V \cap W^{1,2+\varepsilon} \text{ for some } \varepsilon \in (0,1). \quad (3.17)$$

And we require that $\left\{V_h\right\}_{h\to 0}$ is a family of finite-dimensional subspaces of $V \cap C\left(\Omega_i\right)$, which satisfies for any

$$v \in V \cap C\left(\Omega_i\right), \text{ there exists } \left\{v_h\right\}: v_h \in V_h, \left\|v - v_h\right\|_{1,\Omega_i} \to 0, \text{ as } h \to 0, \quad (3.18)$$

$$\left\|v_h\right\|_{1,2+\varepsilon,\Omega_i} \le C(v) \text{ for any } h, \quad (3.19)$$

where $C(v) > 0$ is independent of h.

The continuous piecewise polynomial spaces, such as (3.15), satisfy the condition (3.17). And if we let $v_h = \Pi_h v$, where $\Pi_h : V \to V_h$ is the interpolation operator, then by (3.19), we have

$$\left\|v_h\right\|_{1,2+\varepsilon,\Omega_i} \le \left\|\Pi_h v - v\right\|_{1,2+\varepsilon,\Omega_i} + \left\|v\right\|_{1,2+\varepsilon,\Omega_i} \le C(v).$$

And we can also obtain the following result.

Lemma 3.4 $\lim\limits_{N\to\infty} \left\|u - u^N\right\|_{1,\Omega_i} = 0$.

Proof From the (1.2), (3.12) and Lemma 3.2, we have

$$\begin{aligned} \left\|u^N\right\|_{1,\Omega_i}^2 &\le C\left[D\left(u^N; u^N, u^N\right) + \widehat{D}\left(u^N; u^N, u^N\right)\right] = C\left[F\left(u^N\right) + \widehat{D}\left(u^N; u^N, u^N\right) - \widehat{D}_N\left(u^N; u^N, u^N\right)\right] \\ &\le C\left[\|f\|_{0,\Omega_i} \cdot \left\|u^N\right\|_{1,\Omega_i} + \left|\widehat{D}\left(u^N; u^N, u^N\right) - \widehat{D}_N\left(u^N; u^N, u^N\right)\right|\right]. \end{aligned}$$

For $u^N \in V$, we assume that

$$w^N\left(\mu, \phi'\right) = \int_0^{w^N(r,\phi')} a_0(\mathbf{y})\,\mathrm{d}\mathbf{y} = \sum_{n=-\infty}^{+\infty} a_n e^{|n|(\mu_0-\mu)+in\phi'}, \forall \mu \ge \mu_0,$$

$$u^N\left(\mu_1, \phi\right) = \sum_{n=-\infty}^{+\infty} u_n e^{in\phi},$$

with

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{u(\mu_1,\phi')} a_0(\mathbf{y})\,\mathrm{d}\mathbf{y}\right) e^{-in\phi'}\,\mathrm{d}\phi',$$

and

$$u_n = \frac{1}{2\pi}\int_0^{2\pi} u\left(\mu_1, \phi\right) e^{-in\phi}\,\mathrm{d}\phi.$$

Then we have

$$\begin{aligned} \left|\widehat{D}\left(u^N; u^N, u^N\right) - \widehat{D}_N\left(u^N; u^N, u^N\right)\right| &= \left|\sum_{n=|N+1|}^{+\infty} \frac{\sqrt{\alpha\beta}}{n\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial w\left(\mu_1, \phi'\right)}{\partial \phi'} \frac{\partial v\left(\mu_1, \phi\right))}{\partial \phi} \cos n\left(\phi' - \phi\right)\mathrm{d}\phi'\mathrm{d}\phi\right| \\ &= \left|2\pi \sum_{n=|N+1|}^{+\infty} \mathrm{e}^{|n|(\mu_0-\mu_1)}\right| n\left|a_n \overline{u}_n\right| \\ &\le Ce^{\|N+1|(\mu_0-\mu_1)}\left(\sum_{|n|=N+1}^{+\infty}\left(1+n^2\right)^{\frac{1}{2}} \cdot \left|w_n\right|^2\right)^{\frac{1}{2}}\left(\sum_{|n|=N+1}^{+\infty}\left(1+n^2\right)^{\frac{1}{2}} \cdot \left|u_n\right|^2\right)^{\frac{1}{2}} \\ &\le Ce^{\|N+1|(\mu_0-\mu_1)}\left\|w^N\right\|_{1/2,\Gamma_{\mu_1}}\left\|u^N\right\|_{1/2,\Gamma_{\mu_1}} \le Ce^{|N+1|(\mu_0-\mu_1)}\left\|u^N\right\|_{1,\Omega_i}^2, \end{aligned}$$

From $\mu_1 > \mu_0$, we obtain that $\left\{u^N\right\}$ is bounded in V. Therefore, there exists a subsequence $\left\{u^{N_n}\right\}$ such that $u^{N_n} \rightharpoonup \bar{u} \in V$. Then similar with the proof of Lemma 3.4 of [3], we obtain

$$\lim_{N\to\infty} \left\| u - u^N \right\|_{1,\Omega_i} = 0.$$

By the above lemmas, we get the following convergence result.

Theorem 3.1 Let $u \in H^2(\Omega_i)$, and the assumptions (3.17)-(3.19) be satisfied, then we have

$$\lim_{h\to 0, N\to\infty} \left\| u - u_h^N \right\|_{1,\Omega_i} = 0. \tag{3.20}$$

3.2.2. Error Analysis

In the following, we shall get error estimates for the approximate solution obtained from a FEM-NBEM discrete scheme in the cases $\alpha = \beta = 1$. We assume that the solution u of problem (1.1) satisfies

$$u\big|_{\Omega_i} \in V \cap W^{k,2+\varepsilon}(\Omega_i), \varepsilon > 0, k \ge 2.$$

For simplicity let us define the following notation

$$\bar{A}(u;u,v) = D(u;u,v) + \hat{D}(u;u,v);$$

$$\bar{A}_N\left(u^N;u^N,v\right) = D\left(u^N;u^N,v\right) + \hat{D}_N\left(u^N;u^N,v\right);$$

$$\bar{A}_N\left(u_h^N;u_h^N,v_h\right) = D\left(u_h^N;u_h^N,v_h\right) + \hat{D}_N\left(u_h^N;u_h^N,v_h\right).$$

Then (3.2), (3.12), (3.16) can be replaced by the corresponding simple forms respectively.

Now we introduce the bilinear form $A'(u;\cdot,\cdot)$ and $A'_N\left(u^N;\cdot,\cdot\right)$ defined by

$$\begin{aligned} A'(u;v,z) = &\int_{\Omega_i} \frac{\partial a}{\partial s}(\mathbf{x},u) v \nabla u \cdot \nabla z \mathrm{d}\mathbf{x} + \int_{\Omega_i} a(\mathbf{x},u) \nabla v \cdot \nabla z \mathrm{d}\mathbf{x} \\ &+ \int_0^{2\pi}\int_0^{2\pi} \frac{\partial a_0}{\partial s}(u) v \frac{\partial u}{\partial \phi'}(\mu_1,\phi') \frac{\partial z}{\partial \phi}(\mu_1,\phi) \sum_{n=1}^{+\infty} \frac{\cos n(\phi'-\phi)}{n\pi} \mathrm{d}\phi' \mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_0^{2\pi} a_0(u) \frac{\partial v}{\partial \phi'}(\mu_1,\phi') \frac{\partial z}{\partial \theta}(\mu_1,\phi) \sum_{n=1}^{+\infty} \frac{\cos n(\phi'-\phi)}{n\pi} \mathrm{d}\phi' \mathrm{d}\phi, \end{aligned}$$

$$\begin{aligned} A'\left(u^N;v,z\right) = &\int_{\Omega_i} \frac{\partial a}{\partial s}\left(\mathbf{x},u^N\right) v \nabla u^N \cdot \nabla z \mathrm{d}\mathbf{x} + \int_{\Omega_i} a\left(\mathbf{x},u^N\right) \nabla v \cdot \nabla z \mathrm{d}\mathbf{x} \\ &+ \int_0^{2\pi}\int_0^{2\pi} \frac{\partial a_0}{\partial s}\left(u^N\right) v \frac{\partial u^N}{\partial \phi'}(\mu_1,\phi') \frac{\partial z}{\partial \phi}(\mu_1,\phi) \sum_{n=1}^{+\infty} \frac{\cos n(\phi'-\phi)}{n\pi} \mathrm{d}\phi' \mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_0^{2\pi} a_0\left(u^N\right) \frac{\partial v}{\partial \phi'}(\mu_1,\phi') \frac{\partial z}{\partial \theta}(\mu_1,\phi) \sum_{n=1}^{+\infty} \frac{\cos n(\phi'-\phi)}{n\pi} \mathrm{d}\phi' \mathrm{d}\phi. \end{aligned}$$

Let V' be the dual space of V. By (1.2) and continuity of $\frac{\partial a}{\partial s}(\cdot,u(\cdot))$, we obtain that $A'(u;\cdot,\cdot)$ is bounded in Ω_i. Then there exists an operator $T: V \to V'$ such that

$$(Tv,z) = A'(u;v,z), \forall\, v,z \in V. \tag{3.21}$$

Similar with the proof of [10], we have the lemma as follows

Lemma 3.5 The bilinear form (Tv,v) defined by $A'(u;v,v)$ satisfies the following inequality

$$(Tv,z) + K\left(\|v\|_{0,\Omega_i}^2 + \|v\|_{1/2,\Gamma_{\mu_1}}^2\right) \ge C\|v\|_{1,\Omega_i}^2, \forall\, v \in V, \tag{3.22}$$

where $K \ge 0$ is a sufficient large constant and $C > 0$.

We assume that

$$A'(u;v,z) = 0, \forall\, z \in V \Rightarrow v = 0. \tag{3.23}$$

Let $I: V \to V'$ be the canonical injection. Since V is compactly embedded in $L^2(\Omega_i)$, we have that the operator $J: V \to V'$ defined by $J(v) = (I(v),0)$ is also compact. By (3.21) and (3.23) and T satisfies the property of J, we obtain that $T: V \to V'$ is an isomorphism.

By the conditions (3.2), (3.22), (3.23) and Theorem 10.1.2 of [20], one can get that there exists $h_0 \in (0,1]$, such that the following inequality is satisfied

$$\sup_{x\in V_h} \frac{A'(u;v,z)}{\|z\|_{1,\Omega_i}} \ge \alpha_1 \|v\|_{1,\Omega_i}, \forall\, v \in V_h \tag{3.24}$$

for some constant α_1 independent of $h\,(h < h_0)$.

We define the Galerkin projection with respect to $A'(u;\cdot,\cdot)$, $P_h: V \to V_h$

$$A'(u;P_h v,z) = A'(u;v,z), \forall\, z \in V_h.$$

Then the operator P_h satisfies

$$\begin{aligned} \|v - P_h v\|_{1,p,\Omega_i} &\le C \inf_{v_h \in V_h} \|v - v_h\|_{1,p,\Omega_i} \\ &\le Ch^\sigma, 2 \le p \le \infty, 0 < \sigma < 1. \end{aligned} \tag{3.25}$$

We define the set

$$B_h = \left\{ v \in V_h \mid \|v - P_h v\|_{1,\infty,\Omega_i} \le Ch^\sigma \right\}.$$

Lemma 3.6 $u_h^N \in V_h$ is a solution of (3.14) if and only if the following equation

$$A'_N\left(u^N; u^N - u_h^N, v\right) = R\left(u^N; u^N, v\right), \forall\, v \in V_h,$$

holds, where

$$\begin{aligned} R\left(u^N; u^N, v\right) &= \int_{\Omega_i}\left(\int_0^1\left[\frac{\partial^2 a}{\partial s^2}\left(\mathbf{x}, w_h^N\right)\nabla w_h^N \nabla v\right](1-t)\,\mathrm{d}t\right)\left(\mathrm{d}_h^N\right)^2 \mathrm{d}\mathbf{x} \\ &+2\int_{\Omega_i}\left(\int_0^1\left[\frac{\partial a}{\partial s}\left(\mathbf{x}, w_h^N\right)\nabla d_h^N \nabla v\right](1-t)\,\mathrm{d}t\right) d_h^N \mathrm{d}\mathbf{x} \\ &+\int_0^{2\pi}\int_0^{2\pi}\left(\int_0^1\left[\frac{\partial^2 a_0}{\partial s^2}\left(w_h^N\right)\frac{\partial w_h^N}{\partial \phi'}\frac{\partial v}{\partial \phi}\sum_{n=1}^{N}\frac{\cos n(\phi-\phi')}{n\pi}\right](1-t)\,\mathrm{d}t\right)\left(\mathrm{d}_h^N\right)^2 \mathrm{d}\phi'\mathrm{d}\phi \\ &+2\int_0^{2\pi}\int_0^{2\pi}\left(\int_0^1\left[\frac{\partial a_0}{\partial s}(w_h^N)\frac{\partial d_h^N}{\partial \phi'}\frac{\partial v}{\partial \phi}\sum_{n=1}^{N}\frac{\cos n\left(\phi-\phi'\right)}{n\pi}\right](1-t)\,\mathrm{d}t\right)\mathrm{d}_h^N \mathrm{d}\phi'\mathrm{d}\phi \end{aligned}$$

with $w_h^N = u^N + t\left(u_h^N - u^N\right)$, $d_h^N = u_h^N - u^N$.

Proof. Let $\eta(t) = \overline{A}_N\left(w_h^N; w_h^N, v\right)$, then by

$$\eta(1) = \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1-t)\,\mathrm{d}t$$

and

$$\overline{A}_N\left(u_h^N; u_h^N, v\right) = \overline{A}_N\left(u_h^N; u_h^N, v\right) = F(v), \forall\, v \in V_h.$$

We can get the desired result.

Let $M_h = \left\{ v \in V_h \mid \|v\|_{1,\infty,\Omega_i} \le 1 + \|u^N\|_{1,\infty,\Omega_i} \right\}$. Then following [10,11], we have

Lemma 3.7 There exists a positive constant *C* independent of h, such that

$$\left|R\left(u^N; v, z\right)\right| \le C\left(\|u^N - v\|_{1,\Omega_i}^2 + \|u^N - v\|_{1,\Omega_i}\right)\|z\|_{1,\Omega_i},$$
$$\forall\, v \in M_h, \forall\, z \in V_h.$$

We also have the following result.

Lemma 3.8 $B_h \subset M_h$.

Proof For any $v \in B_h$, we only need to show that $v \in M_h$.

$$\|v\|_{1,\infty,\Omega_i} \le \|u^N - v\|_{1,\infty,\Omega_i} + \|u^N\|_{1,\infty,\Omega_i},$$

$$\|u^N - v\|_{1,\infty,\Omega_i} \le \|u^N - P_h u^N\|_{1,\infty,\Omega_i} + \|P_h u^N - v\|_{1,\infty,\Omega_i},$$

$$\|u^N - P_h u^N\|_{1,\infty,\Omega_i} \le \|u^N - \Pi_h u^N\|_{1,\infty,\Omega_i} + \|\Pi_h u^N - P_h u^N\|_{1,\infty,\Omega_i}.$$

Since T_h is regular and quasi-uniform, referring to [19], we obtain the following inverse inequality

$$\|w\|_{1,\infty,\Omega_i} \le C\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\|w\|_{1,\Omega_i}, \forall\, w \in V_h.$$

Combining the above inequalities with the definition of B_h and (3.26), we obtain

$$\|u^N - v\|_{1,\infty,\Omega_i} \le 1.$$

By the definition of M_h, we get the desired result.

Theorem 3.2 Assume $u \in V \cap W^{k,2+\varepsilon}(\Omega_i)$ be the solution of (1), with $\varepsilon > 0$, $k \ge 2$, and we also assume that $u|_{\Gamma_{\mu_0}} \in H^{k-1/2}\left(\Gamma_{\mu_0}\right)$ and u satisfies (3.23). With sufficiently small h, the finite element Equation (3.16) has the approximate solution $u_h^N \in V_h$ such that

$$\|u - u_h^N\|_{1,\Omega_i} \le C\left[h^\sigma + \frac{e^{(N+1)(\mu_0-\mu_1)}}{(N+1)^{k-1}}\|u\|_{k-1/2,\Gamma_{\mu_0}}\right],$$

where C is a constant independent of *h* and *N*.

Proof Firstly, for any $u^N \in V$, we have

$$\begin{aligned} \left|\widehat{D}\left(u^N; u^N, v\right) - \widehat{D}_N\left(u^N; u^N, v\right)\right| &= \left|\sum_{n=|N+1|}^{+\infty}\frac{1}{n\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\partial w(\mu_1,\phi')}{\partial \phi'}\frac{\partial v(\mu_1,\phi)}{\partial \phi}\cos n(\phi'-\phi)\,\mathrm{d}\phi'\mathrm{d}\phi\right| \\ &\le \frac{Ce^{|N+1|(\mu_0-\mu_1)}}{(N+1)^{k-1}}\left(\sum_{|n|=N+1}^{+\infty}\left(1+n^2\right)^{k-\frac{1}{2}}\cdot|w_n|^2\right)^{\frac{1}{2}}\left(\sum_{|n|=N+1}^{+\infty}\left(1+n^2\right)^{\frac{1}{2}}\cdot|v_n|^2\right)^{\frac{1}{2}} \\ &\le \frac{Ce^{|N+1|(\mu_0-\mu_1)}}{(N+1)^{k-1}}\|w^N\|_{k-1/2,\Gamma_{\mu_0}}\|v^N\|_{1/2,\Gamma_{\mu_0}} \le \frac{Ce^{|N+1|(\mu_0-\mu_1)}}{(N+1)^{k-1}}\|u\|_{k-1/2,\Gamma_{\mu_0}}\|v\|_{1,\Omega_i}^2. \end{aligned}$$

Then by (3.12), we have

$$\bar{A}\left(u^N;u^N,v\right)=D\left(u^N;u^N,v\right)+\hat{D}\left(u^N;u^N,v\right)$$
$$=F(v)+\hat{D}\left(u^N;u^N,v\right)-\hat{D}_N\left(u^N;u^N,v\right).$$

Let $\eta(t)=\bar{A}\left(u+t\left(u^N-u\right);u+t\left(u^N-u\right),v\right)$, we have

$$\int_0^1 A'\left(u+t\left(u^N-u\right);u^N-u,v\right)\mathrm{d}t$$
$$=\bar{A}\left(u^N;u^N,v\right)-\bar{A}(u;u,v).$$

From (3.2), (3.22), (3.23) and [20], we obtain

$$\begin{aligned}&\left\|u-u^N\right\|_{1,\Omega_i}\\&\le C\sup_{v\in V}\left(\frac{1}{\|v\|_{1,\Omega_i}}\int_0^1 A'\left(u+t\left(u^N-u\right);u^N-u,v\right)\mathrm{d}t\right)\\&\le C\frac{\left|\hat{D}\left(u^N;u^N,v\right)-\hat{D}_N\left(u^N;u^N,v\right)\right|}{\|v\|_{1,\Omega_i}}\\&\le Ce^{(N+1)(\mu_0-\mu_1)}\|u\|_{k-\frac{1}{2},\Gamma_{\mu_1}}.\end{aligned}\tag{3.26}$$

We denote a nonlinear mapping $\phi:V_h\to V_h$, which satisfies that for any given $v\in V_h$, $\phi(v)$ is the unique solution of

$$A'\left(u,\varphi(v),z\right)=A'(u,u,z)-R(u,v,z),\ \forall\ z\in V_h.\tag{3.27}$$

Therefore, we have

$$A'\left(u,\varphi(v)-\varphi(v_n),z\right)=R(u,v_n,z)-R(u,v,z).$$

Combining the above equation with (3.25), we obtain the operator ϕ is continuous, *i.e.*,

$$\lim_{v_n\to v}\phi(v_n)=\phi(v).$$

Next, we assume that $v\in B_h$, then by Lemma 3.8, we have that $v\in M_h$. By the definition of P_h, (3.27) can be rewritten as

$$A'\left(u^N,\varphi(v)-P_hu^N,z\right)=-R\left(u^N,v,z\right),\ \forall\ z\in V_h.$$

Then, from (3.24), Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned}&\left\|\varphi(v)-P_hu^N\right\|_{1,\Omega_i}\le C\sup_{z\in V_h}\frac{\left|A'\left(u,\varphi(v)-P_hu^N,z\right)\right|}{\|z\|_{1,\Omega_i}}\\&\le C\left(\left\|u^N-v\right\|^2_{1,\Omega_i}+\left\|u^N-v\right\|_{1,\Omega_i}\right)\\&\le C\left\{\left\|u^N-P_hu^N\right\|^2_{1,\Omega_i}+\left\|P_hu^N-v\right\|^2_{1,\Omega_i}\right.\\&\left.+\left\|u^N-P_hu^N\right\|_{1,\Omega_i}+\left\|P_hu^N-v\right\|_{1,\Omega_i}\right\}\\&\le Ch^\sigma.\end{aligned}$$

This implies that $\phi:B_h\to B_h$. And since ϕ is also continuous, following from Brouwer's fixed theorem, one can obtain that there exists $u_h^N\in V_h$, such that $\varphi\left(u_h^N\right)=u_h^N$. From Lemma 3.6, we deduce that u_h^N is the solution of (3.16). What's more, by (3.25) and the fact $u_h^N\in B_h$, we obtain

$$\begin{aligned}\left\|u^N-u_h^N\right\|_{1,\Omega_i}&\le\left\|u^N-P_hu^N\right\|_{1,\Omega_i}+\left\|P_hu^N-u_h^N\right\|_{1,\Omega_i}\\&\le Ch^\sigma,0<\sigma<1.\end{aligned}\tag{3.28}$$

Combining (3.26) with (3.28), one can obtain

$$\begin{aligned}\left\|u-u_h^N\right\|_{1,\Omega_i}&\le\left\|u-u^N\right\|_{1,\Omega_i}+\left\|u^N-u_h^N\right\|_{1,\Omega_i}\\&\le C\left[h^\sigma+\frac{e^{(N+1)(\mu_0-\mu_1)}}{(N+1)^{k-1}}\|u\|_{k-\frac{1}{2},\Gamma_{\mu_0}}\right].\end{aligned}$$

This completes the proof.

4. Numerical Examples

In this section, we shall give some examples to confirm our theoretical results. In the following, we choose the finite element space as given in (3.16). For simplicity, we let

$$\Delta r=1/m,\Delta\theta=2\pi/M,e_0(h,N)=\left\|u-u_h^N\right\|_{L^2(\Omega_i)}.$$

Example 4.1 We assume the exterior domain Ω^c with elliptical boundary

$$\Gamma_0=\left\{(\mu_0,\phi)\,\middle|\,\mu_0=0.8,0\le\phi\le2\pi\right\},$$
$$\Gamma_{\mu_1}=\left\{(\mu_0,\phi)\,\middle|\,\mu_1>\mu_0,0\le\phi\le2\pi\right\}.$$

Now we consider the problem

$$\begin{cases}-\nabla\cdot\left(a(\mathbf{x},u)\nabla u\right)=f, & \text{in }\Omega_i,\\ u=0, & \text{on }\Gamma_0,\\ a_0(u)\dfrac{\partial u}{\partial n}=K_1\left(u(\mu_1,\phi)\right), & \text{on }\Gamma_{\mu_1},\end{cases}\tag{4.1}$$

when $a(x,u)=\dfrac{1}{1+u^2}$, $f=0$ and $f_0=1.25$.

The exact solution of **Example 4.1** is

$$u=\tan\left(2\sinh\mu\sin\phi/f_0\left(\cosh2\mu+\cos2\phi\right)\right).$$

The numerical results are given in **Figures 1** and **2** and **Table 1**.

Example 4.2 Similar with **Example 4.1**, Γ_0 and $a(x,u)$ are replaced by

$$\Gamma_0=\left\{(\mu_0,\phi)\,\middle|\,\mu_0=0.5,0\le\phi\le2\pi\right\}$$

and $a(\mathbf{x},u)=1/\sqrt{1-u^2}$ respectively.

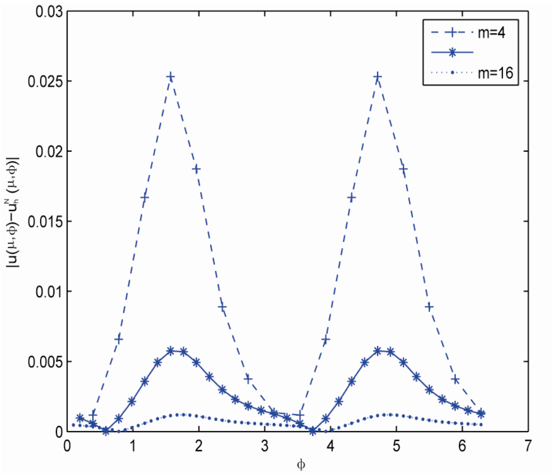

Figure 1. Example 4.1 with $N = 16$, $\mu_1 = 1.7$.

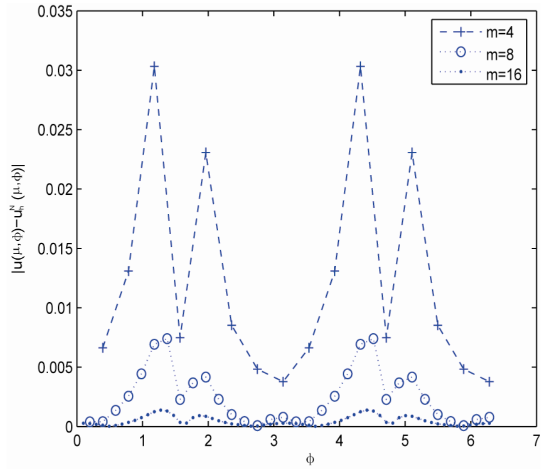

Figure 3. Example 4.2 with $N = 6$, $\mu_1 = 1.0$.

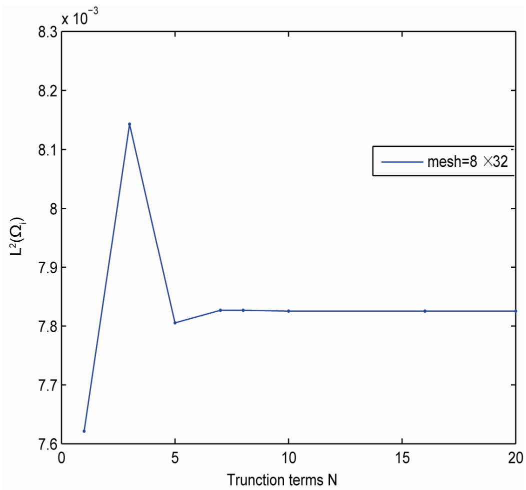

Figure 2. Example 4.1 with different N.

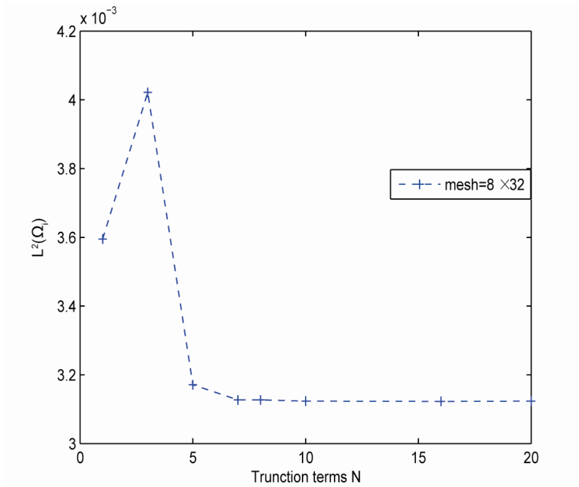

Figure 4. Example 4.2 with different N.

Table 1. The errors with $N = 16$ for Example 4.1.

μ_1	(m, M)	$e_0(h, N)$	ratio
	(4,16)	2.9888E-02	-
1.5	(8,32)	7.1183E-03	4.1987
	(16,64)	1.9991E-03	3.5607
	(4,16)	3.1917E-02	-
1.7	(8,32)	7.8255E-03	4.0786
	(16,64)	2.1387E-03	3.6591
	(4,16)	3.5553E-02	-
2.0	(8,32)	9.0701E-03	3.9198
	(16,64)	2.4284E-03	3.7351

The exact solution of **Example 4.2** is

$$u = \sin\left(2\cosh\mu\cos\phi / f_0\left(\cosh 2\mu + \cos 2\phi\right)\right).$$

The numerical results are given in **Figure 3**, **Figure 4** and **Table 2**.

Table 2. The errors with $N = 6$ for Example 4.2.

μ_1	(m, M)	$e_0(h, N)$	ratio
	(2,8)	3.0471E-02	-
0.8	(4,16)	1.0654E-02	2.8601
	(8,32)	3.1506E-03	3.3816
	(2,8)	4.5002E-02	-
1.0	(4,16)	1.2723E-02	3.5370
	(8,32)	3.1711E-03	4.0122
	(2,8)	8.7937E-02	-
1.5	(4,16)	2.2960E-02	3.8299
	(8,32)	5.5786E-03	4.1157

Example 4.3 We assume the exterior domain Ω^c with elliptical boundary

$$\Gamma_0 = \left\{(\mu_0, \phi) \mid \mu_0 = 0.8, 0 \le \phi \le 2\pi\right\},$$

$$\Gamma_{\mu_1} = \left\{(\mu_1, \phi) \mid \mu_1 = 1.5, 0 \le \phi \le 2\pi\right\}.$$

Now we consider the problem

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\varepsilon a(\mathbf{x},u)\dfrac{\partial u}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(a(\mathbf{x},u)\dfrac{\partial u}{\partial y}\right)\right)=f, & \text{in } \Omega_i, \\ u=0, & \text{on } \Gamma_0, \\ \varepsilon n_x a_0(u)\dfrac{\partial u}{\partial x}+n_y a_0(u)\dfrac{\partial u}{\partial y}=K_1\left(u(\mu_1,\phi)\right), & \text{on } \Gamma_{\mu_1}, \end{cases} \tag{4.2}$$

when $a(\mathbf{x},u)=1/1+u^2$, $f_0=1.25$ and

$$f=\frac{2(1-\varepsilon)\sinh\mu\sin\phi\left(3\cosh^2\mu\cos^2\phi-\sinh^2\mu\sin^2\phi\right)}{\left(\cosh 2\mu+\cos 2\phi\right)^3}.$$

The exact solution of **Example 4.3** is

$$u=\tan\left(2\sinh\mu\sin\phi/f_0\left(\cosh 2\mu+\cos 2\phi\right)\right).$$

The numerical results are given in **Figures 5** and **6** and **Table 3**.

Example 4.4 Similar with **Example 4.3**, Γ_0 and $a(\mathbf{x},u)$ are replaced by

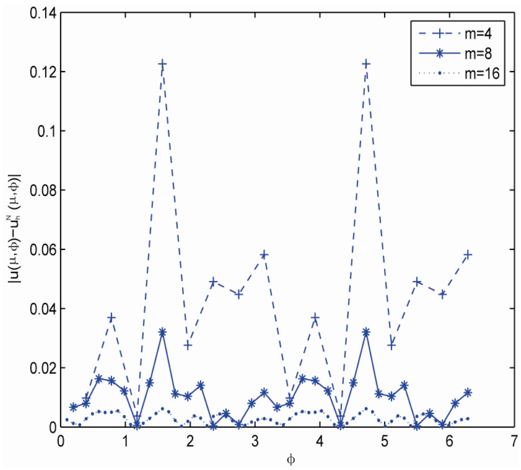

Figure 5. Example 4.3 with N = 10, ε = 0.005.

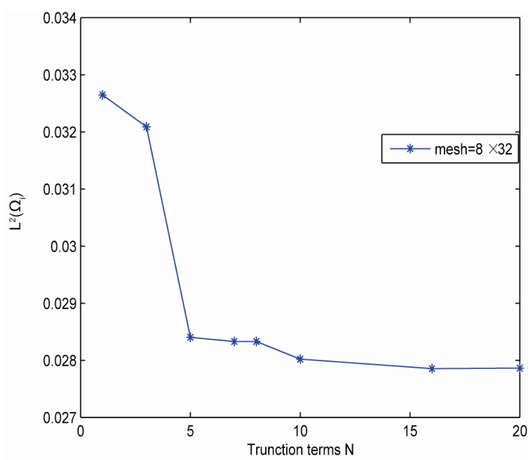

Figure 6. Example 4.3 with different N.

$$\Gamma_0=\left\{(\mu_0,\phi)\middle|\,\mu_0=0.5,0\le\phi\le 2\pi\right\}$$

and $a(\mathbf{x},u)=1/\sqrt{1-u^2}$ respectively. And we take

$$f=\frac{2(1-\varepsilon)\cosh\mu\cos\phi\left(\cosh^2\mu\cos^2\phi-3\sinh^2\mu\sin^2\phi\right)}{\left(\cosh 2\mu+\cos 2\phi\right)^3}.$$

The exact solution of **Example 4.3** is

$$u=\sin\left(2\cosh\mu\cos\phi/f_0\left(\cosh 2\mu+\cos 2\phi\right)\right).$$

The numerical results are given in **Figures 7** and **8** and **Table 4**.

From the numerical results, one obtains that the numerical errors can be affected by the order of artificial boundary condition, the mesh of the domain and the location of the artificial boundary, and it can be reduced by increasing the order of the artificial boundary condition and refining the mesh. What's more, the convergence rate of anisotropic problems can also be affected by the choice of ε as it is shown in **Tables 3** and **4**. The numerical results are in agreement with the error analysis we obtain and show the efficiency of the coupling method.

Table 3. The errors with N = 10 for Example 4.3.

ε	(m,M)	$e_0(h,N)$	ratio
	(4,16)	3.4114E-02	-
0.5	(8,32)	8.9808E-03	3.7985
	(16,64)	2.3296E-03	3.8550
	(4,16)	9.1725E-02	-
0.025	(8,32)	2.2626E-02	4.0539
	(16,64)	5.8604E-03	3.8609
	(4,16)	1.0129E-01	-
0.005	(8,32)	2.8020E-02	3.6149
	(16,64)	1.0173E-02	2.7543

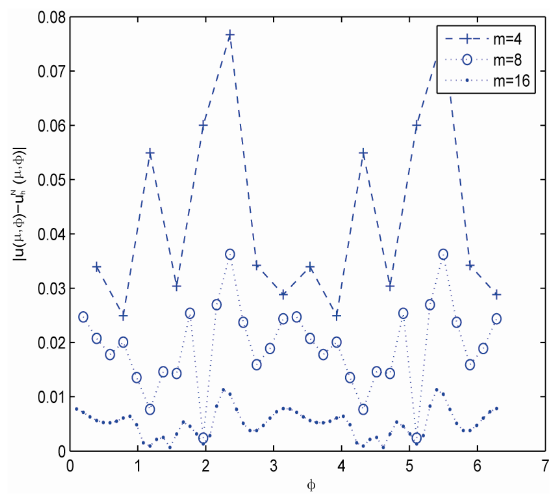

Figure 7. Example 4.4 with N = 5, ε = 0.05.

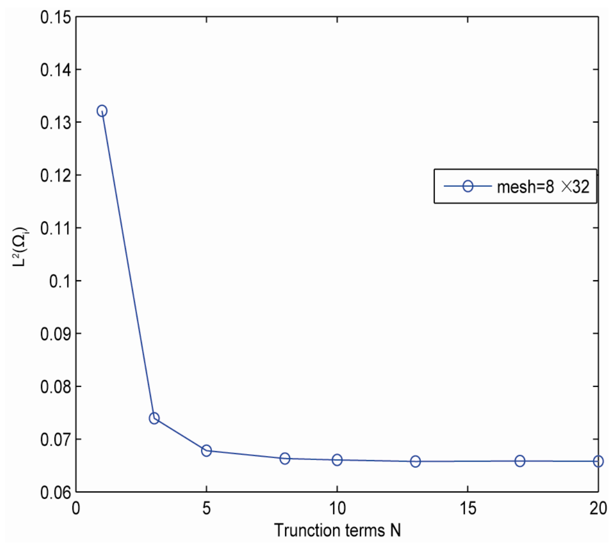

Figure 8. Example 4.4 with different *N*.

Table 4. The errors with $N = 10$ for Example 4.4.

ε	(m,M)	$e_0(h,N)$	ratio
	(4,16)	4.5556E-02	-
0.5	(8,32)	1.1454E-02	3.9772
	(16,64)	2.9414E-03	3.8942
	(4,16)	1.6183E-01	-
0.05	(8,32)	6.7805E-02	2.3867
	(16,64)	1.8030E-02	3.7606
	(4,16)	2.8137E-01	-
	(8,32)	1.4554E-01	1.9332
0.025	(16,64)	4.4792E-02	3.2493

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