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On the Location of Zeros of Polynomials

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Abstract

In this paper, we prove some extensions and generalizations of the classical Eneström-Kakeya theorem.

Keywords: Polynomial, Zeros, Eneström-Kakeya Theorem

1. Introduction and Statement of Results

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_1 \ge a_0 \ge 0,$$

then according to a classical result usually known as Eneström-Kakeya theorem [11], $P(z)$ does not vanish in $|z| > 1$. Applying this result to the polynomial $P(tz)$, the following more general result is immediate.

Theorem A. *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $t > 0$*

$$a_n t^n \ge a_{n-1} t^{n-1} \ge a_{n-2} t^{n-2} \ge \cdots \ge a_1 t \ge a_0 \ge 0,$$

then $P(z)$ has all the zeros in $|z| \le t$.

In the literature, [1-15], there exist extensions and generalizations of Eneström-Kakeya theorem. Joyal, Labelle and Rahman [9] extended this theorem to polynomials whose coefficients are monotonic but not necessarily non negative and the result was further generalized by Dewan and Bidkham [6] to read as:

Theorem B. *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $t > 0$ and $0 < \lambda \le n$,*

$$a_n t^n \le a_{n-1} t^{n-1} \le \cdots \le a_\lambda t^\lambda \ge a_{\lambda-1} t^{\lambda-1} \ge \cdots \ge t a_1 \ge a_0,$$

then $P(z)$ has all the zeros in the circle

$$|z| \le \frac{t}{|a_n|}\left\{\left(\frac{2a_\lambda}{t^{n-\lambda}} - a_n\right) + \frac{1}{t^n}\left(|a_0| - a_0\right)\right\}.$$

Govil and Rahman [8] extended Theorem A to the polynomials with complex coefficients. As a refinement of the result of Govil and Rahman, Govil and Jain [7] proved the following.

Theorem C. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some β*

$$|\arg a_k - \beta| \le \alpha \le \pi/2, \quad k = 0, 1, \cdots, n$$

and

$$|a_n| \ge |a_{n-1}| \ge |a_{n-2}| \ge \ldots \ge |a_1| \ge |a_0|,$$

then $P(z)$ has all its zeros in the ring-shaped region given by

$$R_3 \le |z| \le R_2.$$

Here

$$R_2 = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{\frac{1}{2}},$$

$$R_3 = \frac{1}{2M_2^2}\Big[-R_2^2 \;|b|\left(M_2 - |a_0|\right) + \Big\{4|a_0| R_2^2 M_2^3 + R_2^4 |b|^2 \left(M_2 - |a_0|\right)^2\Big\}^{\frac{1}{2}}\Big],$$

where

$$M_1 = |a_n| R_2,$$

$$M_2 = |a_n| R_2^2 \left[R + R_2 - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)\right],$$

$$c = |a_n - a_{n-1}|,$$

$$b = a_1 - a_0$$

and

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|$$

By using Schwarz's Lemma, Aziz and Mohammad [1] generalized Eneström-Kakeya theorem in a different way and proved:

Theorem D. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real positive coefficients. If* $t_1 > t_2 \ge 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, r = 1, 2, \cdots, n+1$$

where $a_{-1} = a_{n+1} = 0$, *then all the zeros of* $P(z)$ *lie in* $|z| \le t_1$.

In this paper, we also make use of a generalized form of Schwarz's Lemma and prove some more general results which include not only the above theorems as special cases, but also lead to a standard development of interesting generalizations of some well known results. Infact we prove

Theorem 1. *Let* $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial of degree n such that*

$\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0, 1, \cdots, n$ *are real numbers and for certain non negative real numbers* t_1, t_2 *with* $t_1 \ge t_2$ *and* $t_1 \ne 0$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \ge 0, r = 1, 2, \cdots, n+1$$

$$a_{-1} = a_{n+1} = 0 = b_{-1} = b_{n+1},$$

then all the zeros of $P(z)$ *lie in*

$$\min\left(r_2, \frac{1}{t_1}\right) \le |z| \le \max(r_1, t_1).$$

Here

$$r_1 = \frac{-\left(|\alpha_n| - K_1\right)\left|\alpha_n(t_1 - t_2) - \alpha_{n-1}\right| + \left\{\left(|\alpha_n| - K_1\right)^2 \left|\alpha_n(t_1 - t_2) - \alpha_{n-1}\right|^2 + 4K_1^3 t_1^2 |\alpha_n|\right\}^{\frac{1}{2}}}{2K_1|\alpha_n|},$$

$$r_2 = \frac{-\left(|\alpha_0| t_1 t_2 - K_2\right)\left|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\right| t_1^2 + \left\{\left(|\alpha_0| t_1 t_2 - K_2\right)^2 \left|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\right|^2 t_1^4 + 4K_2^3 |\alpha_0| t_1^3 t_2\right\}^{\frac{1}{2}}}{2K_2^2},$$

where

$$K_1 = (a_n + b_n) + \left(|a_0| - a_0\right)\frac{t_2}{t_1^{n+1}} + \left(|b_0| - b_0\right)\frac{t_2}{t_1^{n+1}},$$

$$K_2 = (a_n + b_n)t_1^{n+2} + \left(|a_n| + |b_n|\right)t_1^{n+2} - (a_0 + b_0)t_1 t_2.$$

Assuming that all the coefficients $\alpha_j, j = 0, 1, \cdots, n$ are real, the following result is immediate:

Corollary 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients such that for certain non negative real numbers* t_1, t_2 *with* $t_1 \ge t_2$ *and* $t_1 \ne 0$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0,$$

$$r = 1, 2, \cdots, n+1, \quad (a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ *lie in*

$$\min\left(r_2, \frac{1}{t_1}\right) \le |z| \le \max(r_1, t_1).$$

Here

$$r_1 = \frac{-\left(|a_n| - K_1\right)\left|a_n(t_1 - t_2) - a_{n-1}\right| + \left\{\left(|a_n| - K_1\right)^2 \left|a_n(t_1 - t_2) - a_{n-1}\right|^2 + 4K_1^3 t_1^2 |a_n|\right\}^{\frac{1}{2}}}{2K_1|a_n|},$$

$$r_2 = \frac{-\left(|a_0| t_1 t_2 - K_2\right)\left|a_1 t_1 t_2 + a_0(t_1 - t_2)\right| t_1^2 + \left\{\left(|a_0| t_1 t_2 - K_2\right)^2 \left|a_1 t_1 t_2 + a_0(t_1 - t_2)\right|^2 t_1^4 + 4K_2^3 |a_0| t_1^3 t_2\right\}^{\frac{1}{2}}}{2K_2^2},$$

where

$$K_1 = a_n - a_0\frac{t_2}{t_1^{n+1}} + |a_0|\frac{t_2}{t_1^{n+1}}, K_2 = |a_n| t_1^{n+2} + a_n t_1^{n+2} - a_0 t_1 t_2.$$

If in Corollary 1, we assume that all the coefficients are positive and $t_2 = 0$, then we have the following:

Corollary 2. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some real number* $t > 0$

$$a_n t^n \ge a_{n-1}t^{n-1} \ge a_{n-2}t^{n-2} \ge \cdots \ge a_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$\min\left(\frac{a_0}{2a_n t^{n-1}}, \frac{1}{t}\right) \le |z| \le t.$$

In particular, if $t = 1$, Corollary 2 gives the following improvement of Eneström-Kakeya theorem.

Corollary 3. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$\frac{a_0}{2a_n} \le |z| \le 1.$$

We next prove the following more general result which include many known results as special cases.

Theorem 2. *Let* $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + \mathrm{i}b_j$ *where* a_j *and* b_j, $j = 0,1,\cdots,n$ *are real numbers. If* $t_1 > t_2 \ge 0$ *can be found such that for a certain integer* λ, $0 < \lambda \le n$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,\lambda+1 \\ \le 0, r = \lambda+2,\lambda+3,\cdots,n+1 \end{cases}$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,\lambda+1 \\ \le 0, r = \lambda+2,\lambda+3,\cdots,n+1 \end{cases}$$

$$a_{n+1} = 0 = b_{n+1},$$

then all the zeros of $P(z)$ *lie in*

$$|z| \le R, \tag{1}$$

where

$$\begin{aligned} R = {} & \frac{t_1}{|\alpha_n|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}} - a_n\right) + \frac{1}{t_1^n}\left(|a_0| - a_0\right)\right\} \\ & + \frac{t_2}{|\alpha_n|}\left\{\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n}\left(|a_0| - a_0\right)\right\} \\ & + \frac{t_1}{|\alpha_n|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}} - b_n\right) + \frac{1}{t_1^n}\left(|b_0| - b_0\right)\right\} \\ & + \frac{t_2}{|\alpha_n|}\left\{\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n}\left(|b_0| - b_0\right)\right\} \end{aligned}$$

Remark 1. Theorem B is a special case of Theorem 2, if we take $t_2 = 0$ and assume that all the coefficients α_j, $j = 0,1,2,\cdots,n$ are real.

The following result follows immediately from Theorem 2 by taking $\lambda = n$ and assuming α_j, $j = 0,1,2,\cdots,n$ to be a real.

Corollary 4. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients. If* $t_1 > t_2 \ge 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, r = 2,3,\cdots,n+1,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \le \frac{1}{|a_n| t_1^n}\left[t_1\left(a_n t_1^n + |a_0| - a_0\right) + t_2\left(|a_0| - a_0\right)\right].$$

Remark 2. For $t_1 = 1$ and $t_2 = 0$, Corollary 4 reduces to a result of Joyal, Labelle and Rahman [9].

We also prove the following result which is of independent interest.

Theorem 3. *Let* $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,...,n$ *are real numbers. If* $t_1 > t_2 \ge 0$ *can be found such that for a certain integer* λ, $1 < \lambda \le n-1$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,\lambda+1 \\ \le 0 . r = \lambda+2,\lambda+3,\cdots,n+1 \end{cases}$$

and

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,\lambda+1 \\ \le 0, r = \lambda+2,\lambda+3,\cdots,n+1 \end{cases}$$

$$\left(a_{-1} = a_{-2} = a_{n+1} = 0 = b_{-1} = b_{-2} = b_{n+1}\right),$$

then all the zeros of $P(z)$ *lie in*

$$\left|z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2)\right| \le R_1, \tag{2}$$

where

$$\begin{aligned} R_1 = {} & \frac{t_1}{|\alpha_n|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}} - \frac{a_{n-1}}{t_1}\right) + \frac{1}{t_1^n}\left(|a_0| - a_0\right)\right\} \\ & + \frac{t_2}{|\alpha_n|}\left\{\left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} - a_n\right) + \frac{1}{t_1^n}\left(|a_0| - a_0\right)\right\} \\ & + \frac{t_1}{|\alpha_n|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}} - \frac{b_{n-1}}{t_1}\right) + \frac{1}{t_1^n}\left(|b_0| - b_0\right)\right\} \\ & + \frac{t_2}{|\alpha_n|}\left\{\left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} - b_n\right) + \frac{1}{t_1^n}\left(|b_0| - b_0\right)\right\} . \end{aligned}$$

Remark 3. Theorem 4 of [4] immediately follows from Theorem 3 when $t_1 = 1$, $t_2 = 0$ and the coefficients α_j, $j = 0,1,\cdots,n$ are real.

On combining Theorem 2 and Theorem 3 the following more interesting result is immediate.

Corollary 5. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + \mathrm{i}b_j$ *where* a_j *and* b_j, $j = 0,1,...,n$ *are real. If* $t_1 > t_2 \ge 0$ *can be found such that for a certain integer* λ, $\lambda \le n-1$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,\lambda+1 \\ \le 0, r = \lambda+2,\lambda+3,\cdots,n+1 \end{cases}$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,\lambda+1 \\ \le 0, r = \lambda+2,\lambda+3,\cdots,n+1 \end{cases} \qquad a_{n+1} = 0 = b_{n+1},$$

then all the zeros of $P(z)$ *lie in the intersection of the two circles given by* (1) *and* (2).

If we take $\lambda = n-1$ and the coefficients α_j, $j = 0,1\cdots,n$ are real in Theorem 3, we get the following result.

Corollary 6. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients. If* $t_1 > t_2 \ge 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \begin{cases} \ge 0, r = 2,3,\cdots,n \\ \le 0, r = n+1 \end{cases}$$

then all the zeros of $P(z)$ *lie in*

$$\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \le \frac{1}{|a_n|}\left[t_2 a_n + a_{n-1} + \frac{1}{t_1^n}\left(|a_0| - a_0\right)(t_1 + t_2) \right].$$

The following result also follows from Theorem 3, when $\lambda = n-1$, the coefficients α_j, , $j = 0,1,2,\cdots,n$ are real and $t_2 = 0$.

Corollary 7. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients. If for some* $t > 0$,

$$a_n t^n \ge a_{n-1} t^{n-1} \ge a_{n-2} t^{n-2} \ge \cdots \ge a_1 t \ge a_0,$$

then $P(z)$ *has all the zeros in*

$$\left| z + \frac{a_{n-1}}{a_n} - t \right| \le \frac{1}{|a_n|}\left\{ a_{n-1} + \frac{\left(|a_0| - a_0\right)}{t^{n-1}} \right\}.$$

2. Lemmas

For proving the above theorems, we require the following lemmas. The first Lemma which we need is due to Rahman and Schmeisser [11].

Lemma 1. *If* $f(z)$ *is analytic in* $|z| \le 1$, $f(0) = a$, *where* $|a| < 1$, $f'(0) = b$, $|f(z)| \le 1$ *on* $|z| \le 1$, *then for* $|z| \le 1$,

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

From Lemma 1, one can easily deduce the following :

Lemma 2. *If* $f(z)$ *is analytic in* $|z| \le R$, $f(0) = 0$, $f'(0) = b$ *and* $|f(z)| \le M$ *for* $|z| = R$, *then*

$$|f(z)| \le \frac{M|z|}{R^2}\frac{M|z| + R^2|b|}{M + |b||z|}, \quad |z| \le R.$$

The next Lemma is due to Aziz and Mohammad [2].

Lemma 3. *Let* $f(z) := a_n z^n + \sum_{j=0}^{p} a_j z^j$, $0 \le p \le n-1$ *be a polynomial of degree n with complex coefficients.*

Then for every positive real number r, all the zeros of $f(z)$ *lie in the disk*

$$|z| \le \max\left\{ r, \sum_{k=0}^{p} \frac{|a_k|}{|a_n| r^{n-k-1}} \right\}. \tag{3}$$

3. Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t_2 + z)(t_1 - z)P(z) = -\alpha_n z^{n+2} + \left(\alpha_n(t_1 - t_2) - \alpha_{n-1}\right)z^{n+1} + \left(\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\right)z^n + \cdots \\ &\quad + \left(\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\right)z^2 + \left(\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\right)z + \alpha_0 t_1 t_2 \\ &= -a_n z^{n+2} + \left(a_n(t_1 - t_2) - a_{n-1}\right)z^{n+1} + \cdots + \left(a_1 t_1 t_2 + a_0(t_1 - t_2)\right)z + a_0 t_1 t_2 \\ &\quad + \mathrm{i}\left[-b_n z^{n+2} + \left(b_n(t_1 - t_2) - b_{n-1}\right)z^{n+1} + \cdots + \left(b_1 t_1 t_2 + b_0(t_1 - t_2)\right)z + b_0 t_1 t_2\right]. \end{aligned} \tag{4}$$

Further, let

$$\begin{aligned} G(z) &= z^{n+2}F\left(\frac{1}{z}\right) = -\alpha_n + \left(\alpha_n(t_1 - t_2) - \alpha_{n-1}\right)z + \left(\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\right)z^2 + \cdots \\ &\quad + \left(\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\right)z^n + \left(\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\right)z^{n+1} + \alpha_0 t_1 t_2 z^{n+2} \\ &= -a_n + \left(a_n(t_1 - t_2) - a_{n-1}\right)z + \left(a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\right)z^2 + \cdots + \left(a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\right)z^n \\ &\quad + \left(a_1 t_1 t_2 + a_0(t_1 - t_2)\right)z^{n+1} + a_0 t_1 t_2 z^{n+2} + \mathrm{i}\left[-b_n + \left(b_n(t_1 - t_2) - b_{n-1}\right)z + \left(b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}\right)z^2 + \cdots \right. \\ &\quad \left. + \left(b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0\right)z^n + \left(b_1 t_1 t_2 + b_0(t_1 - t_2)\right)z^{n+1} + b_0 t_1 t_2 z^{n+2}\right] = -\alpha_n + \psi(z) \end{aligned} \tag{5}$$

where

$$\psi(z)=\left(a_n(t_1-t_2)-a_{n-1}\right)z+\left(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2}\right)z^2+\cdots$$
$$+\left(a_2t_1t_2+a_1(t_1-t_2)-a_0\right)z^n+\left(a_1t_1t_2+a_0(t_1-t_2)\right)z^{n+1}+a_0t_1t_2z^{n+2}$$
$$+\mathrm{i}\Big[\left(b_n(t_1-t_2)-b_{n-1}\right)z+\left(b_nt_1t_2+b_{n-1}(t_1-t_2)-b_{n-2}\right)z^2+\cdots$$
$$+\left(b_2t_1t_2+b_1(t_1-t_2)-b_0\right)z^n+\left(b_1t_1t_2+b_0(t_1-t_2)\right)z^{n+1}+b_0t_1t_2z^{n+2}\Big].$$

Now

$$|\psi(z)|\le\left|\left(a_n(t_1-t_2)-a_{n-1}\right)\right||z|+\left|\left(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2}\right)\right|\left|z^2\right|+\cdots+\left|\left(a_2t_1t_2+a_1(t_1-t_2)-a_0\right)\right|\left|z^n\right|$$
$$+\left|\left(a_1t_1t_2\right)+a_0(t_1-t_2)\right|\left|z^{n+1}\right|+\left|a_0t_1t_2\right|\left|z^{n+2}\right|+\left|\left(b_n(t_1-t_2)-b_{n-1}\right)\right||z|+\left|\left(b_nt_1t_2+b_{n-1}(t_1-t_2)-b_{n-2}\right)\right|\left|z^2\right|+\cdots$$
$$+\left|\left(b_2t_1t_2+b_1(t_1-t_2)-b_0\right)\right|\left|z^n\right|+\left|\left(b_1t_1t_2+b_0(t_1-t_2)\right)\right|\left|z^{n+1}\right|+\left|b_0t_1t_2\right|\left|z^{n+2}\right|$$

This gives after using hypothesis, for $|z|=1/t_1$

$$|\psi(z)|\le(a_n+b_n)+\left(|a_0|-a_0\right)\frac{t_2}{t_1^{n+1}}+\left(|b_0|-b_0\right)\frac{t_2}{t_1^{n+1}}=K_1(Say),$$

Clearly, $\psi(0)=0,\psi'(0)=\alpha_n(t_1-t_2)-\alpha_{n-1}$ and $|\psi(z)|\le K_1$ for $|z|=1/t_1$

Thus, it follows by Lemma 2 that

$$|\psi(z)|\le\frac{K_1|z|}{\frac{1}{t_1^2}}\left[\frac{K_1|z|+\frac{1}{t_1^2}\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right|}{K_1+\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|}\right],\ |z|\le\frac{1}{t_1}.$$

From (5), we get

$$|G(z)|\ge|\alpha_n|-\frac{K_1|z|}{\frac{1}{t_1^2}}\left[\frac{K_1|z|+\frac{1}{t_1^2}\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right|}{K_1+\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|}\right]$$
$$=\frac{K_1|\alpha_n|+|\alpha_n|\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|-K_1^2t_1^2|z|^2-K_1|z|\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right|}{K_1+\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|}$$
$$=\frac{-K_1^2t_1^2|z|^2+\left(|\alpha_n|-K_1\right)\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|+K_1|\alpha_n|}{K_1+\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|}>0,$$

if

$$K_1^2t_1^2|z|^2-\left(|\alpha_n|-K_1\right)\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right||z|-K_1|\alpha_n|<0.$$

This gives $|G(z)|>0$, if

$$|z|<\frac{-\left(|\alpha_n|-K_1\right)\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right|+\left\{\left(|\alpha_n|-K_1\right)^2\left|\alpha_n(t_1-t_2)-\alpha_{n-1}\right|^2+4K_1^3t_1^2|\alpha_n|\right\}^{\frac{1}{2}}}{2K_1^2t_1^2}=\frac{1}{r_1}.$$

Consequently, all the zeros of $G(z)$ lie in

$$|z|\ge\min\left(\frac{1}{r_1},\frac{1}{t_1}\right).$$

Since $F(z)=z^{n+2}G(1/z)$, it follows that all the zeros of $F(z)$ and hence all the zeros of $P(z)$ lie in

$$|z|\le\max(r_1,t_1).\tag{6}$$

Again from (4)

$$|F(z)|\ge|\alpha_0|t_1t_2-|\phi(z)|,\tag{7}$$

where

$$|\phi(z)| \le |a_n||z^{n+2}| + |(a_n(t_1-t_2)-a_{n-1})||z^{n+1}| + \cdots + |(a_1t_1t_2 + a_0(t_1-t_2))||z| + |b_n||z^{n+2}| + |(b_n(t_1-t_2)-b_{n-1})||z^{n+1}| + \cdots + |(b_1t_1t_2 + b_0(t_1-t_2))||z|.$$

Therefore, for $|z| = t_1$, we have by using the hypothesis

$$\begin{aligned}|\phi(z)| &\le |a_n||t_1^{n+2}| + |(a_n(t_1-t_2)-a_{n-1})||t_1^{n+1}| + \cdots \\ &\quad + |(a_1t_1t_2 + a_0(t_1-t_2))||t_1| + |b_n||t_1^{n+2}| + |(b_n(t_1-t_2)-b_{n-1})||t_1^{n+1}| + \cdots + |(b_1t_1t_2 + b_0(t_1-t_2))||t_1| \\ &= (a_n+b_n)t_1^{n+2} + (|a_n|+|b_n|)t_1^{n+2} - (a_0+b_0)t_1t_2 = K_2 \text{ (say) for } |z| = t_1\end{aligned}$$

Therefore, it follows again by Lemma 2 that

$$|\phi(z)| \le \frac{K_2|z|}{t_1^2}\left[\frac{K_2|z| + t_1^2|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)|}{K_2 + |\alpha_1t_1t_2 + \alpha_0(t_1-t_2)||z|}\right].$$

Using this result in (7), we get

$$\begin{aligned}|F(z)| &\ge |\alpha_0|t_1t_2 - \frac{K_2|z|}{t_1^2}\left[\frac{K_2|z| + t_1^2|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)|}{K_2 + |\alpha_1t_1t_2 + \alpha_0(t_1-t_2)||z|}\right] \\ &= \frac{|\alpha_0|t_1^3t_2\left[K_2 + |\alpha_1t_1t_2 + \alpha_0(t_1-t_2)||z|\right] - K_2^2|z|^2 - K_2t_1^2|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)||z|}{t_1^2(K_2 + \alpha_1t_1t_2 + \alpha_0(t_1-t_2)|z|)} \\ &= \frac{-K_2^2|z|^2 - \left[K_2 - |\alpha_0|t_1t_2\right]|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)|t_1^2|z| + K_2|\alpha_0|t_1^3t_2}{t_1^2\left(K_2 + |\alpha_1t_1t_2 + \alpha_0(t_1-t_2)||z|\right)} > 0,\end{aligned}$$

if

$$K_2^2|z|^2 + \left[|\alpha_0|t_1t_2 - K_2\right]|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)|t_1^2|z| - K_2|\alpha_0|t_1^3t_2 < 0.$$

Thus $|F(z)| > 0$, if

$$|z| < \frac{-\left[|\alpha_0|t_1t_2 - K_2\right]|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)|t_1^2 + \left\{\left[|\alpha_0|t_1t_2 - K_2\right]^2|\alpha_1t_1t_2 + \alpha_0(t_1-t_2)|^2t_1^4 + 4K_2^3|\alpha_0|t_1^3t_2\right\}^{\frac{1}{2}}}{2K_2^2} = r_2.$$

This shows that all the zeros of $F(z)$ and hence of the polynomial $P(z)$ lie in

$$|z| \ge \min\left(r_2, \frac{1}{t_1}\right). \tag{8}$$

Combining (6) and (8), we get the desired result.

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}f(z) &= (t_2+z)(t_1-z)P(z) \\ &= -\alpha_nz^{n+2} + \sum_{r=0}^{n+1}(\alpha_rt_1t_2 + \alpha_{r-1}(t_1-t_2) - \alpha_{r-2})z^r.\end{aligned}$$

Since $f(z)$ is a polynomial of degree n + 2, it follows by applying Lemma 3 to $f(z)$ with $p = n+1$ and $r = t_1$, that all the zeros of $f(z)$ lie in

$$|z| \le \max\left\{t_1, \sum_{r=0}^{n+1}\frac{|\alpha_rt_1t_2 + \alpha_{r-1}(t_1-t_2) - \alpha_{r-2}|}{|\alpha_n|t_1^{n-r+1}}\right\}. \tag{9}$$

Now

$$t_1=\left|\sum_{r=0}^{n+1}\frac{\left(\alpha_r t_1t_2+\alpha_{r-1}\left(t_1-t_2\right)-\alpha_{r-2}\right)}{\left|\alpha_n\right|t_1^{n-r+1}}\right|=\left|\sum_{r=0}^{n+1}\frac{\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-\alpha_{r-2}+i\left(b_r t_1t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)\right)}{\left|\alpha_n\right|t_1^{n-r+1}}\right|$$

$$\le\sum_{r=0}^{n+1}\frac{\left|\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=0}^{n+1}\frac{\left|\left(b_r t_1t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}$$

$$=\frac{\left|a_0\right|t_1t_2}{\left|\alpha_n\right|t_1^{n+1}}+\left|\frac{a_1t_1t_2+a_0\left(t_1-t_2\right)}{\left|\alpha_n\right|t_1^{n+1}}\right|+\sum_{r=2}^{\lambda+1}\frac{\left|\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=\lambda+2}^{n+1}\frac{\left|\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}$$

$$+\left[\frac{\left|b_0\right|t_1t_2}{\left|\alpha_n\right|t_1^{n+1}}+\left|\frac{b_1t_1t_2+b_0\left(t_1-t_2\right)}{\left|\alpha_n\right|t_1^{n+1}}\right|+\sum_{r=2}^{\lambda+1}\frac{\left|\left(b_r t_1t_2+b_{r-1}(t_1-t_2)-b_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=\lambda+2}^{n+1}\frac{\left|\left(b_r t_1t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}\right]$$

$$\le\frac{\left|a_0\right|t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{\left|a_1t_1-a_0\right|t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{\left|a_0\right|}{\left|\alpha_n\right|t_1^{n-1}}+\sum_{r=2}^{\lambda+1}\frac{\left|\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=\lambda+2}^{n+1}\frac{\left|\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}$$

$$+\left[\frac{\left|b_0\right|t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{\left|b_1t_1-b_0\right|t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{\left|b_0\right|}{\left|\alpha_n\right|t_1^{n-1}}+\sum_{r=2}^{\lambda+1}\frac{\left|(b_r t_1t_2+b_{r-1}(t_1-t_2)-b_{r-2})\right|}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=\lambda+2}^{n+1}\frac{\left|\left(b_r t_1t_2+b_{r-1}(t_1-t_2)-b_{r-2}\right)\right|}{\left|\alpha_n\right|t_1^{n-r+1}}\right].$$

Using the hypothesis, we get

$$t_1\le\frac{\left|a_0\right|t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{a_1t_2}{\left|\alpha_n\right|t_1^{n-1}}-\frac{a_0t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{\left|a_0\right|}{\left|\alpha_n\right|t_1^{n-1}}+\sum_{r=2}^{\lambda+1}\frac{\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=\lambda+2}^{n+1}\frac{\left(a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)}{\left|\alpha_n\right|t_1^{n-r+1}}$$

$$+\left[\frac{\left|b_0\right|t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{b_1t_2}{\left|\alpha_n\right|t_1^{n-1}}-\frac{b_0t_2}{\left|\alpha_n\right|t_1^{n}}+\frac{\left|b_0\right|}{\left|\alpha_n\right|t_1^{n-1}}+\sum_{r=2}^{\lambda+1}\frac{\left(b_r t_1t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)}{\left|\alpha_n\right|t_1^{n-r+1}}+\sum_{r=\lambda+2}^{n+1}\frac{\left(b_r t_1t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)}{\left|\alpha_n\right|t_1^{n-r+1}}\right]$$

$$=\frac{t_1}{\left|\alpha_n\right|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}}-a_n\right)+\frac{1}{t_1^n}\left(\left|a_0\right|-a_0\right)\right\}+\frac{t_2}{\left|\alpha_n\right|}\left\{\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}}+\frac{1}{t_1^n}\left(\left|a_0\right|-a_0\right)\right\}$$

$$+\frac{t_1}{\left|\alpha_n\right|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}}-b_n\right)+\frac{1}{t_1^n}\left(\left|b_0\right|-b_0\right)\right\}+\frac{t_2}{\left|\alpha_n\right|}\left\{\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}}.+\frac{1}{t_1^n}\left(\left|b_0\right|-b_0\right)\right\}.$$

Hence by (9) all the zeros of $f(z)$ lie in the circle $|z|\le R$,
where

$$R=\frac{t_1}{\left|\alpha_n\right|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}}-a_n\right)+\frac{1}{t_1^n}\left(\left|a_0\right|-a_0\right)\right\}+\frac{t_2}{\left|\alpha_n\right|}\left\{\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}}+\frac{1}{t_1^n}\left(\left|a_0\right|-a_0\right)\right\}+\frac{t_1}{\left|\alpha_n\right|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}}-b_n\right)+\frac{1}{t_1^n}\left(\left|b_0\right|-b_0\right)\right\}$$

$$+\frac{t_2}{\left|\alpha_n\right|}\left\{\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}}+\frac{1}{t_1^n}\left(\left|b_0\right|-b_0\right)\right\}.$$

Since every zero of $P(z)$ is also a zero of $f(z)$, the theorem is proved completely.

$$f(z)=(t_2+z)(t_1-z)P(z)=-\alpha_n z^{n+2}+\left(\alpha_n\left(t_1-t_2\right)-\alpha_{n-1}\right)z^{n+1}+\left(\alpha_n t_1t_2+\alpha_{n-1}\left(t_1-t_2\right)-\alpha_{n-2}\right)z^n+\cdots$$

$$+\left(\alpha_2t_1t_2+\alpha_1\left(t_1-t_2\right)-\alpha_0\right)z^2+\left(\alpha_1t_1t_2+\alpha_0\left(t_1-t_2\right)\right)z+\alpha_0t_1t_2$$

$$=-\alpha_n z^{n+2}+\left(\alpha_n\left(t_1-t_2\right)-\alpha_{n-1}\right)z^{n+1}+\sum_{r=0}^{n}\left\{a_r t_1t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right\}z^r+\mathrm{i}\sum_{r=0}^{n}\left\{b_r t_1t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right\}z^r.$$

This gives

$$\begin{aligned}\left|f(z)\right| &\ge \left|-\alpha_n z^{n+2}+\left(\alpha_n\left(t_1-t_2\right)-\alpha_{n-1}\right)z^{n+1}\right|-\left|\sum_{r=0}^{n}\left\{a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right\}z^r\right| \\ &\quad -\left|\sum_{r=0}^{n}\left\{b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right\}z^r\right| \\ &\ge |z|^{n+1}\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\sum_{r=0}^{n}\left|a_r t_1 t_2+a_{r-1}(t_1-t_2)-a_{r-2}\right||z|^r \\ &\quad -\sum_{r=0}^{n}\left|b_r t_1 t_2+b_{r-1}(t_1-t_2)-b_{r-2}\right||z|^r .\end{aligned}$$

Let $|z|>t_1$, we get by using the hypothesis

$$\begin{aligned}\left|f(z)\right| &\ge |z|^{n+1}\Bigg[\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\sum_{r=0}^{n}\left|a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right|\frac{1}{t_1^{n-r+1}}-\sum_{r=0}^{n}\left|b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right|\frac{1}{t_1^{n-r+1}}\Bigg] \\
&= |z|^{n+1}\Bigg[\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\left\{\frac{|a_0|t_2}{t_1^n}+\left|\left(a_1t_1-a_0\right)t_2+a_0t_1\right|\frac{1}{t_1^n}\right\}-\left\{\frac{|b_0|t_2}{t_1^n}+\left|\left(b_1t_1-b_0\right)t_2+b_0t_1\right|\frac{1}{t_1^n}\right\} \\
&\quad -\sum_{r=2}^{\lambda+1}\left|a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right|\frac{1}{t_1^{n-r+1}}-\sum_{r=\lambda+2}^{n}\left|a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right|\frac{1}{t_1^{n-r+1}} \\
&\quad -\sum_{r=2}^{\lambda+1}\left|b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right|\frac{1}{t_1^{n-r+1}}-\sum_{r=\lambda+2}^{n}\left|b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right|\frac{1}{t_1^{n-r+1}}\Bigg] \\
&\ge |z|^{n+1}\Bigg[\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\left\{\frac{|a_0|t_2}{t_1^n}+\frac{|a_1t_1-a_0|t_2}{t_1^n}+\frac{|a_0|t_1}{t_1^n}\right\} \\
&\quad -\left\{\frac{|b_0|t_2}{t_1^n}+\frac{|b_1t_1-b_0|t_2}{t_1^n}+\frac{|b_0|t_1}{t_1^n}\right\}-\sum_{r=2}^{\lambda+1}\left|a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right|\frac{1}{t_1^{n-r+1}} \\
&\quad -\sum_{r=\lambda+2}^{n}\left|a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right|\frac{1}{t_1^{n-r+1}}-\sum_{r=2}^{\lambda+1}\left|b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right|\frac{1}{t_1^{n-r+1}} \\
&\quad -\sum_{r=\lambda+2}^{n}\left|b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right|\frac{1}{t_1^{n-r+1}}\Bigg] \\
&= |z|^{n+1}\Bigg[\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\frac{|a_0|t_2}{t_1^n}-\frac{\left(a_1t_1-a_0\right)t_2}{t_1^n}-\frac{|a_0|t_1}{t_1^n}+\left\{-\frac{|b_0|t_2}{t_1^n}-\frac{\left(b_1t_1-b_0\right)t_2}{t_1^n}-\frac{|b_0|t_1}{t_1^n}\right\} \\
&\quad -\sum_{r=2}^{\lambda+1}\left(a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\frac{1}{t_1^{n-r+1}}-\sum_{r=\lambda+2}^{n}\left(a_r t_1 t_2+a_{r-1}\left(t_1-t_2\right)-a_{r-2}\right)\frac{1}{t_1^{n-r+1}} \\
&\quad -\sum_{r=2}^{\lambda+1}\left(b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)\frac{1}{t_1^{n-r+1}}-\sum_{r=\lambda+2}^{n}\left(b_r t_1 t_2+b_{r-1}\left(t_1-t_2\right)-b_{r-2}\right)\frac{1}{t_1^{n-r+1}}\Bigg] \\
&= |z|^{n+1}\Bigg[\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\frac{|a_0|t_2}{t_1^n}-\frac{a_1t_2}{t_1^{n-1}}+\frac{a_0t_2}{t_1^n}-\frac{|a_0|t_1}{t_1^n} \\
&\quad +\left\{-\frac{|b_0|t_2}{t_1^n}-\frac{b_1t_2}{t_1^{n-1}}+\frac{b_0t_2}{t_1^n}-\frac{|b_0|t_1}{t_1^n}\right\}-\left\{-\left(\frac{a_1t_2}{t_1^{n-1}}+\frac{a_0}{t_1^{n-1}}\right)+\left(\frac{a_{\lambda+1}t_2+a_\lambda}{t_1^{n-\lambda-1}}\right)\right\}+\left\{-\left(\frac{a_{\lambda+1}t_2+a_\lambda}{t_1^{n-\lambda-1}}\right)+a_nt_2+a_{n-1}\right\} \\
&\quad -\left\{-\left(\frac{b_1t_2}{t_1^{n-1}}+\frac{b_0}{t_1^{n-1}}\right)+\left(\frac{b_{\lambda+1}t_2+b_\lambda}{t_1^{n-\lambda-1}}\right)\right\}+\left\{-\left(\frac{b_{\lambda+1}t_2+b_\lambda}{t_1^{n-\lambda-1}}\right)+b_nt_2+b_{n-1}\right\}\Bigg] \\
&= |z|^{n+1}\Bigg[\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|-\frac{|a_0|t_2}{t_1^n}+\frac{a_0t_2}{t_1^n}-\frac{|a_0|t_1}{t_1^n}+\frac{a_0}{t_1^{n-1}}-2\left(\frac{a_{\lambda+1}t_2+a_\lambda}{t_1^{n-\lambda-1}}\right)+a_nt_2+a_{n-1} \\
&\quad +\left\{-\frac{|b_0|t_2}{t_1^n}+\frac{b_0t_2}{t_1^n}-\frac{|b_0|t_1}{t_1^n}+\frac{b_0}{t_1^{n-1}}-2\left(\frac{b_{\lambda+1}t_2+b_\lambda}{t_1^{n-\lambda-1}}\right)+b_nt_2+b_{n-1}\right\}\Bigg]>0,\end{aligned}$$

if

$$\left|\alpha_n z+\alpha_{n-1}-\alpha_n\left(t_1-t_2\right)\right|>2\left(\frac{a_{\lambda+1}t_2+a_\lambda}{t_1^{n-\lambda-1}}\right)-a_nt_2-a_{n-1}+\frac{|a_0|t_2-a_0t_2}{t_1^n}+\frac{|a_0|t_1-a_0t_1}{t_1^n}$$
$$+\left\{2\left(\frac{b_{\lambda+1}t_2+b_\lambda}{t_1^{n-\lambda-1}}\right)-b_nt_2-b_{n-1}+\frac{|b_0|t_2-b_0t_2}{t_1^n}+\frac{|b_0|t_1-b_0t_1}{t_1^n}\right\}.$$

Thus $|z|>0$, if

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}-\left(t_1-t_2\right)\right|>\frac{t_1}{|\alpha_n|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}}-\frac{a_{n-1}}{t_1}\right)+\frac{1}{t_1^n}\left(|a_0|-a_0\right)\right\}+\frac{t_2}{|\alpha_n|}\left\{\left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}}-a_n\right)+\frac{1}{t_1^n}\left(|a_0|-a_0\right)\right\}$$
$$+\left[\frac{t_1}{|\alpha_n|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}}-\frac{b_{n-1}}{t_1}\right)+\frac{1}{t_1^n}\left(|b_0|-b_0\right)\right\}+\frac{t_2}{|\alpha_n|}\left\{\left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}}-b_n\right)+\frac{1}{t_1^n}\left(|b_0|-b_0\right)\right\}\right].$$

This shows that those zeros of $f(z)$ whose modulus is greater than t_1, lie in the circle

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}-\left(t_1-t_2\right)\right|\le\frac{t_1}{|\alpha_n|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}}-\frac{a_{n-1}}{t_1}\right)+\frac{1}{t_1^n}\left(|a_0|-a_0\right)\right\}+\frac{t_2}{|\alpha_n|}\left\{\left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}}-a_n\right)+\frac{1}{t_1^n}\left(|a_0|-a_0\right)\right\}$$
$$+\frac{t_1}{|\alpha_n|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}}-\frac{b_{n-1}}{t_1}\right)+\frac{1}{t_1^n}\left(|b_0|-b_0\right)\right\}+\frac{t_2}{|\alpha_n|}\left\{\left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}}-b_n\right)+\frac{1}{t_1^n}\left(|b_0|-b\right)_0\right\}.$$

It can be easily verified that those zeros of $f(z)$ whose modulus is less than t_1, lie in the circle as well. Therefore, we conclude that all zeros of $f(z)$ and hence $P(z)$ lie in

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}-\left(t_1-t_2\right)\right|\le\frac{t_1}{|\alpha_n|}\left\{\left(\frac{2a_\lambda}{t_1^{n-\lambda}}-\frac{a_{n-1}}{t_1}\right)+\frac{1}{t_1^n}\left(|a_0|-a_0\right)\right\}+\frac{t_2}{|\alpha_n|}\left\{\left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}}-a_n\right)+\frac{1}{t_1^n}\left(|a_0|-a_0\right)\right\}$$
$$+\frac{t_1}{|\alpha_n|}\left\{\left(\frac{2b_\lambda}{t_1^{n-\lambda}}-\frac{b_{n-1}}{t_1}\right)+\frac{1}{t_1^n}\left(|b_0|-b_0\right)\right\}+\frac{t_2}{|\alpha_n|}\left\{\left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}}-b_n\right)+\frac{1}{t_1^n}\left(|b_0|-b_0\right)\right\}.$$

This completes the proof of the theorem.

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