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An Upwind Finite Volume Element Method for Nonlinear Convection Diffusion Problem

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Abstract

A class of upwind finite volume element method based on tetrahedron partition is put forward for a nonlinear convection diffusion problem. Some techniques, such as calculus of variations, commutating operators and the a priori estimate, are adopted. The a priori error estimate in L^2-norm and H^1-norm is derived to determine the error between the approximate solution and the true solution.

Keywords: Nonlinear, Convection-Diffusion, Tetrahedron Partition, Error Estimates

1. Introduction

Consider the following nonlinear convection-diffusion problem:

$$u_t - \mu\Delta u + \nabla\cdot\mathbf{F}(\mathbf{x},u) = g(\mathbf{x},u), \quad \mathbf{x}\in\Omega, t\in J \tag{1}$$

$$u(\mathbf{x},t) = 0, \quad \mathbf{x}\in\Gamma, t\in J, \tag{2}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{3}$$

where $\Omega\subset R^3$ is a bounded region with piecewise smooth boundary Γ. μ is a small positive constant and $\mathbf{F}(\mathbf{x},u) = (f_1(\mathbf{x},u), f_2(\mathbf{x},u), f_3(\mathbf{x},u))$ is a smooth vector function on $\bar{\Omega}\times R$, $\mathbf{F}(\mathbf{x},0) = 0$.

The finite volume element method (FVEM) is a discrete technique for partial differential equations, especially for those arising from physical conservation laws, including mass, momentum and energy. This method has been introduced and analyzed by R. Li and his collaborators since 1980s, see [1] for details. The FVEM uses a volume integral formulation of the original problem and a finite partitioning set of covolumes to discretize the equations.The approximate solution is chosen out of a finite element spaces [1-3] The FVEM is widely used in computational fluid mechanics and heat transfer problems [2-5]. It possesses the important and crucial property of inheriting the physical conservation laws of the original problem locally. Thus it can be expected to capture shocks, or to study other physical phenomena more effectively.

On the other hand, the convection-dominated diffusion problem has strong hyperbolic characteristics, and therefore the numerical method is very difficult in mathematics and mechanics. when the central difference method, though it has second-order accuracy, is used to solve the convection-dominated diffusion problem, it produces numerical diffusion and oscillation near the discontinuous domain, making numerical simulation failure. The case usually occurs when the finite element methods (FEM) and FVEM are used for solve the convection-dominated diffusion problem.

For the two-phase plane incompressible displacement problem which is assumed to be Ω-periodic, J. Douglas, Jr., and T.F.Russell have published some articles on the characteristic finite difference method and FEM to solve the convection-dominated diffusion problems and to overcome oscillation and faults likely to occur in the traditional method [6]. Tabata and his collaborators have been studying upwind schemes based triangulation for convection-diffusion problem since 1977 [7-11]. Yuan, starting from the practical exploration and development of oil-gas resources, put forward the upwind finite difference fractional steps methods for the two-phase three-dimensional compressible displacement problem [12].

Most of the papers known concern on the FVEM for one- and two-dimensional linear partial differential equations [1-4,13,14]. In recent years, M. Feistauer [15,16], by introducing lumping operator, constructed finite volume-finite element method for nonlinear convection-diffusion problems. On the other hand, because the FEM costs great expense to solve the three-dimensional problems, finite difference methods (FDM) are usually used to approximate the problems [12]. These works inspire us to look into the subject how to use the upwind FVEM to solve three-dimensional nonlinear convection-dominated diffusion problems. In this article, we continue to our work [17] and put forward an upwind FVEM for three-dimensional nonlinear convection-dominated diffusion problems based on tetrahedron partition and its dual partition of Ω. Some techniques, such as calculus of variations, commutating operator and the a priori error estimate, are adopted. The a priori error estimate in L^2-norm and H^1-norm is derived to determine the error between the approximate solution and the true solution.

The remainder of this paper is organized as follows. In Section 2, we put forward the upwind FVEM for problem (1). In this section, we introduce notations, construct tetrahedron mesh partition T_h of Ω and its dual partition. Some auxiliary lemmas and the a priori error estimate in L^2-norm and H^1-norm of the scheme are shown In Section 3 and Section 4, respectively. In Section 5, some concluding remarks are presented.

Throughout this paper we use C (without or with subscript) to denote a generic constant independent of discrete parameters. We also adopt the standard notations of Sobolev spaces and norms and semi-norms as in [18, 19].

2. The UFVE Method

Suppose problem (1) satisfy condition (A1):

(C_1) Continuity condition: $g(\mathbf{x},u)\in L^2(\Omega\times R)$ is Lipschitz continuous w.r.t. the second variable u.

(C_2) The vector function $\mathbf{F}(\mathbf{x},u)$ has 1-order continuous partial derivative w.r.t. $\mathbf{x}$ and u.

Suppose the true solution of problem (1) possess certain smooth and satisfy:

(R) Regular condition:

$$u\in W^{2,\infty}(L^\infty)\cap H^1(L^2)\cap L^\infty(H^2).$$

Before presenting the numerical scheme we introduce some notations. For simplicity we assume Ω is the domain $\Omega=(0,X_L)\times(0,Y_L)\times(0,Z_L)$. Firstly, Let us consider a family of regular tetrahedron partition T_h in the domain $\bar{\Omega}$, which is a closure of Ω. Let h be maximum diameter of cell of T_h. For a fixed tetrahedron partition $T_h=\{K\}$, we define a closed tetrahedron set $\{K_i\}_{i=1}^{N_K}$ and node set

$$\bar{\Omega}_h=\Omega_0\bigcup\Gamma_h=\{P_i\}_{i=1}^{M_1}\bigcup\{P_i\}_{i=M_1+1}^{M_2}=\{P_i\}_{i=1}^{M_2},$$

where Ω_0 is inner nodes set of Ω and Γ_h boundary nodes set on Γ. Let $E_h=\{e_i:1\le i\le M_E\}$ be all edges set.

Definition 2.1. Suppose that $T=\{T_h:0<h\le h_0\}$ is a set of tetrahedron partition of Ω, the set T_h is called regular, if there exists a positive constant σ_1 independent of h, such that

$$\max_{K\in T_h} h_K\rho_K\le\sigma_1,\forall h\in\{0,h_0\}$$

where h_K and ρ_K are the diameter of K and the maximum diameter of circumscribing sphere of tetrahedron K, respectively.

Definition 2.2. The two tetrahedron cells are called face-adjacent if they have one common face, while edge-adjacent if they have one common edge.

Definition 2.3. The two nodes are called adjacent if they form one edge which belongs to E_h. Denote by $\Lambda_i=\{j:P_j$ is adjacent to $P_i,P_i,P_j\in\Omega_h\}$.

For a given tetrahedron partition T_h with nodes $\{P_i\}\in\Omega_h$ and edges $\{e_i\}\in E_h$, we construct two kind of dual partitions. First, we will construct the circumcenter dual partition of T_h. $\forall P_i\in\Omega_h$, let $\Omega_h(P_i)=\{K:K\in T_h,P_i$ is a vertex of $K\}$. Let Q_j be a circumcenter of $K(\in\Omega_h(P_i))$. Connecting Q_j of the two face-adjacent tetrahedron cells which belong to $\Omega_h(P_i)$, then we can derive a polyhedron $K_{P_i}^*$ which surrounds the node P_i. Q_j are vertices of the polyhedron $K_{P_i}^*$ which is called circumcenter dual partition corresponding to node P_i. $T_h^*=\{K_{P_i}^*:P_i\in\Omega_h\}$ is the circumcenter dual partition of T_h. Denote by P_{ij} the midpoint of P_i and its adjacent node P_j.

The other dual partition as follows. $\forall e_k\in E_h$, let $\Omega_h(e_k)=\{K:K\in T_h$ and e_k is a edge of $K\}$. Denote by P_{k_1} and P_{k_2} the vertices of the edge e_k and Q_j the circumcenter of the $K\in\Omega_h(e_k)$. Suppose $K_{e_k}^*$ is a polyhedron whose vertices are P_{k_1}, P_{k_2}, Q_js. $K_{e_k}^*$ is called dual cell for edge e_k. $\bar{T}_h^*=(K_{e_k}^*)_{k=1}^{M_E}$ is the other dual partition to T_h.

Let Ω_h^* be the node set of dual partition. For $Q\in\Omega_h^*$, let K_Q be tetrahedron cell which includes Q. Let $|K_P^*|$ and $|K_Q|$ be volumes of dual cell K_P^* and tetrahedron cell K_Q, respectively. Let h be the diameter of tetrahedron cell K_Q. As follows, we assume that the partition family T_h is regular, *i.e.*, there exist positive constants C_1,C_2 independent of h, such that the following condition (A2) satisfies:

$$\begin{cases}C_1h^3\le|K_p^*|\le C_2h^3, & P\in\bar{\Omega}_h,\\ C_1h^3\le|K_Q|\le C_2h^3, & Q\in\Omega_h^*.\end{cases}\quad(4)$$

Suppose that a trial function space $U_h\subset H_0^1(\Omega)$, whose basis functions are $\{\phi(P_i)\}_{i=1}^{M_1}$ possessing the form $(\alpha_0+\alpha_1x+\alpha_2y+\alpha_3z)$ based on T_h [15], and $\phi(P_i)=0,P_i\in\Gamma_h$. Test function space $V_h\subset L^2(\Omega)$ is a piecewise constant function space corresponding to the dual partition T_h^*, whose basis functions are $\{\psi(P),P\in\Omega_h\}$.

$$\psi(P)=\begin{cases}1, & P\in K_P^*,\\ 0, & \text{otherwise},\end{cases}$$

and $\psi(P)=0, P\in\Gamma_h$.

For the following analysis, we introduce two interpolation operators. Suppose that Π_h and Π_h^* are interpolation operators from H_0^1 to U_h and V_h, respectively, satisfying

$$\Pi_h u=\sum_{i=1}^{M_1} u(P_i)\phi(P_i). \tag{5}$$

$$\Pi_h^* u=\sum_{K_P^*\in T_h^*} u(P)\psi(P). \tag{6}$$

Multiplying both sides of (1) by v, integrating on dual partition cell $K_{P_i}^*$, using Green formula, and summing with respect to $P_i\in\Omega_h$, we have

$$(u_t,v)+a(u,v)+b(u,v)=(g,v), v\in H_0^1(\Omega), \tag{7}$$

where

$$a(u,v)=\sum_{P_i\in\Omega_h}\left[\mu\int_{K_{P_i}^*}\nabla u\cdot\nabla v\mathrm{d}\mathbf{x}-\mu\int_{\partial K_{P_i}^*}\partial u\partial v v\mathrm{d}s\right], \tag{8}$$

$$b(u,v)=-\sum_{P_i\in\Omega_h}\left(\int_{K_{P_i}^*}\mathbf{F}\cdot\nabla v\mathrm{d}\mathbf{x}-\int_{\partial K_{P_i}^*}\mathbf{F}\cdot v v\mathrm{d}s\right). \tag{9}$$

Converting $\mathbf{F}$ into [1]

$$\mathbf{F}(\mathbf{x},u)=\int_0^u \partial\mathbf{F}(\mathbf{x},\bar{u})\partial\bar{u}\mathrm{d}\bar{u}. \tag{10}$$

Let

$$\begin{aligned}\beta_{ij}^+(\mathbf{x},u)&=\int_0^u \max\left(0,\partial\mathbf{F}(\mathbf{x},\bar{u})\partial\bar{u}\cdot\nu_{ij}\right)\mathrm{d}\bar{u},\\ \beta_{ij}^-(\mathbf{x},u)&=\int_0^u \max\left(0,-\partial\mathbf{F}(\mathbf{x},\bar{u})\partial\bar{u}\cdot\nu_{ij}\right)\mathrm{d}\bar{u},\end{aligned} \tag{11}$$

where ν_{ij} is the unit outward normal vector of $\Gamma_{ij}\subset\partial K_{P_i}^*$. For $u_h\in U_h, v_h\in V_h$, we introduce bilinear form

$$\begin{aligned}b_h(u_h,v_h)=&\sum_{P_i\in\Omega_h} v_h(P_i)\sum_{j\in\Lambda_i}\left|\Gamma_{ij}\right|\\ &\cdot\left[\beta_{ij}^+\left(P_{ij},u_h(P_i)\right)-\beta_{ij}^-\left(P_{ij},u_h(P_j)\right)\right],\end{aligned} \tag{12}$$

where $|\Gamma_{ij}|$ is the area of Γ_{ij}.

So far, we can obtain the semi-discrete upwind finite volume element scheme: Find $u_h\in U_h$, such that

$$(u_{h,t},v_h)+a(u_h,v_h)+b_h(u_h,v_h)=(g(\mathbf{x},u_h),v_h), \tag{13}$$

where

$$a(u_h,v_h)=-\sum_{P_i\in\Omega_h}\mu\int_{\partial K_{P_i}^*}\partial u_h\partial v v_h\mathrm{d}s.$$

Let $\Delta t=T/N$, denote by $t^n=n\Delta t, u^n=u(t^n)$, $u_h^n=u_h(t^n), n=1,2,\cdots,N$, $\partial_t u_h^{n-1}=\left(u_h^n-u_h^{n-1}\right)/\Delta t$.

If approximate solution $u_h^{n-1}\in U_h$ is known, then u_h^n can be found by the following full-discrete upwind finite volume element scheme.

$$\begin{aligned}&\left(\Pi_h^*\partial_t u_h^{n-1},v_h\right)+a\left(u_h^{n-1},v_h\right)+b_h\left(u_h^{n-1},v_h\right)\\ &=\left(g\left(\mathbf{x},u_h^{n-1}\right),v_h\right).\end{aligned} \tag{14}$$

3. Auxiliary Lemmas

Define the discrete norm and the discrete semi-norm [1] as follows.

$$\|u_h\|_{0,h}^2=\|\Pi_h^* u_h\|_0^2=\sum_{K_{P_i}^*\in T_h^*}\left(u_h(P_i)\right)^2\left|K_{P_i}^*\right|, \tag{15}$$

$$|u_h|_{1,h}^2=\sum_{k=1}^{M_E}\left(u_h(P_{k2})-u_h(P_{k1})|e_k|\right)^2\left|K_{e_k}^*\right|, \tag{16}$$

$$\|u_h\|_{1,h}^2=\|u_h\|_{0,h}^2+|u_h|_{1,h}^2, \tag{17}$$

obviously, the discrete norm and the discrete semi-norm are equivalent to the continuous norm and the full-norm on U_h, respectively.

Lemma 1. Suppose all cells K_Q of the partition T_h satisfy conditions (A2), T_h^* is a circumcenter dual partition. $\forall u_h, u_h\in U_h$, there exist positive constants γ, C_0 independent of h, such that

$$a\left(u_h,\Pi_h^* u_h\right)\ge\gamma\|u_h\|_1^2,\quad\forall u_h\in U_h. \tag{18}$$

$$a\left(u_h,\Pi_h^*\bar{u}_h\right)\le C_0\|u_h\|_1\|\bar{u}_h\|_1, \forall u_h,\bar{u}_h\in U_h, \tag{19}$$

$$a\left(u_h,\Pi_h^* u_h\right)=a\left(u_h,\Pi_h^* u_h\right),\quad\forall u_h,\bar{u}_h\in U_h. \tag{20}$$

Remarks:

1) From Lemma 1, we know that $a(\cdot,\cdot)$ is symmetrical and positive definite in U_h.

2) Let $|||u_h|||_1=\left[a\left(u_h,\Pi_h^* u_h\right)\right]^{\frac{1}{2}}$, then $|||\cdot|||_1$ is equivalent to $|\cdot|_1$ in U_h.

Lemma 2. Let $|||u_h|||_0=\left(\Pi_h^* u_h,\Pi_h^* u_h\right)^{\frac{1}{2}}$, $|||\cdot|||_0$ is equivalent to $\|\cdot\|_0$ in U_h.

The proof of lemma 2 can be completed by computing integral on cell K_Q, directly.

Theorem 1. (Trace Theorem) [20]. Suppose that Ω has a piecewise Lipschitz boundary, and that p is a real number in range $1\le p\le\infty$. Then there exists a constant C, such that

$$\|v\|_{L^p(\partial\Omega)}\le C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p},\quad\forall v\in W_p^1(\Omega).$$

Lemma 3. For h small enough, suppose P' is a random point in dual partition cell $K_{P_i}^*$,

$$\Gamma_{ij}=K_{P_i}^*\bigcap K_{P_j}^*,$$

then

$$\sum_{j\in\Lambda_i}\iint_{\Gamma_{ij}}\left|u(P')-u(\mathbf{x})\right|\mathrm{d}s\le Ch^2\left(\|u\|_{1,K_{P_i}^*}+\|u\|_{2,K_{P_i}^*}\right). \quad (21)$$

Proof. From Hölder inequality, we can get that

$$\sum_{j\in\Lambda_i}\iint_{\Gamma_{ij}}\left|u(P')-u(\mathbf{x})\right|\mathrm{d}s \le Ch\sum_{j\in\Lambda_i}\left(\iint_{\Gamma_{ij}}\left|u(P')-u(\mathbf{x})\right|^2\mathrm{d}s\right)^{\frac{1}{2}}$$

Using Taylor expansion, trace theory in which we choose $p=2$ and Hölder inequality, we can complete the proof of lemma 3.

Lemma 4. For $\forall u_h,\bar{u}_h\in U_h$, there exists a positive constant C, such that

$$\left(u_h,\Pi_h^*\bar{u}_h\right)=\left(u_h,\Pi_h^*\bar{u}_h\right), \quad (22)$$

$$\left(u_h,\Pi_h^*\bar{u}_h\right)\le C\|u_h\|_0\cdot\|\bar{u}_h\|_0. \quad (23)$$

Proof. From the properties of the functions in U_h, for each partition cell $K(\in T_h)$, we know that $u_h|_K$ has the following expression.

$$u_h(x,y,z,t)\big|_K=u\left(x_{i_0},y_{i_0},z_{i_0},t\right)\lambda_{i_0}+u\left(x_{i_1},y_{i_1},z_{i_1},t\right)\lambda_{i_1} +u\left(x_{i_2},y_{i_2},z_{i_2},t\right)\lambda_{i_2}+u\left(x_{i_3},y_{i_3},z_{i_3},t\right)\lambda_{i_3}, \quad (24)$$

where

$$\lambda_l=\frac{1}{6Ve}\left(a_lx+b_ly+c_lz+d_l\right), l=i_0,i_1,i_2,i_3,$$

and Ve is the volume of tetrahedron $P_{i_0}-P_{i_1}P_{i_2}P_{i_3}$, *i.e.*,

$$Ve=\frac{1}{6}\begin{vmatrix}1 & x_{i_0} & y_{i_0} & z_{i_0}\\ 1 & x_{i_1} & y_{i_1} & z_{i_1}\\ 1 & x_{i_2} & y_{i_2} & z_{i_2}\\ 1 & x_{i_3} & y_{i_3} & z_{i_3}\end{vmatrix}.$$

$P_{i_l}\left(l=i_0,i_1,i_2,i_3\right)$, whose coordinates are $\left(x_{i_l},y_{i_l},z_{i_l}\right)$, are four vertices of tetrahedron cell $P_{i_0}-P_{i_1}P_{i_2}P_{i_3}$ which belongs to T_h. $\lambda_l\left(l=i_0,i_1,i_2,i_3\right)$ are the volume coordinates which are corresponding to tetrahedron cell $P_{i_0}-P_{i_1}P_{i_2}P_{i_3}$. For $l=i_0$,

$$a_{i_0}=-\begin{vmatrix}1 & y_{i_1} & z_{i_1}\\ 1 & y_{i_2} & z_{i_2}\\ 1 & y_{i_3} & z_{i_3}\end{vmatrix}, b_{i_0}=\begin{vmatrix}1 & x_{i_1} & z_{i_1}\\ 1 & x_{i_2} & z_{i_2}\\ 1 & x_{i_3} & z_{i_3}\end{vmatrix},$$

$$c_{i_0}=-\begin{vmatrix}1 & x_{i_1} & y_{i_1}\\ 1 & x_{i_2} & y_{i_2}\\ 1 & x_{i_3} & y_{i_3}\end{vmatrix}, d_{i_0}=\begin{vmatrix}x_{i_1} & y_{i_1} & z_{i_1}\\ x_{i_2} & y_{i_2} & z_{i_2}\\ x_{i_3} & y_{i_3} & z_{i_3}\end{vmatrix}.$$

Analogously, we can define the remaining coefficients $a_l,b_l,c_l,d_l\left(l=i_1,i_2,i_3\right)$. Further,

$$\left(u_h,\Pi_h^*u_h\right)=\sum_{K\in T_h}\sum_{l=i_0,i_1,i_2,i_3}u_h\left(P_l\right)\iiint_{K_{P_l}^*\cap K}u_h\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

For simplifying numerical integral, we divide the polyhedron integral domain $K_{P_{i_0}}^*\cap K$ into six tetrahedron integral domains

$$V_1=\text{tetrahedron}P_{i_0}-P_{i_0i_1}P_{i_0i_1i_2}P_{i_0i_1i_2i_3},$$
$$V_2=\text{tetrahedron}P_{i_0}-P_{i_0i_1}P_{i_0i_1i_3}P_{i_0i_1i_2i_3},$$
$$V_3=\text{tetrahedron}P_{i_0}-P_{i_0i_2}P_{i_0i_1i_2}P_{i_0i_1i_2i_3},$$
$$V_4=\text{tetrahedron}P_{i_0}-P_{i_0i_2}P_{i_0i_2i_3}P_{i_0i_1i_2i_3},$$
$$V_5=\text{tetrahedron}P_{i_0}-P_{i_0i_3}P_{i_0i_1i_3}P_{i_0i_1i_2i_3},$$
$$V_6=\text{tetrahedron}P_{i_0}-P_{i_0i_3}P_{i_0i_2i_3}P_{i_0i_1i_2i_3},$$

where $P_{i_0i_1}$ is the midpoint of segment $P_{i_0}P_{i_1}$ while $P_{i_0i_1i_2}$ and $P_{i_0i_1i_2i_3}$ are circumcenters of triangular surface $\Delta P_{i_0}P_{i_1}P_{i_2}$ and tetrahedron $P_{i_0}-P_{i_1}P_{i_2}P_{i_3}$, respectively. Analogously, we can define the remaining points.

Noting the Equality (24), we have that

$$\iiint_{K_{P_l}^*\cap K}u_h\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_{K_{P_l}^*\cap K}u_h|_K\ \mathrm{d}x\mathrm{d}y\mathrm{d}z =\iiint_{K_{P_l}^*\cap K}[u\left(P_{i_0},t\right)\lambda_{i_0}+u\left(P_{i_1},t\right)\lambda_{i_1} +u\left(P_{i_2},t\right)\lambda_{i_2}+u\left(P_{i_3},t\right)\lambda_{i_3}]\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

For simplicity, we will omit the variable t in function $u(x,y,z,t)$. From volume coordinate formula, noting $\lambda_{i_0}+\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3}=1$, we can derive

$$\iiint_{K_{P_{i_0}}^*\cap K}u_h|_K\ \mathrm{d}x\mathrm{d}y\mathrm{d}z =\sum_{j=1}^{6}\iiint_{V_j}\left[u\left(P_{i_0}\right)\lambda_{i_0}+u\left(P_{i_1}\right)\lambda_{i_1} +u\left(P_{i_2}\right)\lambda_{i_2}+u\left(P_{i_3}\right)\lambda_{i_3}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z =\frac{|K|}{48}\left[-7u\left(P_{i_0}\right)+3u\left(P_{i_1}\right)+3u\left(P_{i_2}\right)+3u\left(P_{i_3}\right)\right]$$

Further,

$$\left(u_h,\Pi_h^*u_h\right)=\sum_{K\in T_h}\frac{|K|}{48}\alpha\begin{pmatrix}-7 & 3 & 3 & 3\\ 3 & -7 & 3 & 3\\ 3 & 3 & -7 & 3\\ 3 & 3 & 3 & -7\end{pmatrix}$$

where

$$\alpha=\left(u_h\left(P_{i_0}\right),u_h\left(P_{i_1}\right),u_h\left(P_{i_2}\right),u_h\left(P_{i_3}\right)\right)$$

and

$$\beta=\left(u_h\left(P_{i_0}\right),u_h\left(P_{i_1}\right),u_h\left(P_{i_2}\right),u_h\left(P_{i_3}\right)\right)^T\cdot\beta,$$

From the above equality, we can complete the proof of

Lemma 4 easily.

4. Convergence Analysis

Now we consider the error estimates of the approximate solution. Let

$$u^n - u_h^n = \left(u^n - \Pi_h u^n\right) + \left(\Pi_h u^n - u_h^n\right) = \rho_h^n + e_h^n.$$

Choosing $t = t^{n-1}$ in (7), then we have

$$\left(u_t\left(t^{n-1}\right), v\right) + a\left(u^{n-1}, v\right) + b\left(u^{n-1}, v\right) = \left(g\left(\mathbf{x}, u^{n-1}\right), v\right). \quad (25)$$

Subtracting (14) from (25), we obtain that

$$\begin{aligned} &\left(\Pi_h^* \partial_t e_h^{n-1}, v_h\right) + a\left(e_h^{n-1}, v_h\right) \\ &= \left(r^n, v_h\right) + a\left(-\rho_h^{n-1}, v_h\right) \\ &+ \left(b_h\left(u_h^{n-1}, v_h\right) - b\left(u^{n-1}, v_h\right)\right) \\ &+ \left(g\left(\mathbf{x}, u^{n-1}\right) - g\left(\mathbf{x}, u_h^{n-1}\right), v_h\right), \end{aligned} \quad (26)$$

where $r^n = \Pi_h^* \partial_t \Pi_h u^{n-1} - u_t\left(t^{n-1}\right)$.

Choosing $v_h = \Pi_h^*\left(e_h^{n-1} + e_h^n\right)$ in Equality (26), denote by W_1, W_2 and T_1, T_2, T_3, T_4 the left and right hand side terms of Equality (26), respectively. We will analyze the six terms successively.

For W_1, from the definition of $|||\cdot|||_0$, we have that

$$W_1 = \frac{1}{2\Delta t}\left(|||e_h^n|||_0^2 - |||e_h^{n-1}|||_0^2\right) \quad (27)$$

Rewriting W_2 as

$$\begin{aligned} W_2 &= a\left(e_h^{n-1} + e_h^n, \Pi_h^*\left(e_h^n + e_h^{n-1}\right)\right) \\ &+ \left(a\left(e_h^{n-1}, \Pi_h^* e_h^{n-1}\right) - a\left(e_h^n, \Pi_h^* e_h^n\right)\right) \\ &+ a\left(e_h^{n-1}, \Pi_h^* e_h^n\right) - a\left(e_h^n, \Pi_h^* e_h^{n-1}\right) \\ &= W_{21} + W_{22} + W_{23}. \end{aligned} \quad (28)$$

From (20) of Lemma 1, we can get the estimate to W_{23} as follows.

$$W_{23} = 0. \quad (29)$$

From (27)-(29), we have

$$\begin{aligned} W_1 + W_2 \geq \frac{1}{2\Delta t}\Bigg[&\left(|||e_h^n|||_0^2 - \frac{\Delta t}{2}|||e_h^n|||_1^2\right) \\ &-\left(|||e_h^{n-1}|||_0^2 - \frac{\Delta t}{2}|||e_h^{n-1}|||_1^2\right)\Bigg] + \frac{1}{4}|||e_h^{n-1} + e_h^n|||_1^2 \end{aligned} \quad (30)$$

For each terms of the right hand side of (26). Using interpolation theory, triangulation inequality and lemma 4, we know that

$$\left|T_1\right| \leq C\left(\left\|e_h^{n-1}\right\|_0^2 + \left\|e_h^n\right\|_0^2 + (\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 + h^4\left|u_t^{n-1}\right|_2^2\right). \quad (31)$$

Similarly, we can bound T_2 as

$$\left|T_2\right| \leq Ch\left|u^{n-1}\right|_2 \cdot \left\|e_h^{n-1} + e_h^n\right\|_1.$$

Further, making use of triangulation inequality and important inequality, we have that

$$\left|T_2\right| \leq C\left(\left\|e_h^{n-1}\right\|_1^2 + \left\|e_h^n\right\|_1^2 + h^2\left|u^{n-1}\right|_2^2\right) \quad (32)$$

From the Lipschitz property of $g\left(\mathbf{x}, u\right)$ in condition $\left(C_2\right)$, making use of triangle inequality, important inequality and Lemma 4, we have

$$\left|T_4\right| \leq C\left(h^4\left\|u^{n-1}\right\|_2^2 + \left\|e_h^{n-1}\right\|_0^2 + \left\|e_h^n\right\|_0^2\right), \quad (33)$$

Combining (34),(35) with (36), we know that

$$\left|T_3\right| \leq C\left(\left\|e_h^{n-1}\right\|_1^2 + \left\|e_h^n\right\|_1^2 + h^2\left\|u^{n-1}\right\|_2^2 + h^4\left\|u^{n-1}\right\|_1^2 + h^2\right) \quad (34)$$

Combining (31), (32), (33) with (34) and applying Sobolev space embedding theory, we know that the RHS of (26) satisfies

$$\begin{aligned} RHS \leq C\Big(&\left\|e_h^{n-1}\right\|_0^2 + \left\|e_h^n\right\|_0^2 + (\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 \\ &+ h^2\left(h^2\left|u_t^{n-1}\right|_2^2 + \left\|u^{n-1}\right\|_2^2 + 1\right)\Big). \end{aligned} \quad (35)$$

From (30) and (35), using inverse estimate we know

$$\begin{aligned} &\frac{1}{2\Delta t}\left[\left(|||e_h^n|||_0^2 - \frac{\Delta t}{2}|||e_h^n|||_1^2\right) - \left(|||e_h^{n-1}|||_0^2 - \frac{\Delta t}{2}|||e_h^{n-1}|||_1^2\right)\right] \\ &+ \frac{1}{4}\mu|||e_h^n + e_h^{n-1}|||_1^2 \leq C\left(\left\|e_h^{n-1}\right\|_0^2 + \left\|e_h^n\right\|_0^2 + (\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2\right. \\ &\left.+ h^2\left(h^2\left|u_t^{n-1}\right|_2^2 + \left\|u^{n-1}\right\|_2^2 + 1\right)\right) \end{aligned}$$

Further, we get that

$$\begin{aligned} &\left[\left(|||e_h^n|||_0^2 - \frac{\Delta t}{2}|||e_h^n|||_1^2\right) - \left(|||e_h^{n-1}|||_0^2 - \frac{\Delta t}{2}|||e_h^{n-1}|||_1^2\right)\right] \\ &+ \frac{\Delta t}{2}\mu|||e_h^n + e_h^{n-1}|||_1^2 \leq C\Delta t\Big(\left\|e_h^{n-1}\right\|_0^2 + \left\|e_h^n\right\|_0^2 \\ &+ (\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 + h^2\left(h^2\left\|u_t^{n-1}\right\|_2^2 + \left\|u^{n-1}\right\|_2^2 + 1\right)\Big) \end{aligned} \quad (36)$$

Summing from 1 to N with respect to n in the above inequality, we can obtain that

$$\begin{aligned} &\left[\left(|||e_h^N|||_0^2 - \frac{\Delta t}{2}|||e_h^N|||_1^2\right) - \left(|||e_h^{N-1}|||_0^2 - \frac{\Delta t}{2}|||e_h^{N-1}|||_1^2\right)\right] \\ &+ \frac{\Delta t}{2}\mu\sum_{n=1}^{N}|||e_h^n + e_h^{n-1}|||_1^2 \leq C\Delta t\sum_{n=1}^{N}\left(\left\|e_h^{n-1}\right\|_0^2 + \left\|e_h^n\right\|_0^2\right) \\ &+ C\Delta t\sum_{n=1}^{N}(\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 + C\Delta t\sum_{n=1}^{N}h^2\left(h^2\left\|u_t^{n-1}\right\|_2^2 + \left\|u^{n-1}\right\|_2^2 + 1\right) \end{aligned} \quad (37)$$

Noting the equivalence of $|||\cdot|||_0$ and $|||\cdot|||_1$ with $\|\cdot\|_0$

and $\|\cdot\|_1$, respectively. Using the inverse estimate, we have that there exist three positive constants $\sigma_0, \sigma_1, \sigma_2$ such that

$$\sigma_0 \|\cdot\|_0 \le |||\cdot|||_0^2, |||\cdot|||_1^2 \le \sigma_2 \|\cdot\|_1^2 \le \sigma_1 h^{-2} \|\cdot\|_0^2 .$$

Further, (37) may be rewritten as

$$\begin{aligned}&\left(\sigma_0 - \Delta t 2\sigma_1 h^{-2}\right)\left\|e_h^N\right\|_0^2 + \Delta t 2\sigma_2 \sum_{n=0}^{N}\left\|e_h^n\right\|_1^2 \\ &\le C\Delta t\sum_{n=0}^{N}\|e_h^n\|_0^2 + C\Delta t\sum_{n=0}^{N}(\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 \\ &+C\Delta t\sum_{n=0}^{N}h^2\left(h^2 | u_t^{n-1} |_2^2 + \|u^{n-1}\|_2^2 + 1\right)\end{aligned} \quad (38)$$

Choosing $\Delta t, h$ in such way that

$$\varepsilon_0 = \sigma_0 - \Delta t 2\sigma_1 h^{-2} > 0 ,$$

further, (38) can be rewritten as

$$\begin{aligned}&\left\|e_h^N\right\|_0^2 + \eta\Delta t\sum_{n=0}^{N}\left\|e_h^n\right\|_1^2 \\ &\le C\Delta t\sum_{n=0}^{N}\|e_h^n\|_0^2 + C\Delta t\sum_{n=0}^{N}(\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 \\ &+C\Delta t\sum_{n=0}^{N}h^2\left(h^2 | u_t^{n-1} |_2^2 + \|u^{n-1}\|_2^2 + 1\right),\end{aligned} \quad (42)$$

where $\eta = 12\varepsilon_0$. Using discrete Gronwall's lemma, we know that

$$\begin{aligned}&\left\|e_h^N\right\|_0^2 + \eta\Delta t\sum_{n=0}^{N}\left\|e_h^n\right\|_1^2 \\ &\le C\Delta t\sum_{n=0}^{N}\left((\Delta t)^2\left\|u_{tt}^{n-1}\right\|_0^2 + h^2\left(h^2\left|u_t^{n-1}\right|_2^2 + \|u^{n-1}\|_2^2 + 1\right)\right).\end{aligned} \quad (43)$$

Noting that $N\Delta t \le T$, combining finite element space interpolation theory, we can obtain the resulting error estimates to the approximate solution as follows.

$$\begin{aligned}&\|u-u_h\|_{\bar{L}^\infty\left((0,T],H^2(\Omega)\right)} + \|u-u_h\|_{\bar{L}^2\left((0,T],H^1(\Omega)\right)} \\ &= O(h+\Delta t),\end{aligned} \quad (44)$$

where,

$$\|v\|_{\bar{L}^\infty((0,T],X)} = \sup_{n\Delta t\le T}\|v^n\|_X , \|v\|_{\bar{L}^2((0,T],X)} = \sup_{N\Delta t\le T}\left\{\eta\sum_{n=0}^{N}\|v^n\|_X \Delta t\right\}^{1/2} .$$

Therefore we have the following theory.

Theorem 2. Suppose that the solution to the problem (1) is sufficiently smooth. When h and Δt are small enough and satisfy the relationship $\Delta t = O(h)$. The initial value u_h^0 is chosen as interpolation of u_0, then the Equation (44) holds.

5. Conclusions

In this paper, we continued our work [17] and presented a class of upwind FVEM based on tetrahedron partition for a three dimensional nonlinear convection diffusion equation, analyzed and derived error estimate in L^2-norm and H^1-norm for the method. In the ongoing work, we will discuss how to derive optimal error estimate in L^2-norm and how to code and present numerical results to demonstrate the performance.

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