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具有 n 个小时滞脉冲系统的周期解 *

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摘 要 本文考虑有 n 个小时滞的脉冲系统，利用隐函数存在性定理证明了该系统时滞充分小时，系统的周期解存在性，推广了已有的相关结论.

关键词 小时滞；脉冲；周期解；隐函数存在性定理

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1 引言

许多实际问题的发展过程往往有这样的特征，在发展的某些阶段，会出现快速的变化，为方便起见，在这些过程的数学模拟中，常常会忽略这个快速变化的持续期间而假设这个过程是通过瞬时突变来完成的. 这种瞬时突变现象通常称之为脉冲现象. 脉冲现象在现代科技各领域的实际问题中是普遍存在的，其数学模型往往可归结为脉冲微分系统.

具有时滞的脉冲微分方程描述了过去依赖和可观察到的瞬间扰动的真正过程和现象. 例如，规定的人口数量通常用时滞微分方程来描绘，在某些时刻，人口的数目可能会突然改变. 脉冲扰动和时滞的相互作用使得分析这类方程的性态更加困难.

周期解存在性是微分方程性态理论的经典问题之一, 众多学者已经关注时滞和脉冲的微分方程的周期解的存在性问题，如 [1–7], 解决此类问题的一个传统方法是把系统化为相应的线性系统 (也称为变分系统) 在满足非退化条件下分析其周期解的存在性，见 [1,2]. 在 [1] 中作者运用隐函数存在性定理，证明了具有小时滞的脉冲系统的周期解，而在 [2] 中用压缩映射定理证明了具有小时滞的脉冲中立型微分方程的周期解.

本文考虑具有 n 个小时滞的脉冲系统，如果无时滞的相应系统有孤立的 ω- 周期解，则在时滞充分小的情况下，证明了在充分小的轨道邻域的脉冲系统有唯一的周期

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解，所得结果推广了 [2] 的结论.

2 问题的陈述和主要结果

本文考虑如下脉冲方程

$$\begin{cases} \dot{x} = f(t, x(t), x(t-h_1), x(t-h_2), \cdots, x(t-h_n)), & t \neq t_i, \quad t \neq t_i + h_j, \\ \triangle x(t_i) = I_i(x(t_i), x(t_i - h_1), \cdots, x(t_i - h_n)), & i \in Z, \\ \triangle x(t_i + h_j) = 0, & h_j > 0, \end{cases} \tag{1}$$

其中 $x \in \Omega \subset R^n$, $f: R \times \Omega \times \cdots \times \Omega \to R^n$, $I_i: \Omega \times \cdots \times \Omega \to R^n$ $(i \in Z)$, $\{t_i\}_{i \in Z}$ 是严格增序列，且 $\lim\limits_{i \to \pm\infty} t_i = \pm\infty$, $\triangle x(t_i) = \triangle x(t_i + 0) - \triangle x(t_i - 0)$, $h_j \geq 0$ $(j = 1, 2, \cdots, n)$ 是小时滞.

为了简便，采用下面的记号：

$$\overline{x}_j(t) = x(t - h_j), \qquad x_i = x(t_i), \qquad \overline{x}_{ij} = x(t_i - h_j).$$

令 $|x|$ 表示向量 $x \in R^n$ 的欧氏空间范数，定义 $n \times n$ 矩阵 A 范数为

$$|A| = \sup \big\{ |Ax|/|x|;\ x \in R^n \backslash 0 \big\}.$$

引入下面的条件：

(H$_1$) $f(t, x, \overline{x}_1, \cdots, \overline{x}_n)$ 连续 (或分段连续， t_i 为第一类间断点), 关于 t 是 ω- 周期的，并关于 $x, \overline{x}_1, \cdots, \overline{x}_n$ 连续可微且偏导数满足局部利普希兹条件.

(H$_2$) $I_i(x_i, \overline{x}_{i1}, \cdots, \overline{x}_{in}) \in C^1(\Omega \times \cdots \times \Omega, R^n)$, $i \in Z$.

(H$_3$) 有正整数 m 使得 $t_{i+m} = t_i + \omega, I_{i+m}(x, \overline{x}_1, \cdots, \overline{x}_n) = I_i(x_i, \overline{x}_{i1}, \cdots, \overline{x}_{in})$, $i \in Z$, $x_i, \overline{x}_{i1}, \cdots, \overline{x}_{in} \in \Omega$.

当 $h_j = 0$, 考虑 (1) 的生成系统

$$\begin{cases} \dot{x} = f(t, x(t), x(t), \cdots, x(t)), & t \neq t_i, \\ \triangle x(t_i) = I_i(x(t_i), x(t_i), \cdots, x(t_i)), & i \in Z. \end{cases} \tag{2}$$

(H$_4$) 生成系统 (2) 存在 ω- 周期解 $\psi(t)$ 使得对 $t \in R$, 有 $\psi(t) \in \Omega$.

现定义关于 $\psi(t)$ 的线性系统：

$$\begin{cases} \dot{y} = A(t)y, & t \neq t_i, \quad t \neq t_i + h, \\ \triangle y(t_i) = B_i y_i, & i \in Z, \end{cases} \tag{3}$$

其中

$$A(t) = \frac{\partial}{\partial x} f(t, x, x, \cdots, x)|_{x = \psi(t)}, \qquad B_i = \frac{\partial}{\partial x_i} I_i(x_i, x_i, \cdots, x_i)|_{x_i = \psi_i}.$$

设 $n \times n$ 矩阵 $X(t)$ 为 (3) 的基解阵，即 $X(0) = E$. 另加以下假设：

(H$_5$) 矩阵 $E - X(\omega)$ 是非奇异的.

(H_6) 矩阵 $E+B_i, i\in Z$ 是非奇异的．由上两个条件我们可以定义 (3) 对应的非齐次系统的周期问题的格林矩阵为

$$G(t,\tau)=\begin{cases} X(t)(E-X(\omega))^{-1}X^{-1}(\tau), & 0\le\tau<t\le\omega,\\ X(t+\omega)(E-X(\omega))^{-1}X^{-1}(\tau), & 0\le t\le\tau\le\omega.\end{cases}$$

记 $G(t,t_i+0)=G(t,t_i)(E+B_i)^{-1}$.

下面给出隐函数存在唯一性定理：若

(i) 函数 F 在以点 $P_0(x_0,y_0)$ 为内点的区域 $D\subset R^2$ 上连续;

(ii) $F(x_0,y_0)=0$;

(iii) 偏导数 $F_y(x,y)$ 在 D 内存在且连续;

(iv) $F_y(x_0,y_0)\neq 0$,

则在点 P_0 的某邻域 $U(P_0)\subset D$ 内，方程 $F(x,y)=0$ 惟一确定了一个定义在某区间 (x_0-a,x_0+a) 内的函数 (隐函数)$y=f(x)$, 使得 $f(x_0)=y_0$, $x\in(x_0-a,x_0+a)$ 时 $f\in U(P_0)$ 且 $F(x,f(x))=0$.

定理 2.1 设系统 (1) 满足条件 (H_1)–(H_6), 则轨道 $x=\psi(t)$ 存在邻域 $\overline{\Omega}\subset\Omega$ 和 $h_0>0$ 使得 $h_j<h_0$, 则系统 (1) 存在唯一的周期解 $x(t,h_1,\cdots,h_n)$ 使得 $x(t,0,\cdots,0)\equiv\psi(t)$.

3 主要结果的证明

设存在常数 $\delta_1>0$ 使得 Ω 包含周期轨道 $x=\psi(t)$ 的 δ_1 闭邻域 Ω_1. 对于向量 $x\in R^n$ 定义其范数为 $|x|$ 和对 $n\times n$ 矩阵 A 定义其范数为

$$|A|=\sup\big\{|Ax|/|x|;\ x\in R^n\backslash 0\big\}.$$

由系统 (1) 做变换

$$x=y+\psi(t), \tag{4}$$

得到系统

$$\begin{cases}\dot y=A(t)y+Q(t,y)+\Delta f(t,y+\psi,\overline{y}_1+\overline{\psi}_1,\cdots,\overline{y}_n+\overline{\psi}_n), & t\neq t_i,\\ \Delta y(t_i)=B_iy_i+J_i(y_i)+\Delta I_i(y_i+\psi_i,\overline{y}_{i1}+\overline{\psi}_{i1},\cdots,\overline{y}_{in}+\overline{\psi}_{in}), & i\in Z,\\ \Delta y(t_i+h_j)=0,\quad h_j>0,\ \ i\in Z,\ \ j=1,\cdots,n.\end{cases} \tag{5}$$

其中

$$Q(t,y)=f(t,y(t)+\psi(t),y(t)+\psi(t),\cdots,y(t)+\psi(t))-f(t,\psi(t),\psi(t),\cdots,\psi(t))-A(t)y,$$
$$J_i(y_i)=I_i(y_i+\psi_i,y_i+\psi_i,\cdots,y_i+\psi_i)-I_i(\psi_i,\psi_i,\cdots,\psi_i))-B_iy_i,$$
$$\Delta f(t,x,\overline{x}_1,\cdots,\overline{x}_n)=f(t,x,\overline{x}_1,\cdots,\overline{x}_n)-f(t,x,x,\cdots,x),$$
$$\Delta I_i(x_i,\overline{x}_{i1},\cdots,\overline{x}_{in})=I_i(x_i,\overline{x}_{i1},\cdots,\overline{x}_{in})-I_i(x_i,x_i,\cdots,x_i).$$

为了找到系统 (1) 的一个 ω- 周期解我们将找系统 (5) 的解 $y(t)$ 满足 $|y(t)|\le\delta\le\delta_1$. 根据 [8], 线性系统

$$\begin{cases}\dot{z}=A(t)z+Q(t,y)+\Delta f(t,y+\psi,\overline{y}_1+\overline{\psi}_1,\cdots,\overline{y}_n+\overline{\psi}_n), & t\ne t_i,\\ \Delta z(t_i)=B_iz_i+J_i(y_i)+\Delta I_i(y_i+\psi_i,\overline{y_{i1}}+\overline{\psi}_{i1},\cdots,\overline{y_{in}}+\overline{\psi}_{in}), & i\in Z,\\ \Delta z(t_i+h_j)=0, \quad h_j>0,\ \ i\in Z,\ \ j=1,\cdots,n\end{cases} \tag{6}$$

有解，其形式为

$$\begin{aligned}z(t)=&\int_0^\omega G(t,\tau)\big[Q(\tau,y(\tau))+\Delta f(\tau,y(\tau)+\psi(\tau),\overline{y}_1(\tau)+\overline{\psi}_1(\tau),\cdots,\overline{y}_n(\tau)+\overline{\psi}_n(\tau))\big]\,\mathrm{d}\tau\\ &+\sum_{0<t_i<\omega}G(t,t_i+0)\big[J_i(y_i)+\Delta I_i(y_i+\psi_i,\overline{y}_{i1}+\overline{\psi}_{i1},\cdots,\overline{y}_{in}+\overline{\psi}_{in})\big].\end{aligned} \tag{7}$$

为方便下面定义，假设 $0<t_1<t_2<\cdots<t_m<\omega$. 设 $h^*>0$ 足够小使得 $\forall h_j\in[0,h^*]$ 有 $t_i+h_j<t_{i+1}$, $i=\overline{1,m-1}$, $t_m+h_j<\omega$, $j=1,2,\cdots,n$.

定义空间 $\widetilde{C}_{\omega,n}$ 表示所有周期 ω, 分段连续函数 $w:R\to R^n$, t_i 为第一类间断点，且 $w(t_i-0)=w(t_i)$, $i\in Z$ 其范数为 $\|w\|=\sup\limits_{t\in R}|w(t)|$.

对 $\delta\in(0,\delta_1]$ 定义子集 $T_\delta=\big\{w\in\widetilde{C}_{\omega,n}:\|w\|\le\delta\big\}$, 用 $\widetilde{C}^{(1)}_{\omega,n}$ 表示所有函数 $w\in\widetilde{C}_{\omega,n}$ 在区间 (t_i,t_i+h_j) 或 (t_i+h_j,t_{i+1}) 连续可导，有 $\dot{w}(t_i\pm0),\dot{w}(t_i+h_j\pm0)$ 极限存在且 $\dot{w}(t_i)=\dot{w}(t_i-0),\dot{w}(t_i+h_j)=\dot{w}(t_i+h_j-0)$. 对 $\zeta>0$ 定义子集

$$V_\zeta=\Big\{w\in\widetilde{C}^{(1)}_{\omega,n}:\sup_{t\in[0,\omega]}|\dot{w}(t)|\le\zeta\Big\}.$$

由系统 (6) 的解表达式 (7) 有 $y\in\widetilde{C}_{\omega,n}$ 满足方程

$$y=\mathfrak{U}(h_1,\cdots,h_n,y). \tag{8}$$

如下定义映射

$$\begin{aligned}&\mathfrak{U}:[0,h^*]\times\cdots\times[0,h^*]\times(T_\delta\subset\widetilde{C}_{\omega,n})\longrightarrow\widetilde{C}_{\omega,n},\\ &\mathfrak{U}(h_1,\cdots,h_n,y)=\mathfrak{F}_1(y)+\mathfrak{F}_2(h_1,\cdots,h_n,y)+\mathfrak{F}_3(y)+\mathfrak{F}_4(h_1,\cdots,h_n,y),\end{aligned} \tag{9}$$

其中

$$\begin{aligned}&\mathfrak{F}_1(y)=\int_0^\omega G(t,\tau)Q(\tau,y(\tau))\,\mathrm{d}\tau,\\ &\mathfrak{F}_2(h_1,\cdots,h_n,y)=\int_0^\omega G(t,\tau)\Delta f(t,y(\tau)+\psi(\tau),\overline{y}_1(\tau)+\overline{\psi}_1(\tau),\cdots,\overline{y}_n(\tau)+\overline{\psi}_n(\tau))\,\mathrm{d}\tau,\\ &\mathfrak{F}_3(y)=\sum_{i=1}^m G(t,t_i+0)J_i(y_i),\\ &\mathfrak{F}_4(h_1,\cdots,h_n,y)=\sum_{i=1}^m G(t,t_i+0)\Delta I_i(y_i+\psi_i,\overline{y}_{i1}+\overline{\psi}_{i1},\cdots,\overline{y}_{in}+\overline{\psi}_{in}).\end{aligned}$$

应用隐函数存在唯一性定理.

定义映射

$$\mathfrak{G}:[0,h^*]\times\cdots\times[0,h^*]\times(T_\delta\subset\widetilde{C}_{\omega,n})\to\widetilde{C}_{\omega,n},$$
$$\mathfrak{G}(h_1,\cdots,h_n,y)=y-\mathfrak{U}(h_1,\cdots,h_n,y).$$

找到方程

$$\mathfrak{G}(h_1,\cdots,h_n,y)=0 \tag{10}$$

的一个解 y. 很明显 $\mathfrak{G}(0,\cdots,0)=0$, 我们将证明：

(1) $\mathfrak{G}$ 关于 $h_1,\cdots,h_n$ 在点 $h_1,\cdots,h_n=0$ 处对 y 在 T_δ 连续.

(2) $\mathfrak{G}$ 在 $[0,h^*]\times\cdots\times[0,h^*]\times T_\delta$ 关于 y 连续可微和 $D_y\mathfrak{G}(0,\cdots,0)=Id$.

断言 (1) 意味着 $\mathfrak{G}$ 是考虑只有参数 y 的映射 $\mathfrak{G}_0=\mathfrak{G}(0,\cdots,0,y):T_\delta\to\widetilde{C}_{\omega,n}$.

下面引入 Banach 空间 $\mathfrak{B}_0=C(T_\delta,\widetilde{C}_{\omega,n})$, $\mathfrak{B}_1=\mathfrak{B}_2=\widetilde{C}_{\omega,n}$. 对于 $\mathfrak{B}_0$ 中 $\mathfrak{G}$ 趋向于 $\mathfrak{G}_0$ 也即是 $h_j\to 0$ 和 $\mathfrak{B}_1$ 中 $y\to 0$ 可定义为 $\phi(\mathfrak{G},y)=\mathfrak{G}(h_1,\cdots,h_n,y)\in\mathfrak{B}_2$.

根据断言 (2)ϕ 对于固定的 $\mathfrak{G}$ 关于 y 连续可微. 由隐函数定理，存在 $h_0\in(0,h^*]$ 和连续映射 $y(h_1,\cdots,h_n):[0,h_0)\times\cdots\times[0,h_0)\to T_{\delta_0}\subset\widetilde{C}_{\omega,n}(\delta_0\subset(0,\delta_1])$ 满足方程 (10), 有 $y(0,\cdots,0)=0$ 依赖于 $h_1,\cdots,h_n$ 的 t 函数, 给出系统 (5) 唯一的 ω- 周期解 $y(t,h_1,\cdots,h_n)$ 满足 $\|y\|\le\delta_0$, 则由 (4) 可得系统 (1) 在周期轨道 $x=\psi(t)$ 的 δ_0 邻域 $\overline{\Omega}$ 上唯一的 ω- 周期解 $x(t,h_1,\cdots,h_n)$, 进一步， $x(t,0,\cdots,0)=\psi(t)$.

下面证明断言 (1) 和 (2).

首先引入一些记号：

$$M_1=\sup\big\{|f(t,x,\overline{x}_1,\cdots,\overline{x}_n)|:t\in[0,\omega],\ x,\overline{x}_1,\cdots,\overline{x}_n\in\Omega_1\big\},$$
$$M_2=\sup\big\{|f_x(t,x,\overline{x}_1,\cdots,\overline{x}_n)|:t\in[0,\omega],\ x,\overline{x}_1,\cdots,\overline{x}_n\in\Omega_1\big\},$$
$$M_3=\sup\big\{|f_{\overline{x}_1}(t,x,\overline{x}_1,\cdots,\overline{x}_n)|:t\in[0,\omega],\ x,\overline{x}_1,\cdots,\overline{x}_n\in\Omega_1\big\},$$
$$\vdots$$
$$M_{n+2}=\sup\big\{|f_{\overline{x}_n}(t,x,\overline{x}_1,\cdots,\overline{x}_n)|:t\in[0,\omega],\ x,\overline{x}_1,\cdots,\overline{x}_n\in\Omega_1\big\},$$
$$N_1=\sup\big\{|\partial_{\overline{x}_{ij}}I_i(x_i,\overline{x}_{i1},\cdots,\overline{x}_{in})|:i=\overline{1,m};\ x_i,\overline{x}_{i1},\cdots,\overline{x}_{in}\in\Omega_1;\ j=\overline{1,n}\big\}.$$

此处的 $f_x,f_{\overline{x}_1},\cdots,f_{\overline{x}_n},\partial_{\overline{x}_{ij}}I_i$ 是 f 关于 $x,\overline{x}_1,\cdots,\overline{x}_n$ 和 I_i 关于 $\overline{x}_{ij}$ 的偏导数.

$$M=\sup\big\{|G(t,\tau)|:t,\tau\in[0,\omega]\big\}.$$

根据条件 (H_1) 引入一阶导函数连续模条件：

$$\eta_1(\mu)=\sup\big\{|f_x(t,x,x+y,\cdots,x+y)-f_x(t,x,x,\cdots,x)|:t\in[0,\omega],\ x\in\Omega_1,\ \|y\|\le\mu\big\},$$
$$\eta_2(\mu)=\sup\big\{|f_{\overline{x}_1}(t,x,x+y,\cdots,x+y)-f_{\overline{x}_1}(t,x,x,\cdots,x)|:t\in[0,\omega],\ x\in\Omega_1,\ \|y\|\le\mu\big\},$$
$$\vdots$$
$$\eta_{n+1}(\mu)=\sup\big\{|f_{\overline{x}_n}(t,x,x+y,\cdots,x+y)$$

$$- f_{\overline{x}_n}(t,x,x,\cdots,x)| : t\in[0,\omega],\ x\in\Omega_1,\ \|y\|\le\mu\Big\},$$

$$\eta(\mu)=\sup\Big\{\Big|\frac{\partial}{\partial x}\big(f(t,x+y',x+y',\cdots,x+y')-f(t,x+y'',x+y'',\cdots,x+y'')\big)\Big|$$
$$: t\in[0,\omega],\ x=\psi(t),\ |y'|\le\delta_1,\ |y''|\le\delta_1,\ |y'-y''|\le\mu\Big\}.$$

(1) 要证明 μ 关于 h_j 在点 $h_j=0$ 连续且 $y\in T_{\delta_1}$ 由 (9) 有 $\mathfrak{U}(0,\cdots,0,y)=\mathfrak{F}_1(y)+\mathfrak{F}_3(y)$ 知，只需估计 $\mathfrak{F}_2(h_1,\cdots,h_n,y),\mathfrak{F}_4(h_1,\cdots,h_n,y)$ 有 $y\in T_\delta\cap V_\zeta$, $\zeta>0$.

定义集合

$$\triangle_{h_1}=\bigcup_{i=1}^{m}(t_i,t_i+h_1),\quad \triangle_{h_2}=\bigcup_{i=1}^{m}(t_i,t_i+h_2),\quad \cdots,\quad \triangle_{h_n}=\bigcup_{i=1}^{m}(t_i,t_i+h_n),$$
$$I_h=[0,\omega]\setminus(\triangle_{h_1}\cup\cdots\cup\triangle_{h_n}).$$

对 $t\in I_h$ 点 $t,t-h_j$ 属于函数 ψ,y 同一连续区间，且 ψ 是 (2) 的解，因此有

$$|\overline{\psi}_1(t)-\psi(t)|=|\psi(t-h_1)-\psi(t)|\le h_1\sup_{t\in(t-h_1,t)}|\dot\psi(t)|<h_1M_1,$$
$$\vdots$$
$$|\overline{\psi}_n(t)-\psi(t)|=|\psi(t-h_n)-\psi(t)|\le h_n\sup_{t\in(t-h_n,t)}|\dot\psi(t)|<h_nM_1,$$
$$|\overline{y}_1(t)-y(t)|\le h_1\zeta,$$
$$\vdots$$
$$|\overline{y}_n(t)-y(t)|\le h_n\zeta.$$

对于 $t\in I_h$,

$$\begin{aligned}
&\Delta f(t,y(t)+\psi(t),\overline{y}_1(t)+\overline{\psi}_1(t),\cdots,\overline{y}_n(t)+\overline{\psi}_n(t))\\
=&\big|f(t,y(t)+\psi(t),\overline{y}_1(t)+\overline{\psi}_1(t),\cdots,\overline{y}_n(t)+\overline{\psi}_n(t))\\
&-f(t,y(t)+\psi(t),y_1(t)+\psi_1(t),\cdots,y_n(t)+\psi_n(t))\big|\\
\le&M_3|\overline{y}_1-y_1+\overline{\psi}_1(t)-\psi_1(t)|+\cdots+M_{n+2}|\overline{y}_n-y_n+\overline{\psi}_n(t)-\psi_n(t)|\\
\le&(M_3h_1+\cdots+M_{n+2}h_n)(\zeta+M_1).
\end{aligned}$$

对于 $t\in\triangle_{h_j}$, $j=1,2,\cdots,n$ 粗略估计得

$$\Delta f(t,y(t)+\psi(t),\overline{y}_1(t)+\overline{\psi}_1(t),\cdots,\overline{y}_n(t)+\overline{\psi}_n(t))\le 2M_1,$$

$$\begin{aligned}
&\big|\mathfrak{F}_2(h_1,\cdots,h_n,y)\big|\\
\le&\int_0^\omega|G(t,\tau)|\,\big|\Delta f(t,y(\tau)+\psi(\tau),\overline{y}_1(\tau)+\overline{\psi}_1(\tau),\cdots,\overline{y}_n(\tau)+\overline{\psi}_n(\tau))\big|\,\mathrm{d}\tau\\
=&\int_{I_h}|G(t,\tau)|\,\big|\Delta f(t,y(\tau)+\psi(\tau),\overline{y}_1(\tau)+\overline{\psi}_1(\tau),\cdots,\overline{y}_n(\tau)+\overline{\psi}_n(\tau))\big|\,\mathrm{d}\tau
\end{aligned}$$

$$\begin{aligned}
&+\sum_{i=1}^{m}\int_{t_i}^{t_i+h_1}|G(t,\tau)|\left|\Delta f(t,y(\tau)+\psi(\tau),\overline{y}_1(\tau)+\overline{\psi}_1(\tau),\cdots,\overline{y}_n(\tau)+\overline{\psi}_n(\tau))\right|\mathrm{d}\tau\\
&+\cdots+\sum_{i=1}^{m}\int_{t_i}^{t_i+h_n}|G(t,\tau)|\left|\Delta f(t,y(\tau)+\psi(\tau),\overline{y}_1(\tau)+\overline{\psi}_1(\tau),\cdots,\right.\\
&\left.\overline{y}_n(\tau)+\overline{\psi}_n(\tau))\right|\mathrm{d}\tau\\
\le& M\big[(\omega-m(h_1+\cdots+h_n))(M_3h_1+\cdots+M_{n+2}h_n)(\zeta+M_1)\\
&+2M_1m(h_1+\cdots+h_n)\big]. \qquad (11)
\end{aligned}$$

同样，

$$\begin{aligned}
&|\overline{\psi}_{i1}-\psi_i|\le h_1M_1,\cdots,|\overline{\psi}_{in}-\psi_i|\le h_nM_1,\\
&|\overline{y}_{i1}-y_i|\le h_1\zeta,\cdots,|\overline{y}_{in}-y_i|\le h_n\zeta,\\
&\left|\Delta I_i(y_i+\psi_i,\overline{y}_{i1}+\overline{\psi}_{i1},\cdots,\overline{y}_{in}+\overline{\psi}_{in})\right|\le N_1(M_1+\zeta)(h_1+h_2+\cdots+h_n),\\
&|\mathfrak{F}_4(h_1,\cdots,h_n,y)|\le\sum_{i=1}^{m}|G(\cdot,t_i+0)|\left|\Delta I_i(y_i+\psi_i,\overline{y}_{i1}+\overline{\psi}_{i1},\cdots,\overline{y}_{in}+\overline{\psi}_{in})\right|\\
&\qquad\le mMN_1(M_1+\zeta)(h_1+h_2+\cdots+h_n). \qquad (12)
\end{aligned}$$

由 (11) 和 (12) 估计式知，映射 $\mathfrak{U}(h_1,\cdots,h_n,y)$ 在点 $h_1,\cdots,h_n=0$ 处连续.

(2) 偏导数 $D_y\mathfrak{F}(h_1,\cdots,h_n,y)$ 是 $\widetilde{C}_{\omega,n}\to\widetilde{C}_{\omega,n}$ 的线性映射定义为：

$$D_y\mathfrak{F}(h_1,\cdots,h_n,y)z=\lim_{\mu\to0}\big[\mathfrak{F}(h_1,\cdots,h_n,y+\mu z)-\mathfrak{F}(h_1,\cdots,h_n,y)\big].$$

$\mathfrak{F}(h_1,\cdots,h_n,y)$ 关于 y 连续可微也即是 $\mathfrak{F}(h_1,\cdots,h_n,y)z:[0,h^*]\times\cdots\times[0,h^*]\times(T_\delta\subset\widetilde{C}_{\omega,n})\to\widetilde{C}_{\omega,n}$ 连续，很明显有

$$\begin{aligned}
&D_y\mathfrak{F}(h_1,\cdots,h_n,y)=Id-D_y\mathfrak{U}(h_1,\cdots,h_n,y),\\
&D_y\mathfrak{U}(h_1,\cdots,h_n,y)=D_y\mathfrak{F}_1(y)+D_y\mathfrak{F}_2(h_1,\cdots,h_n,y)+D_y\mathfrak{F}_3(y)+D_y\mathfrak{F}_4(h_1,\cdots,h_n,y).
\end{aligned}$$

假设上式右端偏导数存在，即上式右端在 $h_1,\cdots,h_n=0,\ y=0$ 处连续.

$$\begin{aligned}
&D_y\mathfrak{F}_1(y)z=\lim_{\mu\to0}\mu^{-1}\big[\mathfrak{F}_1(y+\mu z)-\mathfrak{F}_1(y)\big]\\
=&\lim_{\mu\to0}\mu^{-1}\int_0^\omega G(t,\tau)\big[Q(\tau,y(\tau)+\mu z(\tau))-Q(\tau,y(\tau))\big]\,\mathrm{d}\tau\\
=&\int_0^\omega G(t,\tau)\lim_{\mu\to0}\mu^{-1}\big[f(\tau,y(\tau)+\mu z(\tau)+\psi(\tau),y(\tau)+\mu z(\tau)+\psi(\tau),\\
&\cdots,y(\tau)+\mu z(\tau)+\psi(\tau))-f(\tau,y(\tau)+\psi(\tau),\cdots,y(\tau)+\psi(\tau))-\mu A(\tau)z(\tau)\big]\,\mathrm{d}\tau\\
=&\int_0^\omega G(t,\tau)\Big(\frac{\partial}{\partial x}f(\tau,x,x,\cdots,x)\big|_{x=\psi(\tau)+y(\tau)}-\frac{\partial}{\partial x}f(\tau,x,x,\cdots,x)\big|_{x=\psi(\tau)}\Big)z(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

很明显，$D_y\mathfrak{F}_1(0)=0$. 为了证明导数的连续性，取 $y',y'',z',z''\in\widetilde{C}_{\omega,n}$, $\|y'\|\le\delta$, $\|y''\|\le\delta$,

则有

$$
\begin{aligned}
&\big|(D_y\mathfrak{F}_1(y')z'-(D_y\mathfrak{F}_1(y'')z'')(t)\big|\\
\le&\Big|\int_0^\omega G(t,\tau)\Big(\frac{\partial}{\partial x}f(\tau,x,x,\cdots,x)|_{x=\psi(\tau)+y'(\tau)}\\
&-\frac{\partial}{\partial x}f(\tau,x,x,\cdots,x)|_{x=\psi(\tau)+y''(\tau)}\Big)z'(\tau)\,\mathrm{d}\tau\Big|\\
&+\Big|\int_0^\omega G(t,\tau)\Big(\frac{\partial}{\partial x}f(\tau,x,x,\cdots,x)|_{x=\psi(\tau)+y''(\tau)}\\
&\quad-\frac{\partial}{\partial x}f(\tau,x,x,\cdots,x)|_{x=\psi(\tau)}\Big)\big(z'(\tau)-z''(\tau)\big)\,\mathrm{d}\tau\Big|\\
\le&M\omega\big(\eta_3(\|y^{'}-y''\|)\|z'\|+\eta_3(\delta)\|z^{'}-z''\|\big).
\end{aligned}
$$

类似地,

$$
D_y\mathfrak{F}_3(y)z=\sum_{i=1}^m G(\cdot,t_i+0)\Big(\frac{\partial}{\partial x_i}I_i(x_i,x_i,\cdots,x_i)|_{x_i=\psi_i+y_i}-\frac{\partial}{\partial x_i}I_i(x_i,x_i,\cdots,x_i)|_{x_i=\psi_i}\Big)z_i.
$$

得到 $D_y\mathfrak{F}_3(0)=0$ 且易证导数的连续性.

$$
\begin{aligned}
&D_y\mathfrak{F}_2(h_1,\cdots,h_n,y)z\\
=&\int_0^\omega G(t,\tau)\big\{\big[f_x(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)z(\tau)+f_{\overline{x}_1}(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)\overline{z}_1(\tau)\\
&+\cdots+f_{\overline{x}_n}(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)\overline{z}_n(\tau)\big]\\
&-\big[f_x(\tau,x,x,\cdots,x)+f_{\overline{x}_1}(\tau,x,x,\cdots,x)+\cdots+f_{\overline{x}_n}(\tau,x,x,\cdots,x)\big]z(\tau)\big\}\,\mathrm{d}\tau.
\end{aligned}
$$

很明显，有 $D_y\mathfrak{F}_2(0,\cdots,0,y)=0$ 上式充分表明关于 $h_1,\cdots,h_n$ 连续可导，下面考虑两种情况：

(a) 在 $h_1,\cdots,h_n=0$ 处连续，取 $y\in T_\delta\cap V_\zeta, z\in V_\zeta$,

$$
\begin{aligned}
D_y\mathfrak{F}_2(h_1,\cdots,h_n,y)z=&\int_0^\omega G(t,\tau)\big\{\big[f_x(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)-f_x(\tau,x,x,\cdots,x)\big]\,z(\tau)\\
&+\big[f_{\overline{x}_1}(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)\overline{z}_1(\tau)-f_{\overline{x}_1}(\tau,x,x,\cdots,x)z(\tau)\big]+\cdots\\
&+\big[f_{\overline{x}_n}(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)\overline{z}_n(\tau)-f_{\overline{x}_n}(\tau,x,x,\cdots,x)z(\tau)\big]\big\}\,\mathrm{d}\tau.
\end{aligned}
$$

对 $\tau\in I_h$, $\tau-h,\tau$ 都属于函数 ψ,y,z 的同一连续区间．因此有

$$
\begin{aligned}
&|\overline{\psi_1}(\tau)-\psi(\tau)|\le h_1M_1,\cdots,|\overline{\psi_1}(\tau)-\psi(\tau)|\le h_nM_1,\\
&|\overline{y_1}(\tau)-y(\tau)|\le\zeta h_1,\cdots,|\overline{y_n}(\tau)-y(\tau)|\le\zeta h_n,\\
&|\overline{z}_1(\tau)-z(\tau)|\le\zeta h_1,\cdots,|\overline{z}_n(\tau)-z(\tau)|\le\zeta h_n\\
&\big|f_x(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))\\
&\ -f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau),\cdots,\psi(\tau)+y(\tau))\big|\\
\le&\eta_1(h_1(M_1+\zeta)),
\end{aligned}
$$

$$
\begin{aligned}
&\left|f_{\overline{x}_1}(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))\right.\\
&\left.\quad-f_{\overline{x}_1}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau),\cdots,\psi(\tau)+y(\tau))\right|\\
\le&\left|f_{\overline{x}_1}(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))\right.\\
&\left.\quad-f_{\overline{x}_1}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau),\cdots,\psi(\tau)+y(\tau))\right|\left|\overline{z}_1(\tau)\right|\\
&\quad+\left|f_{\overline{x}_1}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau),\cdots,\psi(\tau)+y(\tau))\right|\left|\overline{z}_1(\tau)-z(\tau)\right|\\
\le&\eta_2(h_1(M_1+\zeta))\left\|z\right\|+M_3h_1\zeta,\\
&\quad\vdots\\
&\left|f_{\overline{x}_n}(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))\right.\\
&\left.\quad-f_{\overline{x}_n}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau),\cdots,\psi(\tau)+y(\tau))\right|\\
\le&\eta_{n+1}(h_n(M_1+\zeta))\left\|z\right\|+M_{n+2}h_n\zeta.
\end{aligned}
$$

当 $\tau\in\triangle_h$ 时，可估计

$$
\begin{aligned}
&\left|f_x(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)-f_x(\tau,x,x,\cdots,x)\right|\left|z(\tau)\right|\le 2M_2\|z\|,\\
&\left|f_{\overline{x}_1}(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)\overline{z}_1(\tau)-f_{\overline{x}_1}(\tau,x,x,\cdots,x)z(\tau)\right|\le 2M_3\|z\|,\\
&\left|f_{\overline{x}_n}(\tau,x,\overline{x}_1,\cdots,\overline{x}_n)\overline{z}_n(\tau)-f_{\overline{x}_1}(\tau,x,x,\cdots,x)z(\tau)\right|\le 2M_{n+2}\|z\|.
\end{aligned}
$$

因此有

$$
\begin{aligned}
&\|D_y\mathfrak{F}_2(h_1,\cdots,h_n,y)z\|\\
\le&M\big\{(\omega-m(h_1+\cdots+h_n))\big[\eta_1(h_1(M_1+\zeta))\\
&\quad+\eta_2(h_1(M_1+\zeta))\left\|z\right\|+M_3h_1\zeta+\cdots+\eta_{n+1}(h_n(M_1+\zeta))\left\|z\right\|+M_{n+2}h_n\zeta\big]\\
&\quad+2m(h_1+\cdots+h_n)(M_2+\cdots+M_{n+2})\left\|z\right\|\big\},
\end{aligned}
$$

即有当 $h_1,\cdots,h_n=0,\ D_y\mathfrak{F}_2(0,\cdots,0,y)=0.$

(b) 下面证明 $D_y\mathfrak{F}_2(h_1,\cdots,h_n,y)z$ 在 $h_1,\cdots,h_n>0$ 时连续，也即是考虑如下的几个方程

$$\int_0^\omega G(t,\tau)f_x(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))z(\tau)\,\mathrm{d}\tau,\tag{13}$$

$$\int_0^\omega G(t,\tau)f_{\overline{x}_1}(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))\overline{z}_1(\tau)\,\mathrm{d}\tau,\tag{14}$$

$$\vdots$$

$$\int_0^\omega G(t,\tau)f_{\overline{x}_n}(\tau,\psi(\tau)+y(\tau),\overline{\psi}_1(\tau)+\overline{y}_1(\tau),\cdots,\overline{\psi}_n(\tau)+\overline{y}_n(\tau))\overline{z}_n(\tau)\,\mathrm{d}\tau.\tag{15}$$

我们只需考虑 (13) 式，取 $h_j^{'},h_j^{''},\ j=1,\cdots,n$ 使得 $0<h_j^{'}<h_j^{''}\le h^*$ 估计方程

$$\int_0^\omega G(t,\tau)\big[f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h_1^{'})+y(\tau-h_1^{'}),\cdots,\psi(\tau-h_n^{'})+y(\tau-h_n^{'}))$$

$$- f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1'') + y(\tau - h_1''), \cdots, \psi(\tau - h_n'') + y(\tau - h_n''))\big]\,\overline{z}(\tau)\,\mathrm{d}\tau.$$

对 $y \in T_\delta \cap V_\zeta$, 如果对任意 $i, \tau \in (t_i, t_i + h_j'')$, 或者对某些 i 有 $\tau \in (t_i, t_i + h_j')$, 那么 $\tau - h_j', \tau - h_j''$ 都属于函数 ψ, y 的同一连续区间. 因此有

$$|\psi(\tau - h_1') - \psi(\tau - h_1'')| \le M_1(h_1'' - h_1'), \cdots, |\psi(\tau - h_n') - \psi(\tau - h_n'')| \le M_1(h_n'' - h_n'),$$
$$|y(\tau - h_1') - y(\tau - h_1'')| \le \zeta(h_1'' - h_1'), \cdots, |y(\tau - h_n') - y(\tau - h_n'')| \le \zeta(h_n'' - h_n').$$

在集合 I_h, 我们可以估计

$$\begin{aligned}&\big|f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1^{'}) + y(\tau - h_1^{'}), \cdots, \psi(\tau - h_n^{'}) + y(\tau - h_n^{'}))\\&\quad - f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1'') + y(\tau - h_1''), \cdots, \psi(\tau - h_n'') + y(\tau - h_n''))\big|\\\le&\eta_1\big[(M_1 + \zeta)((h_1^{''} - h_1^{'}) + \cdots + (h_n^{''} - h_n^{'}))\big].\end{aligned}$$

另一方面，如果 $\tau \in (t_i + h_j', t_i + h_j'')$ 因此可估计

$$\begin{aligned}&\big|f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1^{'}) + y(\tau - h_1^{'}), \cdots, \psi(\tau - h_n^{'}) + y(\tau - h_n^{'}))\\&\quad - f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1'') + y(\tau - h_1''), \cdots, \psi(\tau - h_n'') + y(\tau - h_n''))\big|\\\le&2M_2.\end{aligned}$$

由上可得

$$\begin{aligned}&\Big|\int_0^\omega G(t,\tau)\big[f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1^{'}) + y(\tau - h_1^{'}), \cdots, \psi(\tau - h_n^{'}) + y(\tau - h_n^{'}))\\&\quad - f_x(\tau, \psi(\tau) + y(\tau), \psi(\tau - h_1'') + y(\tau - h_1''), \cdots, \psi(\tau - h_n'') + y(\tau - h_n''))\big]\,z(\tau)\,\mathrm{d}\tau\Big|\\\le&M\big\{\big[\omega - m((h_1^{''} - h_1^{'}) + \cdots + (h_n^{''} - h_n^{'}))\big]\,\eta_1\,\big[(M_1 + \zeta)((h_1^{''} - h_1^{'}) + \cdots + (h_n^{''} - h_n^{'}))\big]\\&\quad + 2M_2 m((h_1^{''} - h_1^{'}) + \cdots + (h_n^{''} - h_n^{'}))\big\},\end{aligned}$$

其他式同样可讨论. 最后我们有

$$\begin{aligned}&D_y\mathfrak{F}_4(h_1, \cdots, h_n, y)z\\=&\sum_{i\in Z} G(t, t_i + 0)\big[\partial_x I_i(\psi_i + y_i, \overline{\psi}_{i1} + \overline{y}_{i1}, \cdots, \overline{\psi}_{in} + \overline{y}_{in})z_i\\&+ \partial_{\overline{x}_{i1}} I_i(\psi_i + y_i, \overline{\psi}_{i1} + \overline{y}_{i1}, \cdots, \overline{\psi}_{in} + \overline{y}_{in})\overline{z}_{i1}\\&+ \cdots + \partial_{\overline{x}_{in}} I_i(\psi_i + y_i, \overline{\psi}_{i1} + \overline{y}_{i1}, \cdots, \overline{\psi}_{in} + \overline{y}_{in})\overline{z}_{in}\\&- \big[(\partial_x I_i(\psi_i + y_i, \psi_i + y_i, \cdots, \psi_i + y_i) + \partial_{\overline{x}_{i1}} I_i(\psi_i + y_i, \psi_i + y_i, \cdots, \psi_i + y_i)\\&+ \cdots + \partial_{\overline{x}_{in}} I_i(\psi_i + y_i, \psi_i + y_i, \cdots, \psi_i + y_i))z_i\big],\end{aligned}$$

很容易看出关于 $h_1, \cdots, h_n$, y, z 连续可微，且 $D_y\mathfrak{F}_4(0, \cdots, 0, y)z = 0$

由上面得到 (2) 是合理的. 因此定理 2.1 得证.

注 文中系统 (1) 中若 $h_2 = \cdots = h_n = 0$, 则系统即为 [1] 中的系统 (1).

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Periodic Solutions of Impulsive Systems with n Small Delays

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Abstract An impulsive system with n small delays is considered. If the delays are small enough, existence of periodic solutions for impulsive system is proved by using the implicit function theorem. The results generalize the corresponding work of known literature.

Key words small delay; impulsive; periodic solution; implicit function theorem

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