

A Batch Arrival Queue System with Coxian-2 Server Vacations and Admissibility Restricted

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ABSTRACT

The $M/G/1$ classic queueing system were extended by many authors in last two decades. The systems with server's vacation are important models that extend the $M/G/1$ queueing system. Also another conditions such as admissibility restricted may occur in systems. From this motivation, in this system I consider a single server queue with batch arrival Poisson input. There is a restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. At each service completion epoch, the server may opt to take a vacation with probability θ or else with probability $1 - \theta$ may continue to be available in the system for the next service. The vacation period of the server has two heterogeneous phases. Phase one is compulsory, and phase two follows the phase one vacation in such a way that the server may take phase two with probability p or may return back to the system with probability $1 - p$. The vacation times are assumed to be general. All stochastic processes involved in this system (service and vacation times) are independent of each other. We derive the PGF's of the system and by using them the performance measures are obtained. Some numerical approaches are examined the validity of results.

Keywords: $M^x/G/1$ Queue; Restricted Admissibility Policy; Bernoulli Vacation; Optional Vacation; Mean Queue Size; Mean Response Time

1. Introduction

The concept of vacation in the $M/G/1$ queueing system were studied by Keilson and Servi in [1]. They introduced the concept of modified service time which has a main rule in the systems with general service and vacation times.

In many examples such as production system, bank services, computer and communication networks; the system have vacation. For overhauling or maintenance the system the server may go to vacation.

For the systems with batch arrival the vacation time were analyzed by Baba in [2], he derived the queue size distribution, waiting time distribution and busy period distribution of $M^x/G/1$ queue with vacation times using the well known supplementary variable technique.

In many systems with batch arrival there is a restriction such that not all batches are allowed to join the system at all time. This policy is named restricted admissibility. For the first time Madan and Abu-Dayyeh [3,4] proposed an $M^x/G/1$ queueing system with restricted admissibility of arriving batches and Bernoulli schedule server vacation. Alnowibet and Tadj [5], also Madan and Choudhury [6] and Choudhury [7] work on this concepts.

Recently author in [8] studied a batch arrival queueing

system with two phases of heterogeneous service with optional second service and restricted admissibility with single vacation policy.

In fact this paper is a generalization of [9] in which the authors inspected a $M/G/1$ system with coxian-2 server vacation.

This paper addresses issue of model building of manufacturing systems of job-shop type, where the server takes vacations after the end of each busy period. This vacation models can be utilized as a post processing time after clearing the jobs in the system. To be more realistic, we further assume that the arrivals occur in batches of random size instead of single units and it covers many practical situations. For example in manufacturing systems of job-shop type, each job requires to manufacture more than one unit; in digital communication systems, messages which are transmitted could consist of a random number of packets. These manufacturing systems can be modeled by $M^x/G/1$ queue with a single vacation policy and this extends the results of former works, especially Doshi [10]. Here we consider as soon as the system becomes empty, the server leaves for vacations of random length V_1 and V_2 (vacation periods). Upon termination of this vacation period, the server returns to the system and

begins to serve those customers that have arriving during that vacation (busy period), exhaustively *i.e.* once the service is started it goes on until there is no one in the queue. However, if the server does not find any customer in the system after returning from that vacation, he stays in the system waiting for the first one to arrive (dormant period). Thus, in our system, a vacation period, a dormant period and a busy period constitute a cycle. Moreover, the system remains idle during a vacation period and a dormant period and these two period together constitute an idle period.

In Section 2 we deal with the mathematical model and definitions. Steady state conditions and generating functions are discussed in Section 3. Mean queue size and mean response time are computed respectively in Sections 4 and 5. Finally in Section 6 some special cases and numerical results obtained.

2. Mathematical Model and Definitions

Consider an infinite capacity queueing facility where customers arrive at a service facility according to a compound Poisson process. According to natural assumption, an idle period begins when the queue drops below level zero and a busy period as soon as the queue accumulates the same number one. However, after each service completion the server takes vacations. The decisions about taking a vacation after each service completion or vacations completion are independent. Also, the vacations are iid random variables whose length is independent of the length of the service times. The service times are iid random variables independent of the input process. In order to fully describe the model, we use the following notations and definitions:

1) New customers arrive in batches according to a compound Poisson process with rate λ . Let X_k denote the number of customers belonging to the k th arrival batch, where $X_k, k = 1, 2, 3, \dots$ are with a common distribution

$$\Pr[X_k = n] = a_n, n = 1, 2, 3, \dots$$

and $X(z)$ denotes the probability generating function of X .

2) The server provides one phases service to each customer. The service discipline is assumed to be first come first served (FCFS). Further, the service time is random variable B , with distribution function $B(x)$, Laplace transform $B^*(x)$ and finite moments $E(B^l)$ for $l \geq 1$.

3) After completion of any customer's service, the server may take a vacation with probability θ or may continue to be in the system with probability $1 - \theta$.

We assume that the vacation time has two phases with phase one is compulsory. However, after phase 1 vacation, the server takes phase 2 vacation with probability p or may return back to the system with probability $1 - p$. The vacation times are random variables V_i for $i = 1, 2$,

follows a general law of probability with distribution function $V_i(x)$, Laplace transform $V_i^*(s)$ and finite moments $E(V_i^k)$ for $k \geq 1$.

4) There is an admissibility restricted policy for batches in which not all batches are allowed to join the system at all times. Let α ($0 \leq \alpha \leq 1$) and β ($0 \leq \beta \leq 1$) be the probability that an arriving batches will allowed to join the system during the period of server's busy and vacation times respectively.

5) The random variables B, V_1 and V_2 are independent.

Definition 2.1 The modified service time or the time required by a customer to complete the service cycle is given by

$$B_c = \begin{cases} B + V_c & \text{with probability } \theta \\ B & \text{with probability } (1 - \theta) \end{cases} \quad (1)$$

where

$$V_c = \begin{cases} V_1 + V_2 & \text{with probability } p \\ V_1 & \text{with probability } (1 - p) \end{cases} \quad (2)$$

then the LST $B_c^*(s)$ of B_c is given by

$$B_c^*(s) = \theta B^*(s) V_c^*(s) + (1 - \theta) B^*(s) \quad (3)$$

and

$$E(B_c) = E(B) + \theta E(V_c) \quad (4)$$

also

$$E(B_c^2) = E(B^2) + 2\theta E(B) E(V_c) + \theta E(V_c^2) \quad (5)$$

where

$$E(V_c) = E(V_1) + p E(V_2) \quad (6)$$

also

$$E(V_c^2) = E(V_1^2) + 2p E(V_1) E(V_2) + p E(V_2^2) \quad (7)$$

Further we assume $B(0) = 1, B(\infty) = 0$ and $B(x)$ is continuous at $x = 0$, so that

$$\mu(x) = \frac{dB(x)}{1 - B(x)} \quad (8)$$

is the hazard rate function of B .

Also for $i = 1, 2$ we assume $V_i(0) = 1, V_i(\infty) = 0$ and $V_i(x)$ are continuous at $x = 0$. The hazard rate functions of V_i 's are

$$v_i(x) = \frac{dV_i(x)}{1 - V_i(x)} \quad (9)$$

Definition 2.2 The elapsed time of service at time " t " is denoted by $B^0(t)$. For $i = 1, 2$, also $V^0(t)$ denote the elapsed time of vacation time at time " t ", and $N_Q(t)$ denote the queue size at time " t ". For $i = 1, 2$ we introduce the random variable $Y(t)$ as follow:

$$Y(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy at time } t, \\ 2 & \text{if the server is on first phase vacation at time } t, \\ 3 & \text{if the server is on second phase vacation at time } t. \end{cases} \quad (10)$$

Then we have a bivariate Markov process $\{N_Q(t), L(t)\}$ where $L(t) = 0$ if $Y(t) = 0$; $L(t) = B(t)$ if $Y(t) = 1$; $L(t) = V_1^0(t)$ if $Y(t) = 2$ and $L(t) = V_2^0(t)$ if $Y(t) = 3$. Now for $i = 1, 2$ the following probabilities defined as

$$V_{i,n}(x,t) = Prob[N_Q(t) = n, L(t) = V_i^0(t); x < V_i^0(t) \leq x + dx] \quad x > 0, n \geq 0 \quad (11)$$

$$P_n(x,t) = Pro [N_Q(t) = n, L(t) = B^0(t); x < B^0(t) \leq x + d] \quad x > 0, n \geq 0 \quad (12)$$

and

$$R_0(t) = Prob[N_Q(t) = 0, L(t) = 0] \quad (13)$$

With assumption that steady state exist, we let

$$R_0 = \lim_{t \rightarrow \infty} R_0(t) \quad (14)$$

$$P_n(x) dx = \lim_{t \rightarrow \infty} P_n(x,t) dx \quad x > 0, n \geq 0 \quad (15)$$

$$V_{i,n}(x) dx = \lim_{t \rightarrow \infty} V_{i,n}(x,t) dx \quad i = 1, 2 \quad x > 0, n \geq 0 \quad (16)$$

Now for $i = 1, 2$ the PGF of this probabilities are defined as follow:

$$P(x, z) = \sum_{n=0}^{\infty} z^n P_n(x) \quad |z| \leq 1, \quad x > 0 \quad (17)$$

$$P(0, z) = \sum_{n=0}^{\infty} z^n P_n(0) \quad |z| \leq 1 \quad (18)$$

Also

$$V_i(x, z) = \sum_{n=0}^{\infty} z^n V_{i,n}(x) \quad |z| \leq 1, \quad x > 0 \quad (19)$$

$$V_i(0, z) = \sum_{n=0}^{\infty} z^n V_{i,n}(0) \quad (20)$$

3. Steady-State Probability Generating Function

From kolmogorov forward equations, for $i = 1, 2$ the steady-state conditions can be written as follow

$$\begin{aligned} \frac{d}{dx} P_n(x) + [\lambda + \mu(x)] P_n(x) \\ = \lambda(1-\alpha) P_n(x) + \lambda \alpha \sum_{k=1}^n a_k P_{n-k}(x) \end{aligned} \quad (21)$$

$n > 0, \quad x > 0$

$$\frac{d}{dx} P_0(x) + [\lambda + \mu(x)] P_0(x) = \lambda(1-\alpha) P_0(x) \quad (22)$$

$$\begin{aligned} \frac{d}{dx} V_{i,n}(x) + [\lambda + \mu(x)] V_{i,n}(x) \\ = \lambda(1-\beta) V_{i,n}(x) + \lambda \beta \sum_{k=1}^n a_k V_{i,n-k}(x) \end{aligned} \quad (23)$$

$n > 0, \quad x > 0$

and

$$\frac{d}{dx} V_{i,0}(x) + [\lambda + v_i(x)] V_{i,0}(x) = \lambda(1-\beta) V_{i,0}(x) \quad (24)$$

also

$$\begin{aligned} \lambda \alpha \beta R_0 = \beta(1-\theta) \int_0^{\infty} \mu_1(x) P_1(x) dx \\ + (1-p) \alpha \int_0^{\infty} v_1(x) V_{1,0}(x) dx \\ + \alpha \int_0^{\infty} v_2(x) V_{2,0}(x) dx \end{aligned} \quad (25)$$

At $x = 0$ for $n \geq 0$ the boundary conditions are

$$\begin{aligned} \beta P_n(0) = \lambda \alpha \beta a_n R_0 + \beta(1-\theta) \int_0^{\infty} \mu(x) P_{n+1}(x) dx \\ + \alpha(1-p) \int_0^{\infty} v_1(x) V_{1,n}(x) dx \\ + \alpha \int_0^{\infty} v_2(x) V_{2,n}(x) dx \end{aligned} \quad (26)$$

$$V_{1,n}(0) = \beta \theta \int_0^{\infty} \mu(x) P_{n+1}(x) dx, \quad (27)$$

also

$$\alpha V_{2,n}(0) = p \int_0^{\infty} v_1(x) V_{1,n}(x) dx \quad (28)$$

Finally the normalizing condition is

$$R_0 + \sum_{n=0}^{\infty} \int_0^{\infty} P_n(x) dx + \sum_{i=1}^2 \sum_{n=0}^{\infty} \int_0^{\infty} V_{i,n}(x) dx = 1 \quad (29)$$

Lemma 3.1 The solution of first order Equation (21) is

$$P(x, z) = P(0, z) [1 - B(x)] e^{-\lambda \alpha (1-x(z))x} \quad x > 0 \quad (30)$$

and for $i = 1, 2$ the solution of first order Equation (23) is

$$V_i(x, z) = V_i(0, z) [1 - V_i(x)] e^{-\lambda\beta(1-X(z))x} \quad x > 0 \quad (31)$$

Proposition 3.2 If for $i = 1, 2$

$$B^* (\lambda\alpha(1 - X(z))) = \int_0^\infty e^{-\lambda\alpha(1-X(z))x} dB(x) \quad (32)$$

$$V_i^* (\lambda\beta(1 - X(z))) = \int_0^\infty e^{-\lambda\beta(1-X(z))x} dV_i(x) \quad (33)$$

are the z-transforms of B and V_i respectively, then

$$1) P(z) = P(0, z) [1 - B^* (\lambda\alpha(1 - X(z)))] \frac{1}{\lambda(1 - X(z))} \quad (34)$$

$$2) V_i(z) = V_i(0, z) [1 - V_i^* (\lambda\beta(1 - X(z)))] \frac{1}{\lambda(1 - X(z))} \quad (35)$$

$$3) \beta z P(0, z) = z\lambda\alpha\beta R_0 (X(z) - 1) + \beta(1 - \theta)P(0, z)B^* (\lambda\alpha(1 - X(z))) + \alpha(1 - p)zV_1(0, z)V_1^* (\lambda\beta(1 - X(z))) + \alpha zV_2(0, z)V_2^* (\lambda\beta(1 - X(z))) \quad (36)$$

$$4) V_1(0, z) = \frac{\beta\theta}{\alpha z} P(0, z)B^* (\lambda\alpha(1 - X(z))) \quad (37)$$

$$5) V_2(0, z) = pV_1(0, z)V_1^* (\lambda\beta(1 - X(z))) \quad (38)$$

Proof: 1) Since

$$P(z) = \int_0^\infty P(x, z) dx$$

then by integration from (30) by part the result obtained.

2) Since for $i = 1, 2$

$$V_i(z) = \int_0^\infty V_i(x, z) dx$$

then from (31) similarly we have 2).

3) First multiply (26) in z^n and summation from $n = 1$ to ∞ , then using (25) we obtain result.

4) By multiplying (27) in z^n and summation from $n = 0$ to ∞ the result obtained.

5) By using the same technique on (28), we have 5).

In the rest of this section for simplifying we omit $[\lambda\alpha(1 - X(z))]$ from B^* and $[\lambda\beta(1 - X(z))]$ from V_i^* .

Corollary 3.3 1) By using (37), (38) in (36) we have

$$P(0, z) = \frac{\lambda\alpha R_0 z (X(z) - 1)}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (39)$$

2) By using (39) in (37) we have

$$V_1(0, z) = \frac{\lambda\beta\theta R_0 (X(z) - 1) B^*}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (40)$$

3) By using (41) in (38) we have

$$V_2(0, z) = \frac{p\lambda\beta\theta R_0 (X(z) - 1) B^* V_1^*}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (41)$$

Corollary 3.4 1) By substituting (39) in (34) we have

$$P(z) = \frac{R_0 z (B^* - 1)}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (42)$$

and for $i = 1, 2$ substituting (40) and (41) in (35) result

$$V_1(z) = \frac{\theta R_0 B^* (1 - V_1^*)}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (43)$$

$$V_2(z) = \frac{p\theta R_0 B^* V_1^* (1 - V_2^*)}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (44)$$

For calculating R_0 , by using the normalizing condition (29) we have

$$R_0 + P(1) + V_1(1) + V_2(1) = 1 \quad (45)$$

R_0 is the steady-state probability that the server is idle but available in the system, hence if $\rho < 1$ be the stability condition under which the steady state solution exist, then $R_0 = (1 - \rho)$ and by using (42), (43), (44) with L'Hopital we have

$$P(1) = R_0 \frac{\lambda\alpha E(X)E(B)}{1 - \lambda E(X) [\alpha E(B) + \beta\theta E(V_1) + p\beta\theta E(V_2)]}$$

and

$$V_1(1) = R_0 \frac{\theta\beta\lambda E(X)E(V_1)}{1 - \lambda E(X) [\alpha E(B) + \beta\theta E(V_1) + p\beta\theta E(V_2)]}$$

$$V_2(1) = R_0 \frac{p\theta\beta\lambda E(X)E(V_1)}{1 - \lambda E(X) [\alpha E(B) + \beta\theta E(V_1) + p\beta\theta E(V_2)]}$$

By substituting this values in (45) we have $R_0 = 1 - \rho$, where

$$\rho = \lambda E(X) [\alpha E(B) + \beta\theta E(V_1) + p\beta\theta E(V_2)] \quad (46)$$

Now the PGF of the queue size at a random epoch is

$$P_Q(z) = P(z) + V_1(z) + V_2(z) = R_0 \frac{z(1 - B^*) + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)}{z - B^* + \theta B^* (1 - V_1^*) + p\theta B^* V_1^* (1 - V_2^*)} \quad (47)$$

The PGF of the system size is

$$\begin{aligned}
 P_S(z) &= R_0 + zP(z) \\
 &= R_0 \frac{(z-1)[B^* + \theta B^*(1-V_1^*) + p\theta B^*V_1^*(1-V_2^*)]}{z - [B^* + \theta B^*(1-V_1^*) + p\theta B^*V_1^*(1-V_2^*)]}
 \end{aligned} \tag{48}$$

Remark 3.5 In this system we set

$$G(z) = B^* + \theta B^*(1-V_1^*) + p\theta B^*V_1^*(1-V_2^*)$$

then

$$P_S(z) = R_0 \frac{(z-1)G(z)}{z - G(z)}$$

$$L = \rho + \frac{[\lambda E(X)]^2 [\alpha^2 E(B^2) + 2\alpha\beta E(B)E(V_c) + \theta\beta^2 E(V_c^2)]}{2(1-\rho)} \tag{50}$$

where $E(V_c)$ and $E(V_c^2)$ are in (6) and (7).

Proof: $P_S(z)$ has the form $R_0 \frac{f(z)}{g(z)}$, where

$$f(z) = (z-1)G(z)$$

and

$$g(z) = z - G(z)$$

Since $\lim_{z \rightarrow 1} f(z) = \lim_{z \rightarrow 1} g(z) = 0$, then by using L'Hopital's rule, we have

$$g''(1) = -G''(1) = -[\lambda E(X)]^2 [\alpha^2 E(B^2) + 2\alpha\beta E(B)E(V_c) + \theta\beta^2 E(V_c^2)]$$

hence by substitute this value in (51) the result obtain.

5. Mean Response Time

The response time W_R is the time interval from the arrival time of a test customer to the time when it leaves the system after service completion *i.e.*, waiting time plus service time.

By using Little's formula, this measure of system is

$$V_1(1) + V_2(1) = \theta\beta\lambda E(X) [E(V_1) + p E(V_2)] = \theta\beta\lambda E(X) E(V_c)$$

and hence the proportion of non-vacation time is $1 - \theta\beta\lambda E(X) E(V_c)$. Consequently

$$\lambda_x = \lambda\alpha + \beta(\beta - \alpha)\theta\lambda^2 E(X) E(V_c) \tag{52}$$

6. Special Cases and Numerical Results

Analyzing a queueing system via actual cases are very important and an useful way for confirm the models. In this section we chose known distributions for service time and vacation times, so with this and by sum numerical approaches the validity of system examined. Also

This is familiar with famous formula Pollaczek-Khinchin in $M/G/1$ queue for PGF of system's queue size.

4. Mean Queue Size

Let L be the mean number of customers in the system, then we have

$$L = \left. \frac{dP_S(z)}{dz} \right|_{z=1} \tag{49}$$

Proposition 4.1 From (49) and using (48) we have

$$L = R_0 \frac{f''(1)g'(1) - g''(1)f'(1)}{2[g'(1)]^2} \tag{51}$$

where $R_0 = 1 - \rho$ and ρ is in (46).

Also

$$f'(1) = G(1) = 1,$$

$$g'(1) = 1 - G'(1) = 1 - \rho$$

and by using (6) and (7)

$$f''(1) = 2\rho$$

$$W_R = \frac{L}{\lambda_x}$$

where following the admissibility assumption of our model, λ_x the actual arrival rate of batches is given by

$$\begin{aligned}
 \lambda_x &= \lambda\alpha \quad (\text{proportion of non-vacation time}) \\
 &\quad + \lambda\beta \quad (\text{proportion of vacation time}).
 \end{aligned}$$

But the proportion of vacation time is

this approach explain that our model can represent some practical problems reasonably well.

Case 1: Let the distribution of service time is ζ -Erlang as follow

$$dB(x) = \frac{(\zeta\mu)^\zeta x^{\zeta-1} e^{-\zeta\mu x}}{(\zeta-1)!} dx \quad x > 0, \zeta \geq 1$$

hence

$$B^*(\lambda - \lambda C(z)) = \frac{(\zeta\mu)^\zeta}{[\lambda(C(z)-1) + \zeta\mu]^\zeta}$$

so $E(B) = \frac{1}{\mu}$ and $E(B^2) = \frac{\zeta+1}{\zeta\mu^2}$. Also we assume the distribution of vacation times respectively are

$$dV_i(x) = \frac{(\gamma_i v_i)^{v_i} x^{\gamma_i-1} e^{-\gamma_i v_i x}}{(\gamma_i-1)!} dx \quad x > 0, \gamma_i \geq 1$$

hence

$$V_i^*(\lambda - \lambda C(z)) = \frac{(\gamma_i v_i)^{\gamma_i}}{[\lambda(C(z)-1) + \gamma_i v_i]^{\gamma_i}}$$

so $E(V_i) = \frac{1}{v_i}$ and $E(V_i^2) = \frac{\gamma_i+1}{\gamma_i v_i^2}$. If we chose geometric distribution for batch size, i.e. $a_n = (1-a)a^{n-1}$,

$$0 < a < 1, \text{ then } E(X) = \frac{a}{1-a}.$$

Now for numerical result we assume the following values for parametrs such that the steady state condition ($\rho < 1$) obtained.

In this case using above values and (46), if $\lambda = 2$ then ρ with respect θ and hence steady state condition is $\rho = 0.2 + 0.63\theta < 1$, so $\theta < 1.26$. By using (50)

$$L = 0.2 + 0.63\theta + \frac{0.15 + 0.292\theta}{0.8 - 0.63\theta}$$

Table 1 shows some values of L with respect θ , in which L increases with respect θ with a mild gradient.

Also **Figure 1** show the mild curve of L with respect θ .

Now we analyze L against λ . Using the values of **Table 2**, if $\theta = 0.5$ then steady state condition is $\rho = 0.257\lambda < 1$ or $\lambda < 3.8$.

Also L with respect λ is

$$L = 0.257\lambda + \frac{0.075\lambda^2}{1 - 0.257\lambda}$$

Figure 2 shows L agains λ . Near $\lambda = 4$ the system blowing up. Untile 2 customers the system is stable.

Case 2: In this case we assume the distribution of service time and for $i = 1, 2$ vacation times are exponential as follow

$$dB(x) = \mu e^{-\mu x} dx, E(B) = \frac{1}{\mu}, E(B^2) = \frac{2}{\mu^2}$$

$$dV_i(x) = v_i e^{-v_i x} dx, E(V_i) = \frac{1}{v_i}, E(V_i^2) = \frac{2}{v_i^2}$$

With geometric distribution for batch size and following values for parameters in **Table 4** the steady state condition is $\rho = \frac{1}{\mu} + 0.12 < 1$ or $\mu > 1.13$. Also

$$L = 0.12 + \frac{1}{\mu} + \frac{1.44 + 0.72\mu + 0.064\mu^2}{0.88\mu^2 - \mu}$$

and the graph of model is in **Figure 3**. According to this curve, L decreases with respect μ , and after $\mu = 2$ the system is stable.

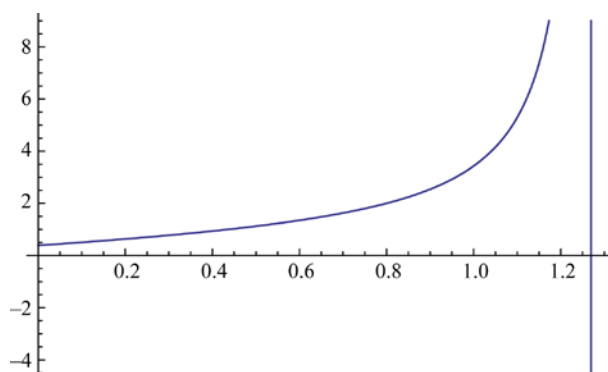


Figure 1. L vis-a-vis θ .

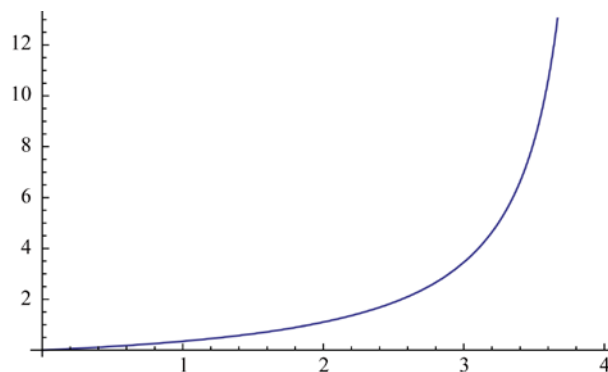


Figure 2. L vis-a-vis λ .

Table 1. Values of L with respect θ .

θ	0.1	0.3	0.5	0.7	0.9	1
L	0.5	0.77	1.12	1.62	2.53	3.43

Table 2. Values of parameters.

a	ζ	μ	γ_1	v_1	γ_2	v_2	p	α	β
0.5	2	1	2	1	2	2	0.1	0.1	0.3

Table 3. Values of L with respect λ .

λ	1	2	3	3.5
L	0.35	1.1	3.45	8.2

Table 4. Values of parameters.

λ	a	α	β	v_1	v_2	p	θ
2	0.5	0.6	0.2	1	1	0.5	0.2

Table 5. Values of L with respect μ .

μ	1.5	2	3	4	5
L	6.33	2.68	1.3	0.9	0.7

Table 6. Values of parameters.

λ	a	α	β	μ	v_1	v_2	p
2	0.5	0.7	0.2	3	1	1	0.1

Table 7. Values of L with respect θ .

θ	0.1	0.3	0.5	0.7	0.9	1
L	2.46	3.23	4.49	7.11	17	60

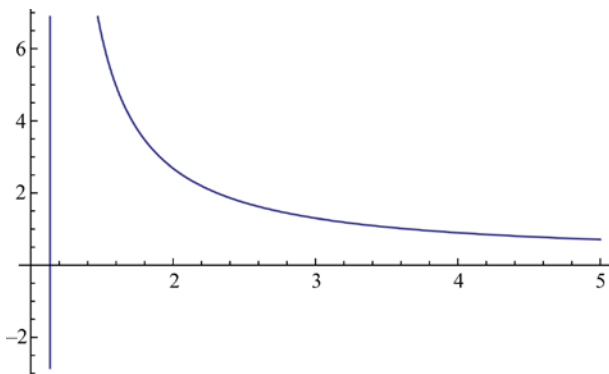


Figure 3. L vis-a-vis μ .

Also **Table 5** shows some values of L against μ . When μ increases from 1.5 to 2, then system changed in stable form.

In this case we assume θ is unknown. With the values in **Table 6** the steady state condition is $\rho = 0.46 + 0.52\theta < 1$ or $\theta < 1.02$.

Also

$$L = 0.46 + 0.52\theta + \frac{0.928 + 0.256\theta}{0.54 - 0.52\theta}$$

Table 7 shows some values of L against θ . After $\theta = 0.5$, the system blowing up.

Case 3: In this case we assume the service time has deterministic distribution. Hence it is sufficient $\zeta \rightarrow \infty$

in case 1, and distribution degenerates in $d = \frac{1}{\mu}$. Also if

$E(X) = 1$ and $\alpha = \beta = 1$ then steady state is

$$\rho = \lambda \left[\frac{1}{\mu} + \theta \left(\frac{1}{v_1} + \frac{p}{v_2} \right) \right]$$

or

$$\lambda \{ d v_1 v_2 + \theta (v_2 + p v_1) \} < v_1 v_2.$$

In this case we obtain the results of [9].

7. Concluding Remarks

In this paper we have studied a batch arrival queueing system with admissibility restricted and optional server's vacation which generalized classical $M/G/1$ queue. An application of this model can be found in mobile network where the messages are in batch form, the system may have two phases vacation such that first phase is essential but the second phase may chosen randomly and have optional cases. Also because of admissibility restriction in service or system all batches don't enter in service. Our investigations concerned with not only queue size distribution but also waiting time distribution. Also by some numerical approaches the validity of results are examined. A practical generalization for this system is to consider optional service.

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