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Small-amplitude limit cycles of some non-smooth Liénard systems

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Abstract: Based on the results by HAN Mao-an, et al. for computing some focus values of non-smooth Liénard systems, the number of limit cycles bifurcated from the origin of some more general non-smooth Liénard systems were given by using maple process.

Key words: Liénard systems; focus values; Hopf cyclicity

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一些非光滑 Liénard 系统的小扰动极限环

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摘要: 根据韩茂安等所得到的计算非光滑 Liénard 系统的焦点量的方法, 应用 maple 程序, 给出一些较一般的非光滑 Liénard 系统从原点处分支出的极限环数目.

关键词: Liénard 系统; 焦点量; 环性数

0 Introduction

Several papers discussed the Hopf bifurcations problems of some non-smooth systems, see [1-4] for example. In [1] the authors considered the non-smooth Liénard systems

$$\dot{x} = p(y) - F(x, a), \quad \dot{y} = -g(x), \tag{0.1}$$

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where $a\in\mathbf{R}^m$, and $F(x,a)=\begin{cases}F^+(x,a), & x>0,\\ F^-(x,a), & x\leqslant 0,\end{cases}$ $g(x)=\begin{cases}g^+(x), & x>0,\\ g^-(x), & x\leqslant 0.\end{cases}$
Here, $F^{\pm}$ and $g^{\pm}$ are all C^{∞} functions and satisfy

$$\begin{aligned} F^{\pm}(0,a)=0,\quad p(0)=0,\quad g^{\pm}(0)=0,\quad (g^{\pm})'(0)=g_1^{\pm}>0,\\ p'(0)=p_0>0,\quad (F_x^{\pm}(0,a_0))^2-4p_0g_1^{\pm}<0,\quad a_0\in\mathbf{R}^m.\end{aligned} \tag{0.2}$$

Let

$$G^{\pm}(x)=\int_0^x g^{\pm}(t)\mathrm{d}t,\quad \alpha(x)=-\sqrt{g_1^+}\Big/\sqrt{g_1^-}x+O(x^2),$$

where $\alpha(x)$ satisfies $G^-(\alpha(x))\equiv G^+(x)$ for $0<x\ll 1$. Suppose formally for $0<x\ll 1$

$$F(\alpha(x),a)-F(x,a)=F^-(\alpha(x),a)-F^+(x,a)=\sum_{i\geqslant 1}B_i(a)x^i.$$

Then the following results about System (0.1) were obtained in [1].

Lemma 0.1 Let (0.2) hold. Then the origin of (0.1) is a fine or weak focus for $a=a_0$ if and only if

$$\sqrt{g_1^-}F_x^+(0,a_0)+\sqrt{g_1^+}F_x^-(0,a_0)=0. \tag{0.3}$$

Corollary 0.1 Let (0.2) and (0.3) hold. If there exists $k\geqslant 1$ such that

$$F^-(\alpha(x),a)\equiv F^+(x,a)\quad \text{when}\quad B_{j+1}=0, j=0,\cdots,k \tag{0.4}$$

for all $a\in\mathbf{R}^m$, then the origin is a focus of order at most $k+1$ of System (0.1) unless it is a center.

Theorem 0.1 Suppose (0.2) and (0.3) are satisfied, and let (0.4) hold for some $k\geqslant 1$. If further

$$B_{j+1}(a_0)=0, j=0,\cdots,k,\quad \operatorname{rank}\frac{\partial(B_1,\cdots,B_{k+1})}{\partial(a_1,\cdots,a_n)}\Big|_{a=a_0}=k+1, \tag{0.5}$$

for some $a_0\in\mathbf{R}^m$, then System (0.1) has Hopf cyclicity k at the origin for $|a-a_0|$ small.

Theorem 0.2 Let (0.2) and (0.3) hold. Suppose there exists $k\geqslant 1$ such that (0.4) holds for all $a\in\mathbf{R}^m$ and (0.5) holds for some $a_0\in\mathbf{R}^m$. If F is linear in a then for any constant $N>|a_0|$, System (0.1) has Hopf cyclicity k for all $|a|\leqslant N$.

Note that Hopf cyclicity denotes the maximum number of small-amplitude limit cycles bifurcated from the origin of the system.

Using Theorem 0.2, the following special cases of System (0.1)

$$\dot{x}=y-\begin{cases}\sum\limits_{i=1}^n a_i^+x^i, & x>0,\\ \sum\limits_{i=1}^n a_i^-x^i, & x\leqslant 0,\end{cases}\qquad \dot{y}=-\begin{cases}x+g_2^+x^2, & x>0,\\ x+g_2^-x^2, & x\leqslant 0\end{cases} \tag{0.6}$$

have been considered in [1-3], where $a_i^{\pm}$ $i=1,2,\cdots,n$ are parameters. When $g_2^+=g_2^-=1$, [2] obtained that the Hopf cyclicity of (0.6) is $\left[\frac{5n-1}{3}\right]$. When g_2^+, g_2^- are constants, System (0.6) has Hopf cyclicity $1,3,4,6,8$ at the origin for $n=1,2,3,4,5$ respectively, see the results in [1,3].

Using Theorem 0.2 and maple process, we will consider small-amplitude limit cycles of some more general non-smooth systems than (0.6) of the form

$$\dot{x}=y-\begin{cases}\sum\limits_{i=1}^{n}a_i^{+}x^i, & x>0,\\ \sum\limits_{i=1}^{n}a_i^{-}x^i, & x\leqslant 0,\end{cases}\qquad \dot{y}=-\begin{cases}x+x^2+\sum\limits_{i=3}^{m}g_ix^i, & x>0,\\ \sum\limits_{i=1}^{m}x^i, & x\leqslant 0,\end{cases}\tag{0.7}$$

where $g_i=1$ for $i\leqslant m-1$, $m\geqslant 3$ and $g_m=2$.

1 Main result and proof

Using Maple process and Theorem 0.2, we obtain the following results.

Theorem 1.1 Table 1. The Hopf cyclicity for System (0.7) when F and g are of varying degrees.

		deg $F(n)$																	
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	3	3	5	6	8	10	12	13	15	17	19	20	22	24	26	27	29	31	33
	4	3	5	7	8	10	12	14	16	17	19	21	23	25	26	28	30	32	
deg $g(m)$	5	3	5	7	9	10	12	14	16	18	20	21							
	6	3	5	7	9	11	12	14											
	7	3	5	7	9	11	13												
	8	3	5	7	9	11													

Proof To use the conclusion of Theorem 0.2, we need to compute the corresponding B_i, $i=1,2,\cdots,2n$. See Appendix.

(I) $m=3$. When $n=2,3$, one obtains

$$\det\frac{\partial(B_1,B_2,B_3,B_4)}{\partial(a_1^+,a_1^-,a_2^+,a_2^-)}=\frac{817}{648}\neq 0,\quad \det\frac{\partial(B_1,B_2,B_3,B_4,B_5,B_6)}{\partial(a_1^+,a_1^-,a_2^+,a_2^-,a_3^+,a_3^-)}=-\frac{16\ 295}{23\ 328}\neq 0.$$

It follows from Theorem 0.2, the Hopf cyclicity of system (0.7) at the origin is $2n-1$ by taking $a_0=(0,0,0,0)$ and $a_0=(0,0,0,0,0,0)$, respectively.

When $n=4$, we have

$$\text{rank}\,\frac{\partial(B_1,B_2,B_3,B_4,B_5,B_6,B_7,B_8)}{\partial(a_1^+,a_1^-,a_2^+,a_2^-,a_3^+,a_3^-,a_4^+,a_4^-)}=7,\quad \det\frac{\partial(B_1,B_2,B_3,B_4,B_5,B_6,B_7)}{\partial(a_1^+,a_1^-,a_2^+,a_2^-,a_3^+,a_3^-,a_4^+)}=-\frac{635\ 317}{497\ 664}\neq 0.$$

Solve $B_i(a)=0$, $i=1,2,\cdots,7$, we obtain $a_1^{\pm}=0$, $a_2^{\pm}=2a_4^-$, $a_3^{\pm}=\frac{4}{3}a_4^-$, $a_4^+=2a_4^-$, then by $G^-(\alpha(x))=G^+(x)$, we have $F^-(\alpha(x))=F^+(x)$. Thus, when $B_i(a)=0$, $i=1,2,\cdots,7$, one has $F(\alpha(x))=F(x)$. The conclusion follows from Theorem 0.2 for $n=4$ by taking $a_0=\left(0,0,2a_4^-,2a_4^-,\frac{4}{3}a_4^-,\frac{4}{3}a_4^-,2a_4^-,a_4^-\right)$.

Similarly, for $n=5,6,7$, we obtain

$$\text{rank}\,\frac{\partial(B_1,B_2,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,a_2^+,a_2^-,\cdots,a_n^+,a_n^-)}=2n-1,\quad \det\frac{\partial(B_1,B_2,\cdots,B_{2n-1})}{\partial(a_1^+,a_1^-,\cdots,a_3^+,a_3^-,a_4^+,\cdots,a_n^+,a_n^-)}\neq 0.$$

The Hopf cyclicity is $2n-2$ by taking

$$a_0=\Big(0,0,2a_4^-,2a_4^-,\frac{4}{3}a_4^-,\frac{4}{3}a_4^-,2a_4^-,a_4^-,0,0\big|_{n=5},0,0\big|_{n=6}0,0\big|_{n=7}\Big).$$

For $n=8,\cdots,11$, we have rank $\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n-2$ and

$$\det\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{14})}{\partial(a_1^+,a_1^-,a_2^+,a_2^-,a_3^+,a_3^-,a_4^+,a_5^+,a_5^-,a_6^+,a_6^-,a_7^+,a_7^-,a_8^+)}\approx 60.199\ 611\ 53\neq 0,(n=8),$$

$$\det\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n-2})}{\partial(a_1^+,a_1^-,a_2^+,a_2^-,a_3^+,a_3^-,a_4^+,a_5^+,a_5^-,a_6^+,a_6^-,a_7^+,a_7^-,a_8^+,\cdots,a_n^+,a_n^-)}\neq 0,(n=9,\cdots,11).$$

The Hopf cyclicity is $2n-3$ by taking

$$a_0=\Big(0,0,2a_4^--8a_8^-,2a_4^--8a_8^-,\frac{4}{3}a_4^--\frac{16}{3}a_8^-,\frac{4}{3}a_4^--\frac{16}{3}a_8^-,2a_4^--4a_8^-,a_4^-,$$
$$\frac{16}{3}a_8^-,\frac{16}{3}a_8^-,\frac{88}{9}a_8^-,\frac{52}{9}a_8^-,\frac{16}{3}a_8^-,\frac{8}{3}a_8^-,4a_8^-,a_8^-\big|_{n=8},0,0\big|_{n=9},0,0\big|_{n=10},0,0\big|_{n=11}\Big).$$

For $n=12,\cdots,15$, we have rank $\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n-3$, and

$$\det\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n-3})}{\partial(a_1^+,a_1^-,\cdots,a_3^+,a_3^-,a_4^+,a_5^+,a_6^+,a_7^+,a_7^-,\cdots,a_n^+,a_n^-)}\neq 0,$$

Hence, the Hopf cyclicity is $2n-4$ by taking

$$a_0=\Big(0,0,2a_4^--\frac{3}{2}a_5^-,2a_4^--\frac{3}{2}a_5^-,\frac{4}{3}a_4^--a_5^-,\frac{4}{3}a_4^--a_5^-,2a_4^--\frac{3}{4}a_5^-,a_4^-,a_5^-,a_5^-,$$
$$\frac{3}{4}a_5^-+a_6^-,a_6^-,2a_6^--\frac{7}{6}a_5^-,2a_6^--\frac{5}{3}a_5^-,\frac{13}{3}a_6^--\frac{71}{18}a_5^-,\frac{17}{6}a_6^--\frac{415}{144}a_5^-,$$
$$\frac{116}{27}a_6^--\frac{377}{81}a_5^-,\frac{62}{27}a_6^--\frac{403}{162}a_5^-,\frac{13}{3}a_6^--\frac{169}{36}a_5^-,\frac{17}{12}a_6^--\frac{221}{144}a_5^-,2a_6^--\frac{13}{6}a_5^-,$$
$$\frac{1}{2}a_6^--\frac{13}{24}a_5^-,a_6^--\frac{13}{12}a_5^-,\frac{1}{8}a_6^--\frac{13}{96}a_5^-\big|_{n=12},0,0\big|_{n=13},0,0\big|_{n=14},0,0\big|_{n=15}\Big).$$

For $n=16,\cdots,19$, we have rank $\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n-4$, and

$$\det\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n-4})}{\partial(a_1^+,a_1^-,\cdots,a_3^+,a_3^-,a_4^+,a_5^+,a_6^+,a_7^+,a_7^-,a_8^+,a_9^+,a_9^-,\cdots,a_n^+,a_n^-)}\neq 0.$$

The Hopf cyclicity is $2n-5$ by taking

$$a_0=\Big(0,0,2a_4^--\frac{3}{2}a_5^-,2a_4^--\frac{3}{2}a_5^-,\frac{4}{3}a_4^--a_5^-,\frac{4}{3}a_4^--a_5^-,2a_4^--\frac{3}{4}a_5^-,a_4^-,a_5^-,a_5^-,$$
$$\frac{3}{4}a_5^-+a_6^-,a_6^-,2a_6^--\frac{7}{6}a_5^-,2a_6^--\frac{5}{3}a_5^-,a_8^-+\frac{3}{2}a_6^--\frac{17}{16}a_5^-,a_8^-,$$
$$\frac{8}{3}a_8^--\frac{88}{27}a_6^-+\frac{491}{162}a_5^-,\frac{8}{3}a_8^--\frac{142}{27}a_6^-+\frac{421}{81}a_5^-,\frac{20}{3}a_8^--\frac{131}{9}a_6^-+\frac{392}{27}a_5^-,$$
$$\frac{14}{3}a_8^--\frac{425}{36}a_6^-+\frac{5\ 147}{432}a_5^-,\frac{248}{27}a_8^--\frac{1\ 946}{81}a_6^-+\frac{5\ 906}{243}a_5^-,\frac{140}{27}a_8^--\frac{2\ 299}{162}a_6^-+$$

$$\frac{27\,997}{1944}a_5^-,\frac{934}{81}a_8^- - \frac{7\,696}{243}a_6^- + \frac{187\,487}{5\,832}a_5^-,\frac{707}{162}a_8^- - \frac{23\,795}{1\,944}a_6^- + \frac{145\,123}{11\,664}a_5^-,$$
$$\frac{248}{27}a_8^- - \frac{2\,108}{81}a_6^- + \frac{12\,865}{486}a_5^-,\frac{70}{27}a_8^- - \frac{595}{81}a_6^- + \frac{14\,525}{1\,944}a_5^-,\frac{20}{3}a_8^- - \frac{170}{9}a_6^- + \frac{2\,075}{108}a_5^-,$$
$$\frac{7}{6}a_8^- - \frac{119}{36}a_6^- + \frac{2\,905}{864}a_5^-,\frac{8}{3}a_8^- - \frac{68}{9}a_6^- + \frac{415}{54}a_5^-,\frac{1}{3}a_8^- - \frac{17}{18}a_6^- + \frac{415}{432}a_5^-,$$
$$a_8^- - \frac{17}{6}a_6^- + \frac{415}{144}a_5^-,\frac{1}{16}a_8^- - \frac{17}{96}a_6^- + \frac{415}{2\,304}a_5^-\Big|_{n=16},0,0\Big|_{n=17},0,0\Big|_{n=18},0,0\Big|_{n=19}\Big).$$

(II) $m=4$. When $n=2,3$ and 4, rank $\frac{\partial(B_1,B_2,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n$, hence, the Hopf cyclicity is $2n-1$ by taking $a_0=(0,0,0,0)$, $a_0=(0,0,0,0,0,0)$ and $a_0=(0,0,0,0,0,0,0,0)$.

For $n=5,\cdots,9$, rank$\frac{\partial(B_1,B_2,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n-1$,

$$\det\frac{\partial(B_1,B_2,\cdots,B_{2n-1})}{\partial(a_1^+,a_1^-,\cdots,a_3^+,a_3^-,a_4^+,\cdots,a_n^+,a_n^-)}\neq 0,$$

hence, the Hopf cyclicity is $2n-2$ by taking

$$a_0=\Big(0,0,2a_4^-,2a_4^-,\frac{4}{3}a_4^-,\frac{4}{3}a_4^-,a_4^-,a_4^-,\frac{8}{5}a_4^-,\frac{4}{5}a_4^-\Big|_{n=5},0,0\Big|_{n=6},0,0\Big|_{n=7},0,0\Big|_{n=8},0,0\Big|_{n=9}\Big).$$

When $n=10,\cdots,14$, we have rank$\frac{\partial(B_1,B_2,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n-2$ and

$$\det\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n-2})}{\partial(a_1^+,a_1^-,\cdots,a_3^+,a_3^-,a_4^+,a_5^+,a_6^+,a_6^-,\cdots,a_n^+,a_n^-)}\neq 0.$$

The Hopf cyclicity is $2n-3$ by taking

$$a_0=\Big(0,0,5a_4^- - \frac{15}{4}a_5^-,5a_4^- - \frac{15}{4}a_5^-,\frac{10}{3}a_4^- - \frac{5}{2}a_5^-,\frac{10}{3}a_4^- - \frac{5}{2}a_5^-,a_4^-,a_4^-,$$
$$2a_4^- - \frac{1}{2}a_5^-,a_5^-,-\frac{13}{6}a_4^- + \frac{65}{24}a_5^-,-\frac{13}{6}a_4^- + \frac{65}{24}a_5^-,\frac{17}{4}a_5^- - \frac{17}{5}a_4^-,\frac{11}{4}a_5^- - \frac{11}{5}a_4^-,$$
$$\frac{79}{32}a_5^- - \frac{79}{40}a_4^-,\frac{47}{32}a_5^- - \frac{47}{40}a_4^-,\frac{3}{2}a_5^- - \frac{6}{5}a_4^-,\frac{3}{4}a_5^- - \frac{3}{5}a_4^-,\frac{6}{5}a_5^- - \frac{24}{25}a_4^-,$$
$$\frac{3}{10}a_5^- - \frac{6}{25}a_4^-\Big|_{n=10},0,0\Big|_{n=11},0,0\Big|_{n=12},0,0\Big|_{n=13},0,0\Big|_{n=14}\Big).$$

For $n=15,\cdots,18$, one obtains rank$\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n})}{\partial(a_1^+,a_1^-,\cdots,a_n^+,a_n^-)}=2n-3$,

$$\det\frac{\partial(B_1,B_2,B_3,B_4,\cdots,B_{2n-3})}{\partial(a_1^+,a_1^-,\cdots,a_3^+,a_3^-,a_4^+,a_5^+,a_6^+,a_7^+,a_7^-,\cdots,a_n^+,a_n^-)}\neq 0.$$

Hence, the Hopf cyclicity is $2n-4$ by taking

$$a_0=\Big(0,0,5a_4^- - \frac{15}{4}a_5^-,5a_4^- - \frac{15}{4}a_5^-,\frac{10}{3}a_4^- - \frac{5}{2}a_5^-,\frac{10}{3}a_4^- - \frac{5}{2}a_5^-,a_4^-,a_4^-,$$

$$2a_4^- - \frac{1}{2}a_5^-, a_5^-, a_6^-, a_6^-, \frac{14}{15}a_4^- - \frac{7}{6}a_5^- + 2a_6^-, \frac{32}{15}a_4^- - \frac{8}{3}a_5^- + 2a_6^-, \frac{1\,499}{360}a_4^- - \frac{1\,499}{288}a_5^- +$$
$$\frac{17}{6}a_6^-, \frac{1\,787}{360}a_4^- - \frac{1\,787}{288}a_5^- + \frac{17}{6}a_6^-, \frac{727}{81}a_4^- - \frac{3\,635}{324}a_5^- + \frac{634}{135}a_6^-, \frac{565}{81}a_4^- - \frac{2\,825}{324}a_5^- +$$
$$\frac{472}{135}a_6^-, \frac{16\,277}{1\,800}a_4^- - \frac{16\,277}{1\,440}a_5^- + \frac{277}{60}a_6^-, \frac{11\,333}{1\,800}a_4^- - \frac{11\,333}{1\,440}a_5^- + \frac{181}{60}a_6^-,$$
$$\frac{1\,547}{180}a_4^- - \frac{1\,547}{144}a_5^- + \frac{119}{30}a_6^-, \frac{871}{180}a_4^- - \frac{871}{144}a_5^- + \frac{67}{30}a_6^-, \frac{1\,547}{180}a_4^- - \frac{1\,547}{144}a_5^- + \frac{119}{30}a_6^-,$$
$$\frac{1\,547}{180}a_4^- - \frac{1\,547}{144}a_5^- + \frac{119}{30}a_6^-, \frac{47}{25}a_6^- - \frac{611}{120}a_5^- + \frac{611}{150}a_4^-, \frac{31}{50}a_6^- - \frac{403}{240}a_5^- + \frac{403}{300}a_4^-,$$
$$\frac{24}{25}a_6^- - \frac{13}{5}a_5^- + \frac{52}{25}a_4^-, \frac{6}{25}a_6^- - \frac{13}{20}a_5^- + \frac{13}{25}a_4^-, \frac{64}{125}a_6^- - \frac{104}{75}a_5^- + \frac{416}{375}a_4^-,$$
$$\frac{8}{125}a_6^- - \frac{13}{75}a_5^- + \frac{52}{375}a_4^-\big|_{n=15}, 0, 0\big|_{n=16}, 0, 0\big|_{n=17}, 0, 0\big|_{n=18}\Big).$$

(III) $m = 5$, when $n = 2, \cdots, 5$, rank $\frac{\partial(B_1, B_2, \cdots, B_{2n})}{\partial(a_1^+, a_1^-, \cdots, a_n^+, a_n^-)} = 2n$, hence, the Hopf cyclicity is $2n-1$; when $n = 6, \cdots, 11$, rank $\frac{\partial(B_1, B_2, \cdots, B_{2n})}{\partial(a_1^+, a_1^-, a_2^+, a_2^-, \cdots, a_n^+, a_n^-)} = 2n-1$, and

$$\det \frac{\partial(B_1, B_2, \cdots, B_{2n-1})}{\partial(a_1^+, a_1^-, a_2^+, a_2^-, a_3^+, a_3^-, a_4^+, a_5^+, a_5^-, \cdots, a_n^+, a_n^-)} \neq 0,$$

one can obtain that the Hopf cyclicity is $2n-2$ by taking

$$a_0 = \Big(0, 0, 2a_4^-, 2a_4^-, \frac{4}{3}a_4^-, \frac{4}{3}a_4^-, a_4^-, a_4^-, \frac{4}{5}a_4^-, \frac{4}{5}a_4^-, \frac{4}{3}a_4^-,$$
$$\frac{2}{3}a_4^-\big|_{n=6}, 0, 0\big|_{n=7}, 0, 0\big|_{n=8}, 0, 0\big|_{n=9}, 0, 0\big|_{n=10}, 0, 0\big|_{n=11}\Big).$$

For $n = 12$, rank $\frac{\partial(B_1, B_2, \cdots, B_{24})}{\partial(a_1^+, a_1^-, a_2^+, a_2^-, \cdots, a_{12}^+, a_{12}^-)} = 22$, and

$$\det \frac{\partial(B_1, B_2, \cdots, B_{22})}{\partial(a_1^+, a_1^-, \cdots, a_3^+, a_3^-, a_4^+, a_5^+, a_6^+, a_6^-, \cdots, a_{12}^+, a_{12}^-)} \approx 6.603\,247\,013 \times 10^5 \neq 0.$$

Hence, the Hopf cyclicity is 21 by taking

$$a_0 = \Big(0, 0, 5a_4^- - \frac{15}{4}a_5^-, 5a_4^- - \frac{15}{4}a_5^-, \frac{10}{3}a_4^- - \frac{5}{2}a_5^-, \frac{10}{3}a_4^- - \frac{5}{2}a_5^-, a_4^-, a_4^-, a_5^-, a_5^-,$$
$$\frac{7}{6}a_4^- + \frac{5}{24}a_5^-, \frac{35}{24}a_5^- - \frac{1}{2}a_4^-, -\frac{11}{5}a_4^- + \frac{11}{4}a_5^-, -\frac{11}{5}a_4^- + \frac{11}{4}a_5^-, \frac{127}{32}a_5^- - \frac{127}{40}a_4^-,$$
$$\frac{87}{32}a_5^- - \frac{87}{40}a_4^-, \frac{29}{12}a_5^- - \frac{29}{15}a_4^-, \frac{19}{12}a_5^- - \frac{19}{15}a_4^-, \frac{31}{20}a_5^- - \frac{31}{25}a_4^-, \frac{37}{40}a_5^- - \frac{37}{50}a_4^-,$$
$$a_5^- - \frac{4}{5}a_4^-, \frac{1}{2}a_5^- - \frac{2}{5}a_4^-, \frac{5}{6}a_5^- - \frac{2}{3}a_4^-, \frac{5}{24}a_5^- - \frac{1}{6}a_4^-\Big).$$

The other values listed in Tab. 1 are calculated using similar arguments to those given above. Notice that although the methods used to compute the hopf cyclicity of System (0.1) have been given by [1], but it is a difficult problem to obtain the corresponding Hopf cyclicity for concrete degrees of $F^{\pm}$ and $g^{\pm}$, for example when $g^{\pm}(x) = x + g_2^{\pm}x^2 + g_3^{\pm}x^3$, we will leave these for further study and try to establish a general formula for Hopf cyclicity as a function of the degrees of $F^{\pm}$ and $g^{\pm}$.

2 Appendix

The maple codes to compute B_i $(i = 1, 2, \cdots, 30)$. See $m = 3$, $n = 15$ for example.

restart; *with*(*Linear Algebra*):
n:=31: H:=$\frac{f^2}{2}+\frac{f^3}{3}+\frac{f^4}{4}-\frac{x^2}{2}-\frac{x^3}{3}-\frac{2x^4}{4}$: phi:=-x: F:=0: for i from 2 to n do phi := phi+e[i]*x^i: od: temp:=expand(subs(f=phi, H)): for i from 3 to n do temp1:=coeff(temp,x^i): e[i-1]:=solve(temp1,e[i-1]): od: for i from 1 to n-16 do F:=F+a_i^-*phi^i-$a_i^+ * x^i$: od: for i from 1 to n-1 do B:=simplify(coeff(F,x^i)): print ((i),B) od:

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