ON IDEMPOTENT D-NORMS

MICHAEL FALK

Abstract. Replacing the spectral measure by a random vector $\boldsymbol{Z}$ allows the representation of a multivariate max-stable distribution with standard negative margins via a norm, called *D-norm*, whose generator is $\boldsymbol{Z}$. We investigate the set of all generators in detail. This approach towards multivariate extreme value distributions entails the definition of a multiplication type operation on the set of D-norms leading to idempotent D-norms. We characterize the set of idempotent D-norms. Iterating the multiplication provides a track of D-norms, whose limit exists and is again a D-norm. If this iteration is repeatedly done on the same D-norm, then the limit of the track is idempotent.

1. Introduction

A random vector (rv) $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$ is called *standard max-stable* (sms) if each component follows the standard negative exponential distribution, i.e., $P(\eta_i \le x) = \exp(x)$, $x \le 0$, $1 \le i \le d$, and if for each $n \in \mathbb{N}$

$$P\left(n \max_{1\le i\le n} \boldsymbol{\eta}^{(i)} \le \boldsymbol{x}\right) = P\left(\boldsymbol{\eta} \le \frac{\boldsymbol{x}}{n}\right)^n = P\left(\boldsymbol{\eta} \le \boldsymbol{x}\right), \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d,$$

where $\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \dots$ are independent copies of $\boldsymbol{\eta}$. All operations on vectors such as max or $\le$ are meant componentwise.

The distribution function (df) $G(\boldsymbol{x}) := P(\boldsymbol{\eta} \le \boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, of a sms rv $\boldsymbol{\eta}$ is called *standard max-stable* as well. The following characterization is a consequence of the de Haan-Resnick-Pickands representation of a sms df, see, e.g., Falk et al. (2010, Sections 4.2, 4.3).

Theorem 1.1. *A function $G : (-\infty, 0]^d \to [0, 1]$ is a sms df $\iff$ there exists a* D-norm $\|\cdot\|_D$ *on $\mathbb{R}^d$ such that*

$$G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right), \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d.$$

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A D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ is defined via a rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ as follows. It is required that $Z_i \geq 0$ a.s., $E(Z_i) = 1$, $1 \leq i \leq d$, together with the condition

$$\|\boldsymbol{Z}\| = \text{const} \quad a.s., \tag{1}$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$. The D-norm corresponding to $\boldsymbol{Z}$ is then defined by

$$\|\boldsymbol{x}\|_D := E\left(\max_{1\leq i\leq d}(|x_i|\, Z_i)\right), \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

and $\boldsymbol{Z}$ is called *generator* of $\|\cdot\|_D$.

The letter D means *dependence* among the components of $\boldsymbol{\eta}$, reflected by the D-norm. If we take for example $Z_i = 1$, $1 \leq i \leq d$, then we obtain

$$\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_\infty := \max_{1\leq i\leq d} |x_i|\,,$$

which is the case of complete dependence $\eta_1 = \dots = \eta_d$ a.s. If $\boldsymbol{Z}$ is a random permutation of the vector $(d, 0, \dots, 0) \in \mathbb{R}^d$ with equal probabilities, then we obtain the L_1-norm

$$\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1 := \sum_{i=1}^d |x_i|\,, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

which characterizes the case of complete independence of $\eta_1, \dots, \eta_d$. These are the two extreme cases of a D-norm and we obviously have

$$\|\cdot\|_\infty \leq \|\cdot\|_D \leq \|\cdot\|_1$$

for each D-norm $\|\cdot\|_D$.

The initial characterization of a sms df G by de Haan and Resnick (1977) and Pickands (1981) is formulated in terms of measure theory and based on a *spectral measure* pertaining to G. This spectral measure is a finite measure on the unit sphere in $\mathbb{R}^d$, taken with respect to an arbitrary norm $\|\cdot\|$. A generator $\boldsymbol{Z}$ is the probabilistic counterpart of the spectral measure, as its distribution equals essentially the spectral measure normed to one; see, e.g., Falk et al. (2010, Sections 4.2, 4.3). Replacing the spectral measure by a generator enables, however, a different perspective on multivariate extreme value theory (EVD); the following remark provides an example.

Remark 1.2. Each sms rv $\boldsymbol{\eta}$ can be generated in the following way. Consider a Poisson point process on $[0, \infty)$ with mean measure $r^{-2}dr$. Let V_i, $i \in \mathbb{N}$, be a realization of this point process. Consider independent copies $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \dots$ of a generator $\boldsymbol{Z}$ of the D-norm corresponding to $\boldsymbol{\eta}$, which are also independent of the Poisson process. Then we have

$$\boldsymbol{\eta} =_D -\frac{1}{\sup_{i\in\mathbb{N}} V_i \boldsymbol{Z}^{(i)}},$$

which is a consequence of de Haan and Ferreira (2006, Lemma 9.4.7) and elementary computations.

The copula of an arbitrary sms df $G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, is given by

$$C(\boldsymbol{u}) = G(\log(\boldsymbol{u})) = \exp\left(-\|\log(\boldsymbol{u})\|_D\right), \qquad \boldsymbol{u} \in (0,1)^d.$$

As each multivariate max-stable df can be obtained from a sms df by just transforming the margins (see, e.g., Falk et al. (2010, Lemma 5.6.8)), the copula of *each* multivariate extreme value distribution is of the preceding form.

We have, moreover, by Taylor expansion of $\log(\cdot)$ and $\exp(\cdot)$ for $\boldsymbol{x} \ge \boldsymbol{0} \in \mathbb{R}^d$

$$\lim_{t\downarrow 0} \frac{1 - C(1 - t\boldsymbol{x})}{t} = \lim_{t\downarrow 0} \frac{1 - \exp\left(-\|\log(1 - t\boldsymbol{x})\|_D\right)}{t} = \|\boldsymbol{x}\|_D,$$

and, thus, $\|\boldsymbol{x}\|_D =: \lambda(\boldsymbol{x})$ is the *stable tail dependence function* introduced by Huang (1992).

The function

$$D(\boldsymbol{t}) := \left\| \left(t_1, \dots, t_{d-1}, 1 - \sum_{i=1}^{d-1} t_i \right) \right\|_D,$$

defined on $\left\{ \boldsymbol{t} \in [0,1]^{d-1} : \sum_{i=1}^{d-1} t_i \le 1 \right\}$ is known as *Pickands dependence functions*, and we have

$$\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1 D\left(\frac{|x_1|}{\|\boldsymbol{x}\|_1}, \dots, \frac{|x_{d-1}|}{\|\boldsymbol{x}\|_1} \right), \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

which offers a different way to represent a sms df; see Falk et al. (2010, Section 4.3).

The generator of a D-norm is not uniquely determined. Take again the D-norm $\|\cdot\|_\infty$, which is generated by the constant rv $\boldsymbol{Z} = (1, \dots, 1)$. But $\|\cdot\|_\infty$ is generated by *any* rv $(\xi, \dots, \xi)$, where $\xi \ge 0$ a.s. is a random variable with $E(\xi) = 1$:

$$E\left(\max_{1\le i\le d} (|x_i|\,\xi) \right) = \left(\max_{1\le i\le d} |x_i| \right) E(\xi) = \|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

Note that the rv $(\xi, \dots, \xi)$ does not necessarily satisfy $\|(\xi, \dots, \xi)\| = \text{const}$ a.s. nor is it necessarily bounded. We, therefore, investigate in Section 2 the set of generators in more detail and extend it to their maximum size.

Based on the componentwise multiplication of their generators, we introduce in Section 3 a multiplication type operation on the set of D-norms. This leads to *idempotent* D-norms, which are characterized in Section 4. Iterating the multiplication provides a *track* of D-norms. We will establish in Section 5 the fact that the limit of a D-norm track is an idempotent D-norm, if the multiplication is repeatedly done with the same D-norm.

The D-norm approach can be extended to functional extreme value theory, see Aulbach et al. (2012). In the present paper, however, we restrict ourself to the finite dimensional space.

2. The Set of Generators

In this section we drop the boundedness condition of a generator $\boldsymbol{Z}$ and, thus, maximally extend the set of generators. In a preparatory step we drop condition (1) on $\boldsymbol{Z}$.

Lemma 2.1. *Let the rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ satisfy $0 \le Z_i \le c$ a.s., $E(Z_i) = 1$, $1 \le i \le d$, for some constant $c \ge 1$. Then $\boldsymbol{Z}$ is the generator of a D-norm $\|\cdot\|_D$, i.e.,*

$$G(\boldsymbol{x}) := \exp\left(-E\left(\|\boldsymbol{x}\boldsymbol{Z}\|_\infty\right)\right) = \exp\left(-\|\boldsymbol{x}\|_D\right), \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d,$$

is a sms df.

Proof. Choose a constant $K < 0$ and a rv U that is on $(0,1)$ uniformly distributed and independent of $\boldsymbol{Z}$. Set

$$\boldsymbol{V} := \left(\max\left(K, -\frac{U}{Z_1}\right), \dots, \max\left(K, -\frac{U}{Z_d}\right)\right).$$

The constant K avoids division by zero. We obtain for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$ and $n \in \mathbb{N}$ large enough

$$\begin{aligned} P\left(\boldsymbol{V} \le \frac{\boldsymbol{x}}{n}\right) &= P\left(\max\left(K, -\frac{U}{Z_i}\right) \le \frac{x_i}{n},\, 1 \le i \le d\right) \\ &= P\left(U \ge \frac{|x_i|}{n} Z_i,\, 1 \le i \le d\right) \\ &= P\left(U \ge \frac{1}{n} \max_{1\le i\le d}\left(|x_i|\, Z_i\right)\right) \\ &= 1 - \frac{1}{n} E\left(\max_{1\le i\le d}\left(|x_i|\, Z_i\right)\right) \end{aligned}$$

by Fubini's theorem. Let now $\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}, \dots$ be independent copies of $\boldsymbol{V}$. Then we obtain for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$ and $n \in \mathbb{N}$ large

$$\begin{aligned} P\left(n \max_{1\le i\le n} \boldsymbol{V}^{(i)} \le \boldsymbol{x}\right) &= P\left(\boldsymbol{V} \le \frac{\boldsymbol{x}}{n}\right)^n \\ &= \left(1 - \frac{1}{n} E\left(\max_{1\le i\le d}\left(|x_i|\, Z_i\right)\right)\right)^n \\ &\to_{n\to\infty} \exp\left(-E\left(\max_{1\le i\le d}\left(|x_i|\, Z_i\right)\right)\right) \\ &=: G(\boldsymbol{x}). \end{aligned}$$

As G is continuous and is the pointwise limit of a sequence of df, with $G(\mathbf{0}) = 1$, $\lim_{\boldsymbol{x}\to-\boldsymbol{\infty}} G(\boldsymbol{x}) = 0$, G is a df. Its max-stability $G(\boldsymbol{x}/n)^n = G(\boldsymbol{x})$ is obvious. □

In the next auxiliary result we drop the boundedness condition of a generator.

Lemma 2.2. *Let the rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ satisfy $Z_i \ge 0$ a.s., $E(Z_i) = 1$, $1 \le i \le d$. Then $\boldsymbol{Z}$ is the generator of a D-norm $\|\cdot\|_D$, i.e.*

$$G(\boldsymbol{x}) = \exp\left(-E\left(\|\boldsymbol{x}\boldsymbol{Z}\|_\infty\right)\right) =: \exp\left(-\|\boldsymbol{x}\|_D\right), \qquad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d,$$

is a sms df.

Proof. Choose a constant $c > 0$ large enough such that $\tilde{Z}_i^{(c)} := \min(c, Z_i)$ satisfies $\mu_i^{(c)} := E\left(\tilde{Z}_i^{(c)}\right) > 0$, $1 \le i \le d$. Note that by the monotone convergence theorem $\mu_i^{(c)} \uparrow 1$ as $c \uparrow \infty$ for $1 \le i \le d$. The rv $\boldsymbol{Z}^{(c)} := (Z_1^{(c)}, \dots, Z_d^{(c)})$ with $Z_i^{(c)} := \tilde{Z}_i^{(c)}/\mu_i^{(c)}$, $1 \le i \le d$, is by Lemma 2.1 the generator of the sms df $G^{(c)}(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_{D^{(c)}}\right)$, $\boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$. As

$$\|\boldsymbol{x}\|_{D^{(c)}} = E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(c)}\right\|_\infty\right) \to_{c\to\infty} E\left(\|\boldsymbol{x}\boldsymbol{Z}\|_\infty\right) =: \|\boldsymbol{x}\|_D, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

by the dominated convergence theorem, we obtain

$$G^{(c)}(\boldsymbol{x}) \to_{c\to\infty} G(\boldsymbol{x}) := \exp\left(-\|\boldsymbol{x}\|_D\right), \qquad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d,$$

and, thus, G is a sms df by repeating the arguments at the end of the proof of Lemma 2.1. □

The set of generators of a D-norm on $\mathbb{R}^d$ is by Lemma 2.2 given by

$$\mathcal{Z} := \left\{\boldsymbol{Z} = (Z_1, \dots, Z_d) \text{ with } Z_i \ge 0 \text{ a.s. and } E(Z_i) = 1,\ 1 \le i \le d\right\}.$$

Two generators $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}$ are said to be *equivalent*, if they generate the same D-norm, i.e., if

$$E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(1)}\right\|_\infty\right) = E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(2)}\right\|_\infty\right) = \|\boldsymbol{x}\|_D, \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

The set $\mathcal{Z}$ then divides into equivalence classes, denoted by $\mathcal{Z}_{\|\cdot\|_D}$, i.e., $\mathcal{Z}_{\|\cdot\|_D}$ is the set of those generators which generate the D-norm $\|\cdot\|_D$.

Remark 2.3. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^d$. By condition (1) on a generator, which uses the initial de Haan-Resnick-Pickands representation of a sms df (see, e.g. Falk et al. (2010, Sections 4.2, 4.3)), each equivalence class $\mathcal{Z}_{\|\cdot\|_D}$ contains a generator $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with the additional property $\|\boldsymbol{Z}\| = \text{const}$ a.s.

If we choose in particular $\|\cdot\| = \|\cdot\|_1$, then $\|\boldsymbol{Z}\| = \sum_{i=1}^d Z_i = \text{const}$ a.s., which, together with $E\left(\sum_{i=1}^d Z_i\right) = d$ implies const $= d$. As a consequence we, thus, obtain in particular that each D-norm has a generator $\boldsymbol{Z}$ with the additional property $\sum_{i=1}^d Z_i = d$. This will in particular be useful in the derivation of Proposition 4.2.

Remark 2.4. The set of D-norms is closely related to the set of copulas. Let the rv $\boldsymbol{U} = (U_1, \dots, U_d)$ follow an arbitrary copula C on $\mathbb{R}^d$, i.e., each component U_i is on $(0,1)$ uniformly distributed. Then

$$\boldsymbol{Z} := 2\boldsymbol{U}$$

is, obviously, the generator of a D-norm. Note, however, that not each D-norm can be generated this way. Take, for example, the bivariate independence D-norm $\|(x,y)\|_1 = |x| + |y|$ and suppose that there exists a rv (U_1, U_2) following a copula such that

$$\|(x,y)\|_1 = 2E\left(\max\left(|x|\,U_1, |y|\,U_2\right)\right), \qquad (x,y) \in \mathbb{R}^2.$$

Choose $x = y = 1$. From the general equation

$$\max(a,b) = \frac{a+b}{2} + \frac{|a-b|}{2}, \qquad a, b \in \mathbb{R}, \tag{2}$$

we obtain

$$\begin{aligned} 2 = 2E\left(\frac{U_1+U_2}{2} + \frac{|U_1-U_2|}{2}\right) &= 1 + E\left(|U_1 - U_2|\right) \\ \iff E\left(|U_1-U_2|\right) &= 1 \\ \iff |U_1 - U_2| &= 1 \quad \text{a.s.} \end{aligned}$$

But as U_1, U_2 realize in $(0,1)$ a.s., we have $|U_1 - U_2| < 1$ a.s. and, thus, a contradiction. The bivariate D-norm $\|\cdot\|_1$, therefore, cannot be generated by $2(U_1, U_2)$. It is obvious that $\|\cdot\|_\infty$ on $\mathbb{R}^d$ with $d \ge 3$ cannot be generated by $2\boldsymbol{U}$, as $\|\boldsymbol{1}\|_1 = d > 2E\left(\|\boldsymbol{U}\|_\infty\right)$.

Each logistic norm $\|\boldsymbol{x}\|_\lambda = \left(\sum_{i=1}^d |x_i|^\lambda\right)^{1/\lambda}$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, $1 \le \lambda \le \infty$, is a D-norm. This is a consequence of the fact that $C(\boldsymbol{u}) = \exp\left(-\|\log(\boldsymbol{u})\|_\lambda\right)$, $\boldsymbol{u} \in (0,1]^d$, defines a copula on $\mathbb{R}^d$, called *Gumbel-Hougaard copula*, see, e.g., Nelsen (2006, Example 4.25), which is the copula of $G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_\lambda\right)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$. Generators of the extreme cases $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are given in Section 1. Explicit generators of the general D-norm $\|\cdot\|_\lambda$, $\lambda \in (0,\infty)$, however, seem to be an open problem.

3. Multiplication of D-Norms

Our approach towards sms df enables the following multiplication-type operation on D-norms. Choose $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)} \in \mathcal{Z}$ with corresponding D-norms $\|\cdot\|_{D^{(1)}}$, $\|\cdot\|_{D^{(2)}}$ and suppose that $\boldsymbol{Z}^{(1)}$, $\boldsymbol{Z}^{(2)}$ are independent. Then

$$\boldsymbol{Z} := \boldsymbol{Z}^{(1)}\boldsymbol{Z}^{(2)}$$

is by Lemma 2.2 again a generator of a D-norm, which we denote by $\|\cdot\|_{D^{(1)}D^{(2)}}$. Recall that all operations on vectors, such as the above multiplication, is meant componentwise. Clearly, the multiplication is commutative $\|\cdot\|_{D^{(1)}D^{(2)}} = \|\cdot\|_{D^{(2)}D^{(1)}}$. The corresponding operation on a sms rv $\boldsymbol{\eta}$ is described in Remark 1.2.

Let, for instance, $\boldsymbol{Z}^{(2)}$ be a generator of the D-norm $\|\cdot\|_\infty$. Then we obtain by conditioning on $\boldsymbol{Z}^{(1)}$

$$\begin{aligned}
\|\boldsymbol{x}\|_{D^{(1)}D^{(2)}} &= E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(1)}\boldsymbol{Z}^{(2)}\right\|_\infty\right)\\
&= \int E\left(\left\|\boldsymbol{x}\boldsymbol{z}^{(1)}\boldsymbol{Z}^{(2)}\right\|_\infty \mid \boldsymbol{Z}^{(1)} = \boldsymbol{z}^{(1)}\right)\left(P * \boldsymbol{Z}^{(1)}\right)\left(d\boldsymbol{z}^{(1)}\right)\\
&= \int E\left(\left\|\boldsymbol{x}\boldsymbol{z}^{(1)}\boldsymbol{Z}^{(2)}\right\|_\infty\right)\left(P * \boldsymbol{Z}^{(1)}\right)\left(d\boldsymbol{z}^{(1)}\right)\\
&= \int \left\|\boldsymbol{x}\boldsymbol{z}^{(1)}\right\|_\infty \left(P * \boldsymbol{Z}^{(1)}\right)\left(d\boldsymbol{z}^{(1)}\right)\\
&= E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(1)}\right\|_\infty\right)\\
&= \|\boldsymbol{x}\|_{D^{(1)}}, \qquad \boldsymbol{x} \in \mathbb{R}^d, \qquad (3)
\end{aligned}$$

i.e., $\|\cdot\|_{D^{(1)}D^{(2)}} = \|\cdot\|_{D^{(1)}}$. The sup-norm $\|\cdot\|_\infty$ is, therefore, the identity element within the set of D-norms, equipped with the above multiplication. There is, clearly, no other D-norm with this property.

When applied to the representation of an arbitrary sms rv $\boldsymbol{\eta}$ in Remark 1.2, this implies that multiplication with an independent rv $\xi \geq 0$, $E(\xi) = 1$, does not alter its distribution:

$$\boldsymbol{\eta} =_D -\frac{1}{\sup_{i\in\mathbb{N}} V_i \boldsymbol{Z}^{(i)}} =_D -\frac{1}{\sup_{i\in\mathbb{N}} V_i \xi^{(i)} \boldsymbol{Z}^{(i)}},$$

where $\xi^{(i)}$, $i \in \mathbb{N}$, are independent copies of ξ, also independent of $\boldsymbol{Z}^{(i)}$, $i \in \mathbb{N}$, and the Poisson process $\{V_i : i \in \mathbb{N}\}$.

Take, on the other hand, as $\boldsymbol{Z}^{(2)}$ a generator of the D-norm $\|\cdot\|_1$. Then we obtain

$$\begin{aligned}
\|\boldsymbol{x}\|_{D^{(1)}D^{(2)}} &= E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(1)}\boldsymbol{Z}^{(2)}\right\|_\infty\right)\\
&= \int E\left(\left\|\boldsymbol{x}\boldsymbol{z}^{(1)}\boldsymbol{Z}^{(2)}\right\|_\infty\right)\left(P * \boldsymbol{Z}^{(1)}\right)\left(d\boldsymbol{z}^{(1)}\right)\\
&= \int \sum_{i=1}^d |x_i|\, z_i^{(1)} \left(P * \boldsymbol{Z}^{(1)}\right)\left(d\boldsymbol{z}^{(1)}\right)\\
&= \sum_{i=1}^d |x_i|\, E\left(Z_i^{(1)}\right)
\end{aligned}$$

$$= \sum_{i=1}^{d} |x_i|, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

i.e., $\|\cdot\|_{D^{(1)}D^{(2)}} = \|\cdot\|_1$. Multiplication with the independence norm $\|\cdot\|_1$ yields the independence norm and thus, $\|\cdot\|_1$ can be viewed as the maximal attractor among the set of D-norms. There is, clearly, no other D-norm with this property.

Applied to the representation of an arbitrary sms rv, this implies that

$$-\frac{1}{\sup_{i\in\mathbb{N}} V_i \boldsymbol{Z}^{(i)} \tilde{\boldsymbol{Z}}^{(i)}} =_D \boldsymbol{\eta},$$

where $\boldsymbol{\eta}$ is a sms rv with independent components, if $\tilde{\boldsymbol{Z}}^{(i)}$, $i \in \mathbb{N}$, are independent copies of a generator of $\|\cdot\|_1$, also independent of $\boldsymbol{Z}^{(i)}$, $i \in \mathbb{N}$, and the Poisson process $\{V_i : i \in \mathbb{N}\}$.

4. Idempotent D-Norms

The maximum-norm $\|\cdot\|_\infty$ and the L_1-norm $\|\cdot\|_1$ both satisfy

$$\|\cdot\|_{D^2} := \|\cdot\|_{DD} = \|\cdot\|_D .$$

Such a D-norm will be called *idempotent*. The problem suggests itself to characterize the set of idempotent D-norms. This will be achieved in the present section. It turns out that in the bivariate case $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are the only idempotent D-norms, whereas in higher dimensions each idempotent D-norm is a certain combination of $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

Speaking in terms of rv, we will characterize in this section the set of generators $\boldsymbol{Z}$ such that

$$\boldsymbol{\eta} = -\frac{1}{\sup_{i\in\mathbb{N}} V_i \boldsymbol{Z}^{(i)}} =_D -\frac{1}{\sup_{i\in\mathbb{N}} V_i \boldsymbol{Z}^{(i)} \tilde{\boldsymbol{Z}}^{(i)}},$$

where $\boldsymbol{Z}^{(i)}$, $\tilde{\boldsymbol{Z}}^{(i)}$, $i \in \mathbb{N}$, are independent copies of $\boldsymbol{Z}$, also independent of the Poisson process $\{V_i : i \in \mathbb{N}\}$ on $[0, \infty)$, with intensity measure $r^{-2}dr$, see Remark 1.2.

The following auxiliary result will be crucial for the characterization of idempotent D-norms.

Lemma 4.1. *Let X be a rv that a.s. attains only values in $[-c, c]$ for some $c > 0$ and $E(X) = 0$. Let Y be an independent copy of X. If*

$$E(|X + Y|) = E(|X|),$$

then either $X = 0$ or $X \in \{-m, m\}$ a.s. with $P(X = -m) = P(X = m) = 1/2$ for some $m \in (0, c]$. The reverse implication is true as well.

Proof. Suppose that $P(X=-m)=P(X=m)=1/2$ for some $m\in(0,c]$. Then, obviously,

$$E(|X|)=m=E(|X+Y|).$$

Next we establish the reverse implication. Suppose that X is not a.s the constant zero. Denote by F the df of X. Without loss of generality we can assume the representation $X=F^{-1}(U_1)$, $Y=F^{-1}(U_2)$, where U_1,U_2 are independent, on $(0,1)$ uniformly distributed rv and $F^{-1}(q):=\inf\{t\in\mathbb{R}:\, F(t)\ge q\}$, $q\in(0,1)$, is the generalized inverse of F. The well known equivalence

$$F^{-1}(q)\le t \iff q\le F(t),\qquad q\in(0,1),\, t\in\mathbb{R},$$

(see, e.g. Reiss (1989, equation (1.2.9))) together with Fubini's theorem implies

$$\begin{aligned}
&E(|X+Y|)\\
&=E\left(\left|F^{-1}(U_1)+F^{-1}(U_2)\right|\right)\\
&=\int_0^1\int_0^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv\\
&=-\int_0^{F(0)}\int_0^{F(0)} F^{-1}(u)+F^{-1}(v)\,du\,dv+\int_{F(0)}^1\int_{F(0)}^1 F^{-1}(u)+F^{-1}(v)\,du\,dv\\
&\qquad+2\int_0^{F(0)}\int_{F(0)}^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv\\
&=-\int_0^{F(0)}\left(F(0)F^{-1}(v)+\int_0^{F(0)}F^{-1}(u)\,du\right)dv\\
&\qquad+\int_{F(0)}^1\left((1-F(0))\,F^{-1}(v)+\int_{F(0)}^1 F^{-1}(u)\,du\right)dv\\
&\qquad+2\int_0^{F(0)}\int_{F(0)}^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv\\
&=-2F(0)\int_0^{F(0)}F^{-1}(v)\,dv+2(1-F(0))\int_{F(0)}^1 F^{-1}(v)\,dv\\
&\qquad+2\int_0^{F(0)}\int_{F(0)}^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv
\end{aligned}$$

and

$$E(|X|)=-\int_0^{F(0)}F^{-1}(u)\,du+\int_{F(0)}^1 F^{-1}(u)\,du.$$

From the assumption $E(|X+Y|)=E(|X|)$ we, thus, obtain the equation

$$0=(1-2F(0))\int_0^{F(0)}F^{-1}(v)\,dv+(1-2F(0))\int_{F(0)}^1 F^{-1}(v)\,dv$$

$$+2\int_0^{F(0)}\int_{F(0)}^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv$$

or

$$0=(1-2F(0))\int_0^1 F^{-1}(v)\,dv+2\int_0^{F(0)}\int_{F(0)}^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv.$$

The assumption $0=E(X)=\int_0^1 F^{-1}(v)\,dv$ now yields

$$\int_0^{F(0)}\int_{F(0)}^1 \left|F^{-1}(u)+F^{-1}(v)\right|\,du\,dv=0$$

and, thus,

$$F^{-1}(u)+F^{-1}(v)=0 \qquad \text{for } \lambda\text{−a.e. } (u,v)\in[0,F(0)]\times[F(0),1], \tag{4}$$

where λ denotes the Lebesgue-measure on $[0,1]$.

If $F(0)=0$, then $P(X>0)=1$ and, thus, $E(X)>0$, which would be a contradiction. If $F(0)=1$, then $P(X<0)>0$ unless $P(X=0)=1$, which we have excluded, and, thus, $E(X)<0$, which would again be a contradiction. We, consequently, have established $0<F(0)<1$.

As the function $F^{-1}(q)$, $q\in(0,1)$, is in general continuous from the left (see, e.g., Reiss (1989, Lemma A.1.2)), equation (4) implies that $F^{-1}(v)$ is a constant function on $(0,F(0)]$ and on $(F(0),1)$, precisely,

$$F^{-1}(v)=\begin{cases}-m, & v\in(0,F(0)],\\ m, & v\in(F(0),1),\end{cases}$$

for some $m\in(0,c]$. Note that the representation $X=F^{-1}(U_1)$ together with the assumption that X is not a.s. the constant zero, implies $m\neq 0$. The condition

$$0=E(X)=\int_0^{F(0)} F^{-1}(v)\,dv+\int_{F(0)}^1 F^{-1}(v)\,dv=m(1-2F(0))$$

implies $F(0)=1/2$ and, thus,

$$X=F^{-1}(U_1)=\begin{cases}m, & U_1>\frac{1}{2},\\ -m, & U_1\le\frac{1}{2},\end{cases}$$

which is the assertion. □

The next Proposition is the first main result of this section.

Proposition 4.2. *A bivariate D-norm* $\|\cdot\|_D$ *is idempotent* $\Leftrightarrow \|\cdot\|_D\in\{\|\cdot\|_1,\|\cdot\|_\infty\}$.

Proof. It suffices to establish the implication

$$\|\cdot\|_{D^2} = \|\cdot\|_D\,,\ \|\cdot\|_D \neq \|\cdot\|_\infty \implies \|\cdot\|_D = \|\cdot\|_1\,.$$

Let $\boldsymbol{Z}^{(1)} = \left(Z_1^{(1)}, Z_2^{(1)}\right)$, $\boldsymbol{Z}^{(2)} = \left(Z_1^{(2)}, Z_2^{(2)}\right)$ be independent and identically distributed generators of $\|\cdot\|_D$. According to Remark 2.3 we can assume that $Z_1^{(1)} + Z_2^{(1)} = 2 = Z_1^{(2)} + Z_2^{(2)}$. Put $X := Z_1^{(1)} - 1$, $Y := Z_1^{(2)} - 1$. Then X, Y are independent and identically distributed with $X \in [-1,1]$, $E(X) = 0$. From equation (2) we obtain the representation

$$\begin{aligned}
&E\left(\max\left(Z_1^{(1)}Z_1^{(2)}, Z_2^{(1)}Z_2^{(2)}\right)\right)\\
&= E\left(\frac{Z_1^{(1)}Z_1^{(2)}}{2} + \frac{Z_2^{(1)}Z_2^{(2)}}{2}\right) + \frac{1}{2}E\left(\left|Z_1^{(1)}Z_1^{(2)} - Z_2^{(1)}Z_2^{(2)}\right|\right)\\
&= 1 + E\left(\left|Z_1^{(1)} - 1 + Z_1^{(2)} - 1\right|\right)\\
&= 1 + E(|X+Y|)
\end{aligned}$$

as well as

$$E\left(\max\left(Z_1^{(1)}, Z_2^{(2)}\right)\right) = 1 + E(|X|).$$

Lemma 4.1 now implies that $P(X = m) = P(X = -m) = 1/2$ for some $m \in (0,1]$. It remains to show that $m = 1$.

Set $x = 1$ and $y = a$, where $0 < a < 1$ satisfies $a(1+m) > 1-m$. Then $a(1+m)^2 > (1-m)^2$ as well, and we obtain by equation 2

$$\begin{aligned}
\|(x,y)\|_{D^2} &= E\left(\max\left(Z_1^{(1)}Z_1^{(2)}, a\left(2 - Z_1^{(1)}\right)\left(2 - Z_1^{(2)}\right)\right)\right)\\
&= \frac{1}{4}\max\left((1-m)^2, a(1+m)^2\right) + \frac{1}{4}\max\left((1+m)^2, a(1-m)^2\right)\\
&\qquad + \frac{1}{2}\max\left(1-m^2, a(1-m^2)\right)\\
&= \frac{1}{4}a(1+m)^2 + \frac{1}{4}(1+m)^2 + \frac{1}{2}(1-m^2)\\
&= \frac{1}{4}(1+m)^2(1+a) + \frac{1}{2}(1-m^2)
\end{aligned}$$

and

$$\begin{aligned}
\|(x,y)\|_D &= E\left(\max\left(Z_1^{(1)}, a\left(2 - Z_1^{(1)}\right)\right)\right)\\
&= \frac{1}{2}\max(1+m, a(1-m)) + \frac{1}{2}\max(1-m, a(1+m))\\
&= \frac{1}{2}(1+m) + \frac{1}{2}a(1+m)\\
&= \frac{1}{2}(1+m)(1+a).
\end{aligned}$$

From the equality $\|(x,y)\|_{D^2} = \|(x,y)\|_D$ and the fact that $1+m>0$ we, thus, obtain

$$\begin{aligned} &\frac{1}{4}(1+m)(1+a)+\frac{1}{2}(1-m)=\frac{1}{2}(1+a)\\ \iff\ &(m-1)(a-1)=0\\ \iff\ &m=1, \end{aligned}$$

which completes the proof. □

Next we will extend Proposition 4.2 to arbitrary dimension $d\ge 2$. Denote by $\boldsymbol{e}_i := (0,\dots,0,1,0,\dots,0)\in\mathbb{R}^d$ the i-th unit vector in $\mathbb{R}^d$, $1\le i\le d$, and let $\|\cdot\|_D$ be an arbitrary D-norm on $\mathbb{R}^d$. Then

$$\|(x,y)\|_{D_{i,j}} := \|x\boldsymbol{e}_i + y\boldsymbol{e}_j\|_D, \qquad (x,y)\in\mathbb{R}^2,\ 1\le i<j\le d,$$

defines a D-norm on $\mathbb{R}^2$, called *bivariate projection* of $\|\cdot\|_D$. If $\boldsymbol{Z}=(Z_1,\dots,Z_d)$ is a generator of $\|\cdot\|_D$, then (Z_i,Z_j) generates $\|\cdot\|_{D_{i,j}}$.

Proposition 4.3. *Let $\|\cdot\|_D$ be a D-norm on $\mathbb{R}^d$ such that each bivariate projection $\|\cdot\|_{D_{i,j}}$ is different from the bivariate sup-norm $\|\cdot\|_\infty$. Then $\|\cdot\|_D$ is idempotent $\Leftrightarrow$ $\|\cdot\|_D=\|\cdot\|_1$.*

Proof. If $\|\cdot\|_D$ is idempotent, then each bivariate projection is an idempotent D-norm on $\mathbb{R}^2$ and, thus, each bivariate projection is by Proposition 4.2 necessarily the bivariate L_1-norm $\|\cdot\|_1$. This implies bivariate independence of the margins of the sms df $G(\boldsymbol{x})=\exp\left(-\|\boldsymbol{x}\|_D\right)$, $\boldsymbol{x}\le\boldsymbol{0}\in\mathbb{R}^d$. It is well-known that bivariate independence of the margins of G implies complete independence (see, e.g., Falk et al. (2010, Theorem 4.3.3)) and, thus, $\|\cdot\|_D=\|\cdot\|_1$ on $\mathbb{R}^d$. □

If we allow bivariate complete dependence, then we obtain the complete class of idempotent D-norms on $\mathbb{R}^d$ as mixtures of lower-dimensional $\|\cdot\|_\infty$- and $\|\cdot\|_1$-norms. To this end we will first introduce the complete dependence frame of a D-norm.

Let D be an arbitrary D-norm on $\mathbb{R}^d$ such that at least one bivariate projection $\|\cdot\|_{D_{i,j}}$ equals $\|\cdot\|_\infty$ on $\mathbb{R}^2$. Then there exist nonempty disjoint subsets $A_1,\dots,A_K$ of $\{1,\dots,d\}$, $1\le K<d$, $|A_k|\ge 2$, $1\le k\le K$, such that

$$\left\|\sum_{i\in A_k} x_i\boldsymbol{e}_i\right\|_D = \max_{i\in A_k}|x_i|, \qquad \boldsymbol{x}\in\mathbb{R}^d,\ 1\le k\le K,$$

and no other projection $\left\|\sum_{i\in B} x_i\boldsymbol{e}_i\right\|_D$, $B\subset\{1,\dots,d\}$, $|B|\ge 2$, $B\ne A_k$, $1\le k\le K$, is the sup-norm $\|\cdot\|_\infty$ on $\mathbb{R}^{|B|}$. We call $A_1,\dots,A_K$ the *complete dependence frame* (CDF) of $\|\cdot\|_D$. If there is no completely dependent bivariate projection of $\|\cdot\|_D$, then we say that its CDF is *empty*.

To illustrate the significance of $A_1,\dots,A_K$, take a sms rv $\boldsymbol{\eta}=(\eta_1,\dots,\eta_d)$ with df $G(\boldsymbol{x})=\exp\left(-\left\|\boldsymbol{x}\right\|_D\right)$, $\boldsymbol{x}\le\mathbf{0}\in\mathbb{R}^d$. Then the sets $A_1,\dots,A_K$ assemble the indices of completely dependent components $\eta_i=\eta_j$ a.s., $i,j\in A_k$, and the sets A_k are maximally chosen, i.e., we *do not* have $\eta_i=\eta_j$ a.s. if $i\in A_k$ for some $j\in A_k^{\complement}$.

The next result characterizes the set of idempotent D-norms with at least one completely dependent bivariate projections.

Theorem 4.4. *Let $\left\|\cdot\right\|_D$ be an idempotent D-norm with non empty CDF $A_1,\dots,A_K$. Then we have*

$$\left\|\boldsymbol{x}\right\|_D=\sum_{k=1}^K\max_{i\in A_k}\left|x_i\right|+\sum_{i\in\{1,\dots,d\}\setminus\cup_{k=1}^d A_k}\left|x_i\right|,\qquad \boldsymbol{x}\in\mathbb{R}^d.$$

On the other hand, the above equation defines for each set of nonempty disjoint subsets $A_1,\dots,A_K$ of $\{1,\dots,d\}$ with $|A_k|\ge 2$, $1\le k\le K<d$, an idempotent D-norm on $\mathbb{R}^d$ with CDF $A_1,\dots,A_K$.

Proof. Let $\boldsymbol{\eta}=(\eta_1,\dots,\eta_d)$ be a sms rv with df $G(\boldsymbol{x})=\exp\left(-\left\|\boldsymbol{x}\right\|_D\right)$, $\boldsymbol{x}\le\mathbf{0}\in\mathbb{R}^d$. Then we have for $\boldsymbol{x}\le\mathbf{0}\in\mathbb{R}^d$

$$\begin{aligned}G(\boldsymbol{x})&=\exp\left(-\left\|\boldsymbol{x}\right\|_D\right)\\&=P(\eta_i\le x_i,\,1\le i\le d)\\&=P\left(\eta_{k^*}\le\min_{i\in A_k}x_i,\,1\le k\le K;\,\eta_j\le x_j,\,j\in\left(\cup_{k=1}^K A_k\right)^{\complement}\right),\end{aligned}$$

where $k^*\in A_k$ is for each $k\in\{1,\dots,K\}$ an arbitrary but fixed element of A_k. The rv $\boldsymbol{\eta}^*$ with joint components η_{k^*}, $1\le k\le K$, and η_j, $j\in\left(\cup_{k=1}^K A_k\right)^{\complement}$, is a sms rv of dimension less than d, and $\boldsymbol{\eta}^*$ has no pair of completely dependent components. The rv $\boldsymbol{\eta}^*$ might be viewed as the rv $\boldsymbol{\eta}$ after the completely dependent components have been removed. Its corresponding D-norm is, of course, still idempotent. From Proposition 4.3 we obtain its df, i.e.,

$$\begin{aligned}G(\boldsymbol{x})&=\exp\left(-\sum_{k=1}^K\left|\min_{i\in A_k}x_i\right|-\sum_{j\in\left(\cup_{k=1}^K A_k\right)^{\complement}}\left|x_j\right|\right)\\&=\exp\left(-\sum_{k=1}^K\max_{i\in A_k}\left|x_i\right|-\sum_{j\in\left(\cup_{k=1}^K A_k\right)^{\complement}}\left|x_j\right|\right),\qquad \boldsymbol{x}\le\mathbf{0}\in\mathbb{R}^d,\end{aligned}$$

which is the first part of the assertion.

Take, on the other hand, a rv U that is on the set of integers $\{k^*:\,1\le k\le K\}\cup\left(\cup_{k=1}^K A_k\right)^{\complement}$ uniformly distributed. Put $m:=K+\left|\left(\cup_{k=1}^K A_k\right)^{\complement}\right|$ and set for $i=$

$1,\dots,d$

$$Z_i := \begin{cases} m, & i \in A_k, \\ 0 & \text{otherwise}, \end{cases}$$

if $U = k^*$, $1 \le k \le K$, and

$$Z_i := \begin{cases} m, & i = j, \\ 0 & \text{otherwise}, \end{cases}$$

if $U = j \in \left(\cup_{k=1}^K A_k\right)^\complement$. Then $E(Z_i) = 1$, $1 \le i \le d$, and

$$\begin{aligned} &E\left(\max_{1\le i\le d}(|x_i|\, Z_i)\right) \\ &= \sum_{j\in\{k^*:\,1\le k\le K\}\cup\left(\cup_{k=1}^K A_k\right)^\complement} E\left(\max_{1\le i\le d}(|x_i|\, Z_i)\, 1(U=j)\right) \\ &= \sum_{k=1}^K \max_{i\in A_k} |x_i| + \sum_{j\in\left(\cup_{k=1}^K A_k\right)^\complement} |x_j|\,, \qquad \boldsymbol{x} \in \mathbb{R}^d. \end{aligned}$$

It is easy to see that this D-norm is idempotent, which completes the proof. □

The set of all idempotent trivariate D-norms is, for example, given by

$$\|(x,y,z)\|_D = \begin{cases} \max(|x|\,,|y|\,,|z|) \\ \max(|x|\,,|y|) + |z| \\ \max(|x|\,,|z|) + |y| \\ \max(|y|\,,|z|) + |x| \\ |x| + |y| + |z| \end{cases},$$

where the three mixed versions are just permutations of the arguments and might be views as equivalent.

5. Tracks of D-Norms

The multiplication of D-norms $D^{(1)}, D^{(2)}, \dots$ on $\mathbb{R}^d$ can obviously be iterated:

$$\|\cdot\|_{\prod_{i=1}^{n+1} D^{(i)}} := \|\cdot\|_{D^{(n+1)}\prod_{i=1}^{n} D^{(i)}}\,, \qquad n \in \mathbb{N}.$$

This operation is commutative as well. In this section we investigate such *D-norm tracks* $\|\cdot\|_{\prod_{i=1}^{n} D^{(i)}}$, $n \in \mathbb{N}$. We will in particular show that each track converges to an idempotent D-norm if $\|\cdot\|_{D^{(i)}} = \|\cdot\|_D$, $i \in \mathbb{N}$, for an arbitrary D-norm D on $\mathbb{R}^d$.

We start by establishing several auxiliary results. The first one indicates in particular that multiplication of D-norms decreases the dependence among the components of the corresponding sms rv.

Lemma 5.1. *We have for arbitrary D-norms $\|\cdot\|_{D^{(1)}}$, $\|\cdot\|_{D^{(2)}}$ on $\mathbb{R}^d$*

$$\|\cdot\|_{D^{(1)}D^{(2)}} \geq \max\left(\|\cdot\|_{D^{(1)}}, \|\cdot\|_{D^{(2)}}\right).$$

Proof. Let $\boldsymbol{Z}^{(1)}$, $\boldsymbol{Z}^{(2)}$ be independent generators of $\|\cdot\|_{D^{(1)}}$, $\|\cdot\|_{D^{(2)}}$. We have for $\boldsymbol{x} \in \mathbb{R}^d$ by conditioning on $\boldsymbol{Z}^{(2)}$ as in equation (3)

$$\|\boldsymbol{x}\|_{D^{(1)}D^{(2)}} = E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(1)}\boldsymbol{Z}^{(2)}\right\|_\infty\right) = E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(2)}\right\|_{D^{(1)}}\right). \tag{5}$$

Note that

$$\|\boldsymbol{x}\|_{D^{(1)}} = \left\|\boldsymbol{x}E\left(\boldsymbol{Z}^{(2)}\right)\right\|_{D^{(1)}} = \left\|E\left(\boldsymbol{x}\boldsymbol{Z}^{(2)}\right)\right\|_{D^{(1)}}. \tag{6}$$

Put

$$T(\boldsymbol{x}) := \|\boldsymbol{x}\|_{D^{(1)}}, \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

Then T is a convex function by the triangle inequality and the homogeneity satisfied by any norm. We, thus, obtain from Jensen's together with equations (5) and (6)

$$\begin{aligned}
\|\boldsymbol{x}\|_{D^{(1)}D^{(2)}} &= E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^{(2)}\right\|_{D^{(1)}}\right)\\
&= E\left(T\left(\boldsymbol{x}\boldsymbol{Z}^{(2)}\right)\right)\\
&\geq T\left(E\left(\boldsymbol{x}\boldsymbol{Z}^{(2)}\right)\right)\\
&= \left\|E\left(\boldsymbol{x}\boldsymbol{Z}^{(2)}\right)\right\|_{D^{(1)}}\\
&= \|\boldsymbol{x}\|_{D^{(1)}}.
\end{aligned}$$

Exchanging $\boldsymbol{Z}^{(1)}$ and $\boldsymbol{Z}^{(2)}$ completes the proof. □

Proposition 5.2. *Let $\|\cdot\|_{D^{(n)}}$, $n \in \mathbb{N}$, be a set of arbitrary D-norms on $\mathbb{R}^d$. Then the limit of the track*

$$\lim_{n\to\infty} \|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}} =: f(x)$$

exists for each $\boldsymbol{x} \in \mathbb{R}^d$ and is a D-norm, i.e., $f(\cdot) = \|\cdot\|_D$.

Proof. From Lemma 5.1 we know that for each $\boldsymbol{x} \in \mathbb{R}^d$ and each $n \in \mathbb{N}$

$$\|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}} \leq \|\boldsymbol{x}\|_{\prod_{i=1}^{n+1} D^{(i)}}.$$

As each D-norm is bounded by the L_1-norm, i.e., $\|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}} \leq \|\boldsymbol{x}\|_1$, the sequence $\|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}}$, $n \in \mathbb{N}$, is monotone increasing and bounded and, thus, the limit

$$\lim_{n\to\infty} \|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}} =: f(x)$$

exists in $[0, \infty)$. The triangle inequality and the homogeneity of $f(\cdot)$ are obvious. The monotonicity of the sequence $\lim_{n\to\infty} \|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}}$ implies that $f(\boldsymbol{x}) = 0 \iff \boldsymbol{x} = \boldsymbol{0}$ and, thus, $f(\cdot)$ is a norm on $\mathbb{R}^d$. The characterization of a D-norm as established by Hofmann (2009) (see Falk et al. (2010, Theorem 4.4.2)) implies that $f(\cdot)$ is a D-norm as well. □

If we set $D^{(n)}$ for each $n \in \mathbb{N}$ equal to a fixed but arbitrary D-norm, then the limit in Proposition 5.2 is an idempotent D-norm.

Theorem 5.3. *Let $\|\cdot\|_D$ be an arbitrary D-norm on $\mathbb{R}^d$. Then the limit*

$$\lim_{n\to\infty} \|\boldsymbol{x}\|_{\prod_{i=1}^n D^{(i)}} =: \|\boldsymbol{x}\|_{D^*}, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

is an idempotent D-norm on $\mathbb{R}^d$.

Proof. We know from Poposition 5.2 that $\|\cdot\|_{D^*}$ is a D-norm on $\mathbb{R}^d$. Let $\boldsymbol{Z}^*$ be a generator of this D-norm and let $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \dots$ be independent copies of the generator $\boldsymbol{Z}$ of $\|\cdot\|_D$, independent of $\boldsymbol{Z}^*$ as well. Then we have for each $\boldsymbol{x} \in \mathbb{R}^d$

$$\|\boldsymbol{x}\|_{D^n} = E\left(\left\|\boldsymbol{x}\prod_{i=1}^n \boldsymbol{Z}^{(i)}\right\|_\infty\right) \uparrow_{n\to\infty} \|\boldsymbol{x}\|_{D^*}$$

by Lemma 5.1, as well as for each $k \in \mathbb{N}$

$$\begin{aligned}
&\|\boldsymbol{x}\|_{D^n}\\
&= E\left(\left\|\boldsymbol{x}\prod_{i=1}^k \boldsymbol{Z}^{(i)} \prod_{j=k+1}^n \boldsymbol{Z}^{(j)}\right\|_\infty\right)\\
&= \int E\left(\left\|\boldsymbol{x}\prod_{i=1}^k \boldsymbol{z}^{(i)} \prod_{j=k+1}^n \boldsymbol{Z}^{(j)}\right\|_\infty\right) \left(P * \left(\boldsymbol{Z}^{(1)}, \dots, \boldsymbol{Z}^{(k)}\right)\right) \left(d\left(\boldsymbol{z}^{(1)}, \dots, \boldsymbol{z}^{(k)}\right)\right)\\
&\to_{n\to\infty} \int \left\|\boldsymbol{x}\prod_{i=1}^k \boldsymbol{z}^{(i)}\right\|_{D^*} \left(P * \left(\boldsymbol{Z}^{(1)}, \dots, \boldsymbol{Z}^{(k)}\right)\right) \left(d\left(\boldsymbol{z}^{(1)}, \dots, \boldsymbol{z}^{(k)}\right)\right)\\
&= E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^*\prod_{i=1}^k \boldsymbol{Z}^{(i)}\right\|_\infty\right)
\end{aligned}$$

by the monotone convergence theorem. We, thus, have

$$\|\boldsymbol{x}\|_{D^*} = E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^*\prod_{i=1}^k \boldsymbol{Z}^{(i)}\right\|_\infty\right)$$

for each $k \in \mathbb{N}$. By letting k tend to infinity and repeating the above arguments we obtain

$$\|\boldsymbol{x}\|_{D^*} = E\left(\left\|\boldsymbol{x}\boldsymbol{Z}^*\prod_{i=1}^k \boldsymbol{Z}^{(i)}\right\|_\infty\right) \uparrow_{k\to\infty} E\left(\|\boldsymbol{x}\boldsymbol{Z}^*\|_{D^*}\right) = \|\boldsymbol{x}\|_{D^*D^*},$$

which completes the proof. □

If the initial D-norm $\|\cdot\|_D$ has no complete dependence structure among its margins, i.e., if its CDF is empty, then the limiting D-norm in Theorem 5.3 is the L_1-norm. Otherwise, the limit has the same CDF as $\|\cdot\|_D$.

The limit of an arbitrary track $\|\cdot\|_{\prod_{i=1}^n D^{(i)}}$, $n \in \mathbb{N}$, is not necessarily idempotent. Take, for example, an arbitrary and non idempotent D-norm $\|\cdot\|_D^{(1)}$ and $\|\cdot\|_D^{(i)} = \|\cdot\|_\infty$, $i \geq 2$. But it is an open problem, whether the limit of a track is again idempotent if $\|\cdot\|_{D^{(i)}} \neq \|\cdot\|_\infty$ for infinitely many $i \in \mathbb{N}$.

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University of Würzburg, Institute of Mathematics, Emil-Fischer-Str. 30, 97074 Würzburg, Germany.

E-mail address: michael.falk@uni-wuerzburg.de