Online Learning in Markov Decision Processes with Adversarially Chosen Transition Probability Distributions

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Abstract

We study the problem of learning Markov decision processes with finite state and action spaces when the transition probability distributions and loss functions are chosen adversarially and are allowed to change with time. We introduce an algorithm whose regret with respect to any policy in a comparison class grows as the square root of the number of rounds of the game, provided the transition probabilities satisfy a uniform mixing condition. Our approach is efficient as long as the comparison class is polynomial and we can compute expectations over sample paths for each policy. Designing an efficient algorithm with small regret for the general case remains an open problem.

1 Notation

Let \mathcal{X} be a finite state space and \mathcal{A} be a finite action space. Let Δ_S be the space of probability distributions over set S. Define a policy π as a mapping from the state space to $\Delta_{\mathcal{A}}, \pi : \mathcal{X} \to \Delta_{\mathcal{A}}$. We use $\pi(a|x)$ to denote the probability of choosing action a in state x under policy π . A random action under policy π is denoted by $\pi(x)$. A transition probability kernel (or transition model) mis a mapping from the direct product of the state and action spaces to $\Delta_{\mathcal{X}}: m : \mathcal{X} \times \mathcal{A} \to \Delta_{\mathcal{X}}$. Let $P(\pi, m)$ be the transition probability matrix of policy π under transition model m. A loss function is a bounded real-valued function over state and action spaces, $\ell : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$. For a vector v, define $\|v\|_1 = \sum_i |v_i|$. For a real-valued function f defined over $\mathcal{X} \times \mathcal{A}$, define $\|f\|_{\infty,1} = \max_{x \in \mathcal{X}} \sum_{a \in \mathcal{A}} |f(x, a)|$. The inner product between two vectors v and w is denoted by $\langle v, w \rangle$.

2 Introduction

Consider the following game between a learner and an adversary: at round t, the learner chooses a policy π_t from a policy class Π . In response, the adversary chooses a transition model m_t from a set of models M and a loss function ℓ_t . The learner takes action $a_t \sim \pi_t(.|x_t)$, moves to state $x_{t+1} \sim m_t(.|x_t, a_t)$ and suffers loss $\ell_t(x_t, a_t)$. To simplify the discussion, we assume that the adversary is oblivious, i.e. its choices do not depend on the previous choices of the learner. We assume that $\ell_t \in [0, 1]$. In this paper, we study the full-information version of the game, where the learner observes the transition model m_t and the loss function ℓ_t at the end of round t. The game is shown in Figure 1. The objective of the learner is to suffer low loss over a period of T rounds, while the performance of the learner is measured using its regret with respect to the total loss he would have achieved had he followed the stationary policy in the comparison class Π minimizing the total loss.

Even-Dar et al. (2004) prove a hardness result for MDP problems with adversarially chosen transition models. Their proof, however, seems to have gaps as it assumes that the learner chooses a deterministic policy before observing the state at each round. Note that an online learning algorithm only needs to choose an action at the current state and does not need to construct a complete deterministic policy at each round. Their hardness result applies to deterministic transition models, while we make a mixing assumption in our analysis. Thus, it is still an open problem whether it is possible to obtain a computationally efficient algorithm with a sublinear regret.

Yu and Mannor (2009a,b) study the same setting, but obtain only a regret bound that scales with the amount of variation in the transition models. This regret bound can grow linearly with time. Initial state: x_0 for t := 1, 2, ... do Learner chooses policy π_t Adversary chooses model m_t and loss function ℓ_t Learner takes action $a_t \sim \pi_t(.|x_t)$ Learner suffers loss $\ell_t(x_t, a_t)$ Update state $x_{t+1} \sim m_t(.|x_t, a_t)$ Learner observes m_t and ℓ_t end for

Figure 1: Online Markov Decision Processes

Even-Dar et al. (2009) prove regret bounds for MDP problems with a fixed and known transition model and adversarially chosen loss functions. In this paper, we prove regret bounds for MDP problems with adversarially chosen transition models and loss functions. We are not aware of any earlier regret bound for this setting. Our approach is efficient as long as the comparison class is polynomial and we can compute expectations over sample paths for each policy.

MDPs with changing transition kernels are good models for a wide range of problems, including dialogue systems, clinical trials, portfolio optimization, two player games such as poker, etc.

3 Online MDP Problems

Let A be an online learning algorithm that generates a policy π_t at round t. Let x_t^A be the state at round t if we have followed the policies generated by algorithm A. Similarly, x_t^{π} denotes the state if we have chosen the same policy π up to time t. Let $\ell(x, \pi) = \ell(x, \pi(x))$. The regret of algorithm A up to round T with respect to any policy $\pi \in \Pi$ is defined by

$$R_T(A,\pi) = \sum_{t=1}^T \ell_t(x_t^A, a_t) - \sum_{t=1}^T \ell_t(x_t^\pi, \pi),$$

where $a_t = \pi_t(x_t^A)$. Note that the regret with respect to π is defined in terms of the sequence of states x_t^{π} that would have been visited under policy π . Our objective is to design an algorithm that achieves low regret with respect to any policy π .

In the absence of state variables, the problem reduces to a *full information online learning problem* (Cesa-Bianchi and Lugosi, 2006). The difficulty with MDP problems is that, unlike the full information online learning problems, the choice of policy at each round changes the future states and losses. The main idea behind the design and the analysis of our algorithm is the following regret decomposition:

$$R_T(A,\pi) = \sum_{t=1}^T \ell_t(x_t^A, a_t) - \sum_{t=1}^T \ell_t(x_t^{\pi_t}, \pi_t) + \sum_{t=1}^T \ell_t(x_t^{\pi_t}, \pi_t) - \sum_{t=1}^T \ell_t(x_t^{\pi}, \pi) .$$
(1)

Let

$$B_T(A) = \sum_{t=1}^T \ell_t(x_t^A, a_t) - \sum_{t=1}^T \ell_t(x_t^{\pi_t}, \pi_t) + C_T(A, \pi) = \sum_{t=1}^T \ell_t(x_t^{\pi_t}, \pi_t) - \sum_{t=1}^T \ell_t(x_t^{\pi_t}, \pi) + \sum_{t=1}^T \ell$$

Notice that the choice of policies has no influence over future losses in $C_T(A, \pi)$. Thus, $C_T(A, \pi)$ can be bounded by a specific reduction to full information online learning algorithms (to be specified later). Also, notice that the competitor policy π does not appear in $B_T(A)$. In fact, $B_T(A)$ depends only on the algorithm A. We will show that if algorithm A and the class of models satisfy the following two "smoothness" assumptions, then $B_T(A)$ can be bounded by a sublinear term.

Assumption A1 Rarely Changing Policies Let α_t be the probability that algorithm A changes its policy at round t. There exists a constant D such that for any $1 \le t \le T$, any sequence of models m_1, \ldots, m_t and loss functions $\ell_1, \ldots, \ell_t, \alpha_t \le D/\sqrt{t}$. N: number of experts, T: number of rounds. Initialize $w_{i,0} = 1$ for each expert *i*. $W_0 = N$. **for** t := 1, 2, ... **do** For any *i*, $p_{i,t} = w_{i,t-1}/W_{t-1}$. Draw I_t such that for any *i*, $\mathbb{P}(I_t = i) = p_{i,t}$. Choose the action suggested by expert I_t . The adversary chooses loss function c_t . The learner suffers loss $c_t(I_t)$. For expert *i*, $w_{i,t} = w_{i,t-1}e^{-\eta c_t(i)}$. $W_t = \sum_{i=1}^N w_{i,t}$. **end for**

Figure 2: The EWA Algorithm

N: number of experts, T: number of rounds. $\eta = \min\{\sqrt{\log N/T}, 1/2\}.$ Initialize $w_{i,0} = 1$ for each expert *i*. $W_0 = N.$ **for** $t := 1, 2, \dots$ **do** For any *i*, $p_{i,t} = w_{i,t-1}/W_{t-1}.$ With probability $\beta_t = w_{I_{t-1},t-1}/w_{I_{t-1},t-2}$ choose the previously selected expert, $I_t = I_{t-1}$ and with probability $1 - \beta_t$, choose I_t based on the distribution $q_t = (p_{1,t}, \dots, p_{N,t}).$ Learner takes the action suggested by expert I_t . The adversary chooses loss function c_t . The learner suffers loss $c_t(I_t)$. For all experts *i*, $w_{i,t} = w_{i,t-1}(1 - \eta)^{c_t(i)}.$ $W_t = \sum_{i=1}^N w_{i,t}.$ **end for**

Figure 3: The Shrinking Dartboard Algorithm

Assumption A2 Uniform Mixing There exists a constant $\tau > 0$ such that for all distributions d and d' over the state space, any deterministic policy π , and any model $m \in M$,

$$||dP(\pi,m) - d'P(\pi,m)||_1 \le e^{-1/\tau} ||d - d'||_1$$
.

As discussed by Neu et al. (2010), if Assumption A2 holds for deterministic policies, then it holds for all policies.

3.1 Full Information Algorithms

We would like to have a full information online learning algorithm that rarely changes its policy. The first candidate that we consider is the well-known Exponentially Weighted Average (EWA) algorithm (Vovk, 1990, Littlestone and Warmuth, 1994) shown in Figure 2. In our MDP problem, the EWA algorithm chooses a policy $\pi \in \Pi$ according to distribution

$$q_t(\pi) \propto \exp\left(-\lambda \sum_{s=1}^{t-1} \mathbb{E}\left[\ell_s(x_s^{\pi}, \pi)\right]\right), \quad \lambda > 0,$$
(2)

The policies that this EWA algorithm generates most likely are different in consecutive rounds and thus, the EWA algorithm might change its policy frequently. However, a variant of EWA, called Shrinking Dartboard (SD) (Geulen et al., 2010) and shown in Figure 3, satisfies Assumption A1. Our algorithm, called SD-MDP, is based on the SD algorithm and is shown in Figure 4. Notice that the algorithm needs to know the number of rounds, T, in advance.

T: number of rounds.

$$\eta = \min\{\sqrt{\log |\Pi|/T}, 1/2\}.$$
For all policies $\pi \in \{1, ..., |\Pi|\}, w_{\pi,0} = 1.$
for $t := 1, 2, ...$ do
For any $\pi, p_{\pi,t} = w_{\pi,t-1}/W_{t-1}.$
With probability $\beta_t = w_{\pi_{t-1},t-1}/w_{\pi_{t-1},t-2}$ choose the previous policy, $\pi_t = \pi_{t-1}$, while with probability $1 - \beta_t$, choose π_t based on the distribution $q_t = (p_{1,t}, \ldots, p_{|\Pi|,t}).$
Learner takes the action $a_t \sim \pi_t(.|x_t)$
Adversary chooses transition model m_t and loss function ℓ_t .
Learner suffers loss $\ell_t(x_t, a_t).$
Learner observes m_t and ℓ_t .
Update state: $x_{t+1} \sim m_t(.|x_t, a_t).$
For all policies $\pi, w_{\pi,t} = w_{\pi,t-1}(1-\eta)^{\mathbb{E}[\ell_t(x_t^{\pi},\pi)]}.$
 $W_t = \sum_{\pi \in \Pi} w_{\pi,t}.$
end for

Figure 4: SD-MDP: The Shrinking Dartboard Algorithm for Markov Decision Processes

Consider a basic full information problem with N experts. Let $R_T(SD, i)$ be the regret of the SD algorithm with respect to expert i up to time T. We have the following results for the SD algorithm.

Theorem 1. For any expert $i \in \{1, \ldots, N\}$,

$$R_T(\mathrm{SD}, i) \le 4\sqrt{T \log N} + \log N$$
,

and also for any $1 \leq t \leq T$,

$$\mathbb{P}\left(Switch \ at \ time \ t\right) \leq \sqrt{\frac{\log N}{T}}$$
.

Proof. The proof of the regret bound can be found in (Geulen et al., 2010, Theorem 3). The proof of the bound on the probability of switch is similar to the proof of Lemma 2 in (Geulen et al., 2010) and is as follows: As shown in (Geulen et al., 2010, Lemma 2), the probability of switch at time t is

$$\alpha_t = \frac{W_{t-1} - W_t}{W_{t-1}}$$

Thus, $W_t = (1 - \alpha_t)W_{t-1}$. Because the loss function is bounded in [0, 1], we have that

$$W_t = \sum_{i=1}^N w_{i,t} = \sum_{i=1}^N w_{i,t-1} (1-\eta)^{c_t(i)} \ge \sum_{i=1}^N w_{i,t-1} (1-\eta) = (1-\eta) W_{t-1} .$$

Thus, $1 - \alpha_t \ge 1 - \eta$, and thus,

$$\alpha_t \le \eta \le \sqrt{\frac{\log N}{T}} \ .$$

3.2 Analysis of the SD-MDP Algorithm

The main result of this section is the following regret bound for the SD-MDP algorithm.

Theorem 2. Let the loss functions selected by the adversary be bounded in [0, 1], and the transition models selected by the adversary satisfy Assumption A2. Then, for any policy $\pi \in \Pi$,

$$\mathbb{E}\left[R_T(\text{SD-MDP},\pi)\right] \le (4+2\tau^2)\sqrt{T\log|\Pi|} + \log|\Pi| .$$

In the rest of this section, we write A to denote the SD-MDP algorithm. For the proof we use the regret decomposition (1):

$$R_T(A,\pi) = B_T(A) + C_T(A,\pi)$$

3.2.1 Bounding $\mathbb{E}[C_T(A, \pi)]$ Lemma 3. For any policy $\pi \in \Pi$,

$$\mathbb{E}[C_T(A,\pi)] = \mathbb{E}\left[\sum_{t=1}^T \ell_t(x_t^{\pi_t},\pi_t) - \sum_{t=1}^T \ell_t(x_t^{\pi},\pi)\right] \le 4\sqrt{T\log|\Pi|} + \log|\Pi|.$$

Proof. Consider the following imaginary game between a learner and an adversary: we have a set of experts (policies) $\Pi = {\pi^1, \ldots, \pi^{|\Pi|}}$. At round t, the adversary chooses a loss vector $c_t \in [0, 1]^{\Pi}$, whose *i*th element determines the loss of expert π^i at this round. The learner chooses a distribution over experts q_t (defined by the SD algorithm), from which it draws an expert π_t . Next, the learner observes the loss function c_t . From the regret bound for the SD algorithm (Theorem 1), it is guaranteed that for any expert π ,

$$\sum_{t=1}^{T} \langle c_t, q_t \rangle - \sum_{t=1}^{T} c_t(\pi) \le 4\sqrt{T \log |\Pi|} + \log |\Pi| .$$

Next, we determine how the adversary chooses the loss vector. At time t, the adversary chooses a loss function ℓ_t and sets $c_t(\pi^i) = \mathbb{E}\left[\ell_t(x_t^{\pi^i}, \pi^i)\right]$. Noting that $\langle c_t, q_t \rangle = \mathbb{E}\left[\ell_t(x_t^{\pi_t}, \pi_t)\right]$ and $c_t(\pi) = \mathbb{E}\left[\ell_t(x_t^{\pi}, \pi)\right]$ finishes the proof.

3.2.2 Bounding $\mathbb{E}[B_T(A)]$

First, we prove the following two lemmas.

Lemma 4. For any state distribution d, any transition model m, and any policies π and π' ,

$$||dP(\pi,m) - dP(\pi',m)||_1 \le ||\pi - \pi'||_{\infty,1}$$
.

Proof. Proof is easy and can be found in (Even-Dar et al., 2009), Lemma 5.1.

Lemma 5. Let α_t be the probability of a policy switch at time t. Then, $\alpha_t \leq \sqrt{\log |\Pi|/T}$. *Proof.* Proof is identical to the proof of Theorem 1.

Lemma 6. We have that

$$\mathbb{E}[B_T(A)] = \mathbb{E}\left[\sum_{t=1}^T \ell_t(x_t^A, a_t) - \sum_{t=1}^T \ell_t(x_t^{\pi_t}, \pi_t)\right] \le 2\tau^2 \sqrt{\log|\Pi|T|}.$$

Proof. Let $\mathcal{F}_t = \sigma(\pi_1, \ldots, \pi_t)$. Notice that the choice of policies are independent of the state variables. We can write

$$\mathbb{E}\left[B_{T}(A)\right] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t}(x_{t}^{A}, a_{t}) - \sum_{t=1}^{T} \ell_{t}(x_{t}^{\pi_{t}}, \pi_{t})\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \sum_{x \in \mathcal{X}} \left(\mathbb{I}_{\{x_{t}^{A}=x\}} - \mathbb{I}_{\{x_{t}^{\pi_{t}}=x\}}\right) \ell_{t}(x, \pi_{t}(x))\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \sum_{x \in \mathcal{X}} \mathbb{E}\left[\left(\mathbb{I}_{\{x_{t}^{A}=x\}} - \mathbb{I}_{\{x_{t}^{\pi_{t}}=x\}}\right) \ell_{t}(x, \pi_{t}(x))\right| \mathcal{F}_{T}\right]\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \sum_{x \in \mathcal{X}} \ell_{t}(x, \pi_{t}(x)) \mathbb{E}\left[\left(\mathbb{I}_{\{x_{t}^{A}=x\}} - \mathbb{I}_{\{x_{t}^{\pi_{t}}=x\}}\right) \left| \mathcal{F}_{T}\right]\right]\right]$$

$$\leq \mathbb{E}\left[\sum_{t=1}^{T} \|\ell_{t}\|_{\infty} \left\|\mathbb{E}\left[\left(\mathbb{I}_{\{x_{t}^{A}=x\}} - \mathbb{I}_{\{x_{t}^{\pi_{t}}=x\}}\right) \left| \mathcal{F}_{T}\right]\right]\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \|\ell_{t}\|_{\infty} \|u_{t} - v_{t,t}\|_{1}\right]$$

(3)

where $u_s = \mathbb{E}\left[\mathbb{I}_{\{x_s^A = x\}} \middle| \mathcal{F}_T\right]$ is the distribution of x_s^A for $s \leq t$ and $v_{s,t} = \mathbb{E}\left[\mathbb{I}_{\{x_s^{\pi_t} = x\}} \middle| \mathcal{F}_T\right]$ is the distribution of $x_s^{\pi_t}$ for $s \leq t$.¹ Let E_t be the event of a policy switch at time t. From inequality

$$\|\pi_{t-k} - \pi_t\|_{\infty,1} \le \|\pi_{t-k} - \pi_{t-k+1}\|_{\infty,1} + \dots + \|\pi_{t-1} - \pi_t\|_{\infty,1} \le 2\sum_{s=t-k+1}^t \mathbb{I}_{\{E_s\}},$$

and Lemma 5, we get that

$$\mathbb{E}\left[\left\|\pi_{t-k} - \pi_t\right\|_{\infty,1}\right] \le 2\sqrt{\frac{\log|\Pi|}{T}}k.$$
(4)

Let $P_t^{\pi} = P(\pi, m_t)$. We have that

$$\mathbb{E}\left[\left\|u_{t}-v_{t,t}\right\|_{1}\right] = \mathbb{E}\left[\left\|u_{t-1}P_{t-1}^{\pi_{t-1}}-v_{t-1,t}P_{t-1}^{\pi_{t}}\right\|_{1}\right] \\
= \mathbb{E}\left[\left\|u_{t-1}P_{t-1}^{\pi_{t-1}}-u_{t-1}P_{t-1}^{\pi_{t}}+u_{t-1}P_{t-1}^{\pi_{t}}-v_{t-1,t}P_{t-1}^{\pi_{t}}\right\|_{1}\right] \\
\leq \mathbb{E}\left[\left\|u_{t-1}P_{t-1}^{\pi_{t-1}}-u_{t-1}P_{t-1}^{\pi_{t}}\right\|_{1}+\left\|u_{t-1}P_{t-1}^{\pi_{t}}-v_{t-1,t}P_{t-1}^{\pi_{t}}\right\|_{1}\right] \\
\leq \mathbb{E}\left[\left\|\pi_{t-1}-\pi_{t}\right\|_{\infty,1}+e^{-1/\tau}\left\|u_{t-1}-v_{t-1,t}\right\|_{1}\right] \\
\leq \mathbb{E}\left[\left\|\pi_{t-1}-\pi_{t}\right\|_{\infty,1}+e^{-1/\tau}\left(\left\|u_{t-2}P_{t-2}^{\pi_{t-2}}-u_{t-2}P_{t-2}^{\pi_{t}}\right\right)\right] \\
+\left\|u_{t-2}P_{t-2}^{\pi_{t}}-v_{t-2,t}P_{t-2}^{\pi_{t}}\right\|_{1}\right)\right] \\
\leq \mathbb{E}\left[\left\|\pi_{t-1}-\pi_{t}\right\|_{\infty,1}+e^{-1/\tau}\left\|\pi_{t-2}-\pi_{t}\right\|_{\infty,1}+e^{-2/\tau}\left\|u_{t-2}-v_{t-2,t}\right\|_{1}\right] \\
\leq \dots \\
\leq \sum_{k=0}^{t}e^{-k/\tau}\mathbb{E}\left[\left\|\pi_{t-k}-\pi_{t}\right\|_{\infty,1}\right]+e^{-t/\tau}\left\|u_{0}-v_{0,t}\right\|_{1} \\
\leq \sum_{k=0}^{t}2e^{-k/\tau}\sqrt{\frac{\log|\Pi|}{T}}k+0 \qquad By (4) \\
\leq 2\sqrt{\frac{\log|\Pi|}{T}}\tau^{2},$$
(5)

where we have used the fact that $||u_0 - v_{0,t}||_1 = 0$, because the initial distributions are identical. By (5) and (3), we get that

$$\mathbb{E}\left[B_T(A)\right] \le 2\tau^2 \sum_{t=1}^T \sqrt{\frac{\log|\Pi|}{T}} = 2\tau^2 \sqrt{\log|\Pi|T} .$$

What makes the analysis possible is the fact that all policies mix no matter what transition model is played by the adversary.

Proof of Theorem 2. The result is obvious by Lemmas 3 and 6.

The next corollary extends the result of Theorem 2 to continuous policy spaces.

Corollary 7. Let Π be an arbitrary policy space, $\mathcal{N}(\epsilon)$ be the ϵ -covering number of space $(\Pi, \|.\|_{\infty,1})$, and $\mathcal{C}(\epsilon)$ be an ϵ -cover. Assume that we run the SD-MDP algorithm on $\mathcal{C}(\epsilon)$. Then, under the same assumptions as in Theorem 2, for any policy $\pi \in \Pi$,

$$\mathbb{E}[R_T(\text{SD-MDP}, \pi)] \le (4 + 2\tau^2)\sqrt{T\log \mathcal{N}(\epsilon)} + \log \mathcal{N}(\epsilon) + \tau T\epsilon .$$

¹Notice that \mathcal{F}_T contains only policies, which are independent of the state variables.

Proof. Let $L_T(\pi) = \mathbb{E}\left[\sum_{t=1}^T \ell_t(x_t^{\pi}, \pi)\right]$ be the value of policy π . Let $u_{\pi,t}(x) = \mathbb{P}(x_t^{\pi} = x)$. First, we prove that the value function is Lipschitz with Lipschitz constant τT . The argument is similar to the argument in the proof of Lemma 6. For any π_1 and π_2 ,

$$|L_T(\pi_1) - L_T(\pi_2)| = \left| \mathbb{E} \left[\sum_{t=1}^T \ell_t(x_t^{\pi_1}, \pi_1) - \sum_{t=1}^T \ell_t(x_t^{\pi_2}, \pi_2) \right] \right|$$
$$\leq 2 \left| \sum_{t=1}^T \|u_{\pi_1, t} - u_{\pi_2, t}\|_1 \|\ell_t\|_{\infty} \right|$$
$$\leq 2 \left| \sum_{t=1}^T \|u_{\pi_1, t} - u_{\pi_2, t}\|_1 \right|.$$

With an argument similar to the one in the proof of Lemma 6, we can show that

$$||u_{\pi_1,t} - u_{\pi_2,t}||_1 \le \tau ||\pi_1 - \pi_2||_{\infty,1}$$
.

Thus,

$$|L_T(\pi_1) - L_T(\pi_2)| \le \tau T \|\pi_1 - \pi_2\|_{\infty, 1} .$$

Given this and the fact that for any policy $\pi \in \Pi$, there is a policy $\pi' \in \mathcal{C}(\epsilon)$ such that $\|\pi - \pi'\|_{\infty,1} \leq \epsilon$, we get that

$$\mathbb{E}[R_T(\text{SD-MDP}, \pi)] \le (4 + 2\tau^2)\sqrt{T\log \mathcal{N}(\epsilon)} + \log \mathcal{N}(\epsilon) + \tau T\epsilon.$$

In particular if Π is the space of all policies, $\mathcal{N}(\epsilon) \leq (|\mathcal{A}|/\epsilon)^{|\mathcal{A}||\mathcal{X}|}$, so regret is no more than

$$\mathbb{E}\left[R_T(\text{SD-MDP}, \pi)\right] \le (4 + 2\tau^2)\sqrt{T|\mathcal{A}||\mathcal{X}|\log\frac{|\mathcal{A}|}{\epsilon}} + |\mathcal{A}||\mathcal{X}|\log\frac{|\mathcal{A}|}{\epsilon} + \tau T\epsilon$$

By the choice of $\epsilon = \frac{1}{T}$, we get that $\mathbb{E}[R_T(\text{SD-MDP}, \pi)] = O(\tau^2 \sqrt{T |\mathcal{A}| |\mathcal{X}| \log(|\mathcal{A}|T)}).$

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