EIGENVALUE OPERATOR SPLIT CONTROL VOLUME METHOD FOR 1D, 2D, 3D FLOW AND TRANSPORT

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ABSTRACT

This paper presented a general, simple, stable, and accurate numerical method for numerical simulation of flow and transport. It could be an explicit or implicit scheme.

As a general numerical scheme, it can handle a complicated flow such as dam break, transcritical flow, supercritical flow, subcritical flow, tide flow, wave flow, free surface flow and non-free surface flow for 1D, 2D, 3D cases.

1. INTRODUCTION

With economic development and living standard rising, the water resource and sedimentation problem is becoming a critical element in the economic feasibility and human activity in China. In west China, more and more, water is becoming a rare resource for drinking, agriculture, and industry. And in coastal region harbors are of vital importance for the economy. The increasing draft of vessels requires dredging of deep-draft channels connecting port to deep water, for example in Yantze river estuary, the dredging ships work day and night to dredge the navigation channel from 8.00(m) depth to 12.50(m) depth. In many situations engineering structures are required to stabilize the shoreline, shoals and inlets, to reduce sedimentation, and to prevent erosion. In a lot of coastal city of China, a lot of human made lands are built up by filling ocean with earth and rock since land is rare resource and very valuable, and its sequence and influence to flood and ecological system should be evaluated. Coastal protection against surges due to windstorms or earthquakes is one of the most basic problems in many estuaries since the recent big Tsunami killed huge number of people in Asian. With the shortage of oil, the hydraulic power generation construction is booming up in China, the side effect attracts lot of attention for dam building. Those side-effects include loss of reservoir storage due to sediment deposition, erosion of the river bed downstream of the reservoir, and damage of downstream river bank due to sediment concentration decline, decline of navigation condition due to navigable water depth unsteadily change and decline of water quality. With more high way constructed, more and more bridges are built up cross the river, bay and coastal region, therefore the local and general scour around bridge pier should be paid attention for bridge safety. The ecology and environment problem, coming from the human interference in hydraulic system, is another big issue for any water and sediment related engineering project. Unfortunately, the turbulence and sedimentation processes in rivers and coastal regions are among the most complicated and least understood phenomena in nature currently. As one of research tool,

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mathematical models are more often used to solve engineering problems combining with field measurement data due to the fact that mathematical models are easy to apply, low cost, and time efficient.

But currently, the numerical models for flow and sedimentation have limitations. The numerical methods, used in the discretion of partial differential equations, are the finite element model, finite different model, finite volume model, finite analytical model, efficient element model, Alternating Direction Implicit (ADI) model, etc. The problem is that for all the numerical methods mentioned above no one can handle all flow and sedimentation processes such as 1D、2D、3D, sub critical flow, supercritical flow, tide and wave flow. The turbulence and sediment transport equations contain the convection and diffusion terms. Generally central difference type scheme is used to discrete the diffusion terms. Convection terms are usually handled by using the upwind scheme for stabilizing the numerical scheme. Analysis shows that upwind scheme is very stable but introduces artificial diffusivity to numerical scheme and makes the numerical solution distorted. Hybrid upwind/central scheme, exponential difference scheme, the finite analytical scheme, upwind interpolation scheme and quick scheme belong to the same type of upwind scheme. Another problem for upwind scheme is that it is hard to keep the conservation of energy, conservation of momentum and conservation of mass when the flow field changes its direction such as in tidal flow. Due to the constrain of mathematical modals for flow and sedimentation, their successful application for engineering problems relies strongly upon experts' experience.

Developing a numerical scheme, a general numerical scheme for all flows and transport situation adopted and numerical diffusion free, is the goal for all researchers working in field of numerical modeling of flow and sedimentation [Yuanya Li, 2005]. It is also the objective of this paper.

2. THE PARTIAL DIFFERENTIAL EQUATIONS

2.1 1D equations

The 1D Saint-Venant equations for flood wave propagation in rivers and channels, where water stages and flow-rates at different locations along the river or channel and at different time instance are the main concern, are

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial z}{\partial x} + g\frac{Q^2}{C^2 AR} = \left(V_x - \frac{Q}{A}\right)g \tag{2}$$

in which A is the river cross section area; Q is the discharge; g is the discharge flowing from a river branch; z is the water level; R is the hydraulic radius; C is Chezy coefficient; V_x is the flowing velocity projected in the main river flow direction for river branch;

The transport equation of sediment or salinity is

$$\frac{\partial (Ac)}{\partial t} + \frac{\partial (Qc)}{\partial x} = S^* \tag{3}$$

in which c is the concentration of sediment or salinity and S^* is the source term.

2.2 2D equations

For $2D$ depth-averaged equations, usually used in lakes and oceans where the horizontal dimensions are much larger than the vertical one, the continuity equation is:

$$\frac{\partial \xi}{\partial t}+\frac{\partial[(\xi+h)u]}{\partial x}+\frac{\partial[(\xi+h)v]}{\partial y}=0 \qquad (4)$$

the dynamic equation are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+g\frac{\partial \xi}{\partial x}+gu\frac{\sqrt{u^2+v^2}}{(\xi+h)c^2}=0 \qquad (5)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fv+g\frac{\partial \xi}{\partial y}+gv\frac{\sqrt{u^2+v^2}}{(\xi+h)c^2}=0 \qquad (6)$$

in which ξ is the water level; h the water depth; u,v is the velocity component in x and y direction respectively, t is time; $f=(2w\sin\varphi)$ is a coefficient, w is the earth angle velocity and φ is geographic latitude.

The 2D transport equation of sediment or salinity is

$$\frac{\partial(Hc)}{\partial t}+\frac{\partial(Huc)}{\partial x}+\frac{\partial(Hvc)}{\partial y}-\varepsilon\left(\frac{\partial^2(Hc)}{\partial x^2}+\frac{\partial^2(Hc)}{\partial y^2}\right)=S^2 \qquad (7)$$

in which c is the concentration of sediment or salinity and S^2 is the source term.

2.3 3D equation

The 3D Continuity equation is

$$\frac{\partial p}{\partial t}+u\frac{\partial D}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}=C^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \qquad (8)$$

The 3D dynamic equation are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon_t\left[\frac{\partial^2 u}{\partial x}+\frac{\partial^2 u}{\partial y}+\frac{\partial^2 u}{\partial z}\right] \qquad (9)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon_t\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right] \qquad (10)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon_t\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right] \qquad (11)$$

On the free surface, there is a kinematic condition or constraint:

$$\frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} - w = 0 \tag{12}$$

in which u, v and w present the velocity components in x, y and z direction respectively; ρ is the fluid density, C is the sound speed in water, p stands for pressure and υ_t is the turbulent viscosity.

The 3D transport equation of sediment or salinity is

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} - \omega\frac{\partial c}{\partial z} = \upsilon_s\left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right] \tag{13}$$

in which c is the concentration of sediment or salinity, ω is the falling velocity and υ_s is the turbulent viscosity.

2.4 Wave equation

For open shallow water the wave impact on sediment movement couldn't be neglected in lakes and coastal regions. It is a critical element for harbors and navigation projects for the seabed composed with fine sand. The non-linear wave propagation equations for shallow water are the modified Boussinesq equations:

$$\eta_t + P_x + Q_y = 0 \tag{14}$$

$$P_t + \left(\frac{P^2}{h}\right)_x + \left(\frac{PQ}{h}\right)_y + gd\eta_x - \left(B + \frac{1}{3}\right)d^2\left(P_{xxt} + Q_{xyt}\right) - Bgd^3\left(\eta_{xxx} + \eta_{xyy}\right) = 0 \tag{15}$$

$$Q_t + \left(\frac{PQ}{h}\right)_x + \left(\frac{Q^2}{h}\right)_y + gd\eta_y - \left(B + \frac{1}{3}\right)d^2\left(Q_{yyt} + P_{xyt}\right) - Bgd^3\left(\eta_{yyy} + \eta_{xxy}\right) = 0 \tag{16}$$

in which η is the wave height, $h = \eta + d$, $P = hU$, $Q = hV$, U and V is the depth-averaged velocity in x and y direction respectively.

3. EIGENVALUE OPERATOR SPLIT CONTROL VOLUME METHOD FOR 1D, 2D AND 3D FLOW AND TRANSPORT

3.1 Operator Split For Partial Differential Equations

The idea of operator split for partial differential equations is dividing a comprehensive multi-direction partial differential equation, containing convective terms, diffusion terms, source terms and cross direction derivative terms, into a group of sub partial differential equations. For each sub partial differential equation, it only has convective terms in one direction or source term and diffusion terms in all direction. The cross direction derivative terms also are treated as convective or diffusion type terms. For all diffusion terms, the central differential scheme is used. The main concern focuses on the Euler type convective equation. As an example the sub partial differential equations related to convective term for equation (9) are

$$\frac{u^{n+1/3^*} - u^n}{\Delta t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{17}$$

$$\frac{u^{n+2/3^*} - u^{n+1/3}}{\Delta t} + v\frac{\partial u}{\partial y} = 0 \tag{18}$$

$$\frac{u^{n+1^*} - u^{n+2/3}}{\Delta t} + w\frac{\partial u}{\partial z} = 0 \tag{19}$$

3.2 Eigenvalue For Convective Equations

For general purpose, a convective term related to 1D, 2D, 3D partial differential equations can be written as [Yuanya Li, 2005]

$$\frac{\partial \overline{U}}{\partial t} + \frac{\partial F}{\partial x_i} = 0 \tag{20}$$

in which $\overline{U}$ is the vector with m elements, and F is also a vector with m elements. Equation (20) can be rewritten as

$$\frac{\partial \overline{U}}{\partial t} + A(\overline{U})\frac{\partial \overline{U}}{\partial x_i} = 0 \tag{21}$$

in which

$$A(\overline{U}) = \frac{\partial F}{\partial U} \tag{22}$$

is Jacobean matrix, the Jacobean matrix has eigenvalue $\lambda_l(\ell = 1,2,\cdots,m)$, and corresponding eigenvector matrix P

$$P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix} \tag{23}$$

So Eq.(21) can be replaced by m independent linear partial differential equations

$$\frac{\partial V_e}{\partial t} + \lambda_1\frac{\partial V_e}{\partial x} = 0 \qquad \ell = 1,2,\cdots,m \tag{24}$$

in which $\overline{V} = (V_1, V_2, V_3, \cdots, V_m)^T P^{-1}\overline{U}$. It is only necessary to analysis the value equation to determine the stability of numerical modeling for partial differential equations (24). Using the backward differential numerical scheme to discrete the convective term in the equation, one obtains

$$\frac{\partial v_j}{\partial t} + \lambda \frac{v_j - v_{j-1}}{\Delta x} = 0 \tag{25}$$

Assuming that the boundary condition is a period, the solution for equation (25) can be written as a Fourier series

$$v_j(t) = v(t)e^{ikj\Delta x} \tag{26}$$

in which $v(t)$ is Fourier coefficient. Replacing equation (25) with equation (26), the common difference equation to satisfy Eq. (25) is

$$\frac{dv}{dt} = \sigma v \tag{27}$$

in which

$$\sigma = -\frac{\lambda}{\Delta x}[2\sin^2\left(\frac{k\Delta x}{2}\right) + i\sin k\Delta x] \tag{28}$$

The solution for Eq. (27) is

$$v = Ce^{\sigma t} \tag{29}$$

If the solution is accountable, then the solution is stable. If the solution is accountable, the real part for σ must be less than zero. It means $\mathrm{Re}\,\sigma \le 0$. Therefore it asks that $\lambda > 0$. It means that when elgenvalue is positive, the backward difference scheme (upwind scheme) makes the solution stable. Using the forward difference scheme, Eq. (24) can be written as

$$\frac{\partial v_j}{\partial t} + \lambda \frac{v_{j+1} - v_j}{\Delta x} = 0 \tag{30}$$

Doing the stability analysis as before, it reads

$$\sigma = \frac{\lambda}{\Delta x}[2\sin^2\left(\frac{k\Delta x}{2}\right) - i\sin k\Delta x] \tag{31}$$

If the solution is accountable, the real part for σ must be less than zero. It means $\mathrm{Re}\,\sigma \le 0$. Therefore, it asks that $\lambda < 0$. It means that when eigenvalue is negative, the forward difference scheme makes the solution stable.

From the analysis mentioned above, a very important conclusion can be obtained that the often used Preismann's four-point implicit scheme, and TVD scheme may have a instable problem for numerical simulation of transcritical flow such as hydropower generation dam adjustment flow, dam beak flow, etc.

3.3 Flux Operator Split

It can be proven that Eq. (23) has the eigenvalues

$$P^{-1}AP = \Lambda = \begin{pmatrix} u & & \\ & u+c & \\ & & u-c \end{pmatrix} \quad (32)$$

For a free surface flow

$$c = \sqrt{gh} \quad (33)$$

Λ can be separated as Λ_1, Λ_2 and Λ_3

$$\Lambda_1 = \begin{pmatrix} u & & \\ & 0 & \\ & & 0 \end{pmatrix} = u\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} = uM_1 \quad (34)$$

$$\Lambda_2 = \begin{pmatrix} 0 & & \\ & u+a & \\ & & 0 \end{pmatrix} = (u+a)\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} = (u+a)M_2 \quad (35)$$

$$\Lambda_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & u-a \end{pmatrix} = (u-c)\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} = (u-c)M_3 \quad (36)$$

and

$$\begin{aligned} A = P\Lambda P^{-1} &= P\Lambda_1 P^{-1} + P\Lambda_2 P^{-1} + P\Lambda_3 P^{-1} = A_1 + A_2 + A_3 \\ &= uPM_1P^{-1} + (u+c)PM_2P^{-1} + (u-c)PM_3P^{-1} \end{aligned} \quad (37)$$

The flux split scheme is

$$F = F_1 + F_2 + F_3 \quad (38)$$

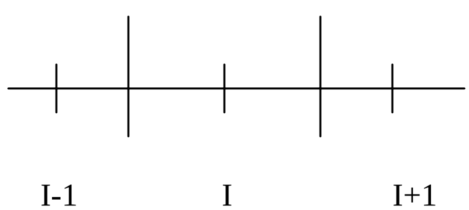

Figure 1. Control volume and its boundary

The first step is integrating the Eq. (20) for the whole control volume by freezing the right boundary

$$\int_{i-\frac{\Delta x}{2}}^{i+\frac{\Delta x}{2}} (U_t + F_x) dx = 0 \tag{39}$$

the result is

$$\Delta x U_t^1(i) + F\left(i - \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \tag{40}$$

Second step, integrating the equation (20) for whole control volume by freezing left boundary

$$\int_{i-\frac{\Delta x}{2}}^{i+\frac{\Delta x}{2}} (U_t + F_x) dx = 0 \tag{41}$$

the result is

$$\Delta x U_t^2(i) + F\left(i + \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \tag{42}$$

the final solution is

$$U^{n+1}(i) = U^n + \frac{1}{2}\left(U^1(i) + U^2(i)\right) \tag{43}$$

It can be proven that the accuracy to $U^{n+1}(i)$ is $O(\Delta x^3, \Delta t^3)$. Using flux split expression of Eq. (38), Eq. (40) and (42) can be written as

$$\Delta x U_t^1(i) + F_1\left(i - \frac{\Delta x}{2}\right) + F_2\left(i - \frac{\Delta x}{2}\right) + F_3\left(i - \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \tag{44}$$

and

$$\Delta x U_t^2(i) + F_1\left(i + \frac{\Delta x}{2}\right) + F_2\left(i + \frac{\Delta x}{2}\right) + F_3\left(i + \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \tag{45}$$

3.4 Determination Of Numerical Schemes By Eigenvalues

1) When the eigenvalue $u > 0$, F_1 uses the upwind differential scheme either first order or second order(QUICK type scheme);
2) When the eigenvalue $u + c > 0$, F_2 uses the upwind differential scheme either first order or second order(QUICK type scheme);
3) When the eigenvalue $u - c > 0$ (supercritical flow), F_3 uses the upwind differential scheme either first order or second order(QUICK type scheme);
4) When the eigenvalue $u - c < 0$ (subcritical flow), F_3 uses the downwind differential scheme either first order or second order(QUICK type scheme);

3.5 Time March By Explicit Scheme Or Implicit Scheme

For an explicit scheme, Eq. (44) and (45) can be written as

$$\Delta x U_t^1(i) + F^n{}_1\left(i - \frac{\Delta x}{2}\right) + F^n{}_2\left(i - \frac{\Delta x}{2}\right) + F^n{}_3\left(i - \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \qquad (46)$$

$$\Delta x U_t^2(i) + F^n{}_1\left(i + \frac{\Delta x}{2}\right) + F^n{}_2\left(i + \frac{\Delta x}{2}\right) + F^n{}_3\left(i + \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \qquad (47)$$

For an implicit scheme, Eq. (44) and (45) can be written as

$$\Delta x U_t^1(i) + F^{n+1}{}_1\left(i - \frac{\Delta x}{2}\right) + F^{n+1}{}_2\left(i - \frac{\Delta x}{2}\right) + F^{n+1}{}_3\left(i - \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \qquad (48)$$

$$\Delta x U_t^2(i) + F^{n+1}{}_1\left(i + \frac{\Delta x}{2}\right) + F^{n+1}{}_2\left(i + \frac{\Delta x}{2}\right) + F^{n+1}{}_3\left(i + \frac{\Delta x}{2}\right) + O(\Delta x^2) = 0 \qquad (49)$$

Assembled the solutions of convective terms, diffusion terms and source terms, the 1D, 2D and 3D flow and transport problems can be solved for all situations.

4. MODEL VERIFICATION AND APPLICATION

4.1 1D Case Verification

The computational domain for a 1D case is from Shuifu Dam to Luzhou city, in the upper Yangtze river where bars and pools are alternatively located and the supercritical and subcritical flow are alternate. The Minjiang river and Jingsha river are the inlet boundary and Luzhou city is the outlet boundary. The total length is 165 kilometers.

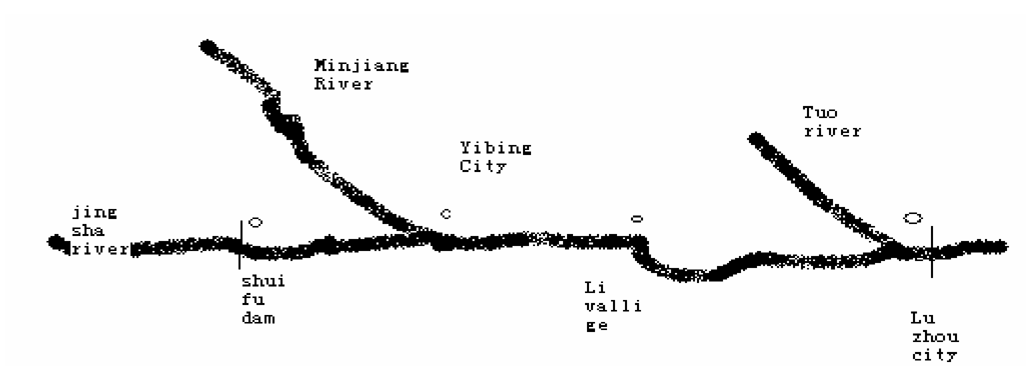

Figure 2. Schematic plan of upper Yangtze River

4.2 2D Case For General Erosion Of Bridge Pier

The 2D numerical model verification case is located in the Jiaozhou bay. A bridge with more than 430 bridge piers is designed to cross the bay. The general erosion around the bridge pier due to the tidal current, wave and water surge of hurricane is predicated by the 2D numerical model.

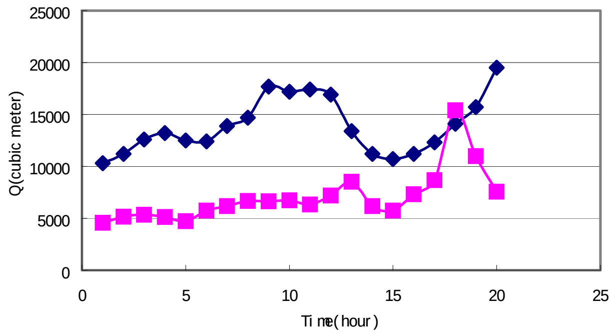

Figure 3. Inlet discharge boundary

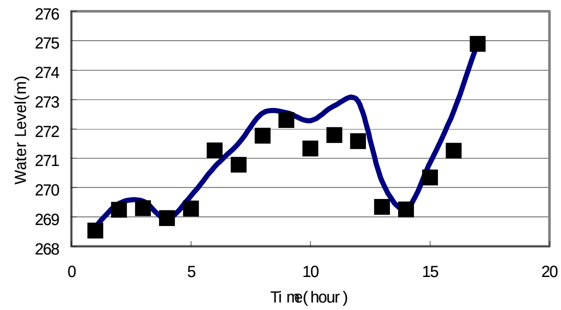

Figure 4. Yibing water level verification

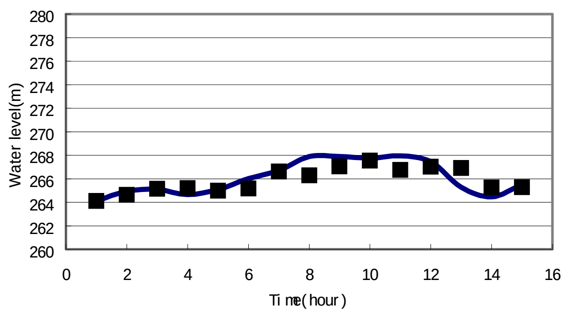

Figure 5. Li village water level verification

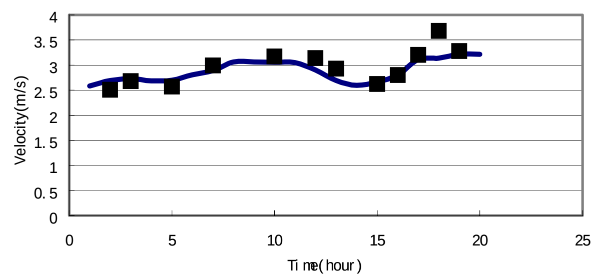

Figure 6. Li village velocity verification

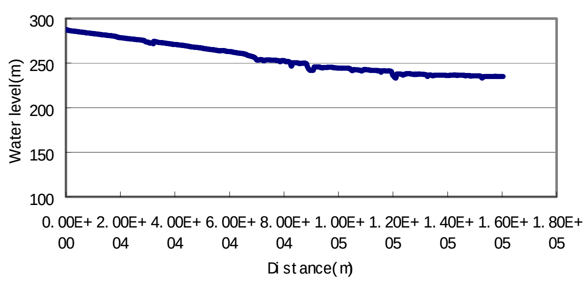

Figure 7. The water surface distribution

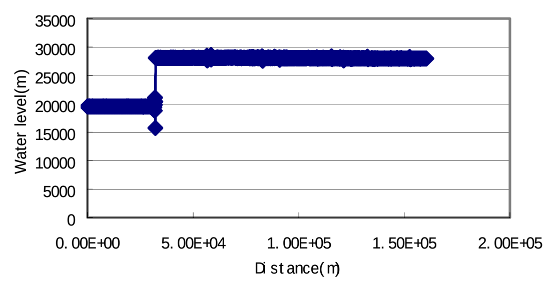

Figure 8. The discharge distribution

4.3 3D Case For Local Scour Of Bridge Pier

Figure 11 shows the simulated local scour pattern around the main pier of Dagu bridge under the action of tidal current and hurricane waves. It is an initial result to simulate local scour for a real bridge pier. The mesh size is 0.5m×0.5m. The wave height is 2.5m and the wave period is 5.4s. The result is reasonable. More sophistical result depends upon finer mesh.

5. SUMMARY

The numerical method presented in this paper is of accuracy in order $O(\Delta x^3, \Delta t^3)$ and based on a solid theoretical analysis. Its time march scheme can be either implicit or explicit, and the space differential scheme could be either first order or second order (QUICK type scheme). Due to its accuracy, stability, simplicity and efficiency, all complicated flow and transport related engineering problems may be solved with one numerical scheme, short computer time and more accuracy.

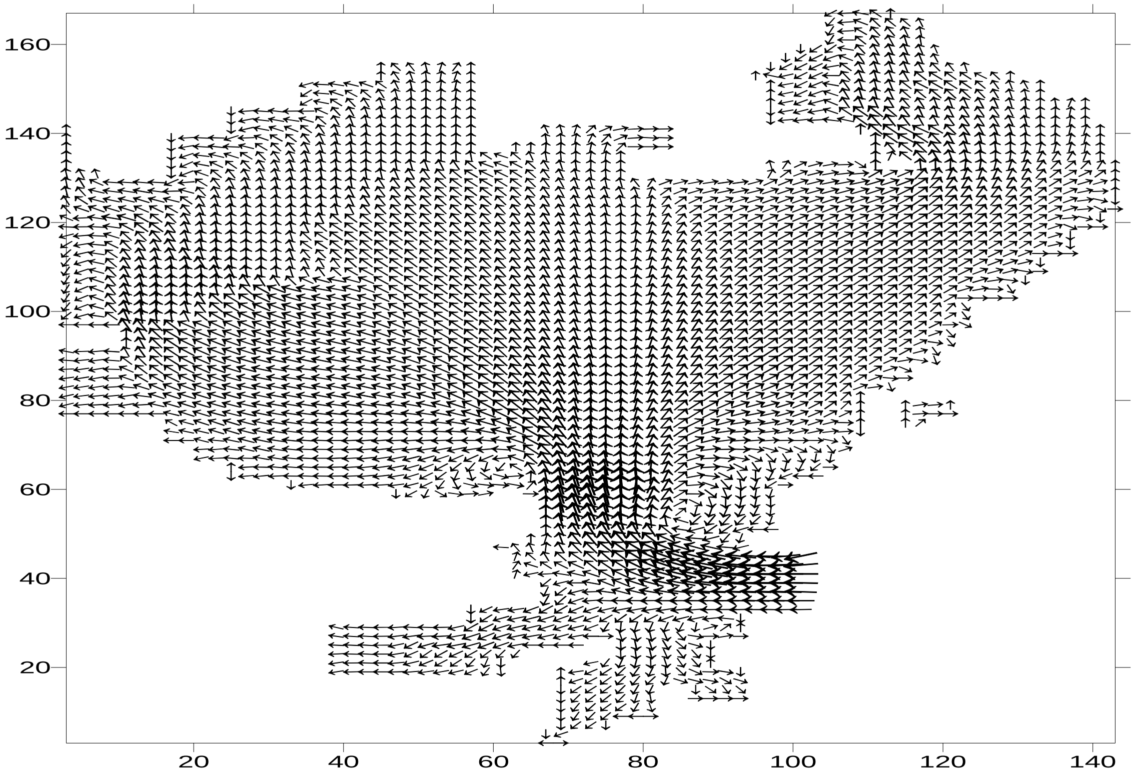

Figure 9. The simulated velocity vector field for Jiaozhou Bay

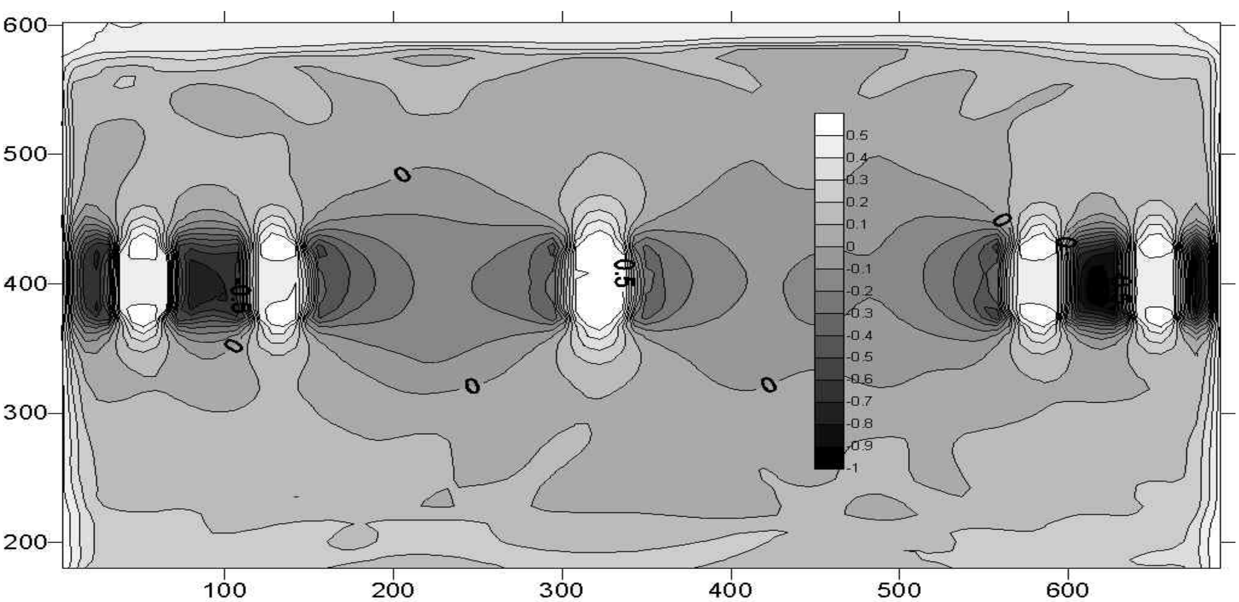

Figure 10. Simulated general erosion around the bridge piers of Dagu bridge

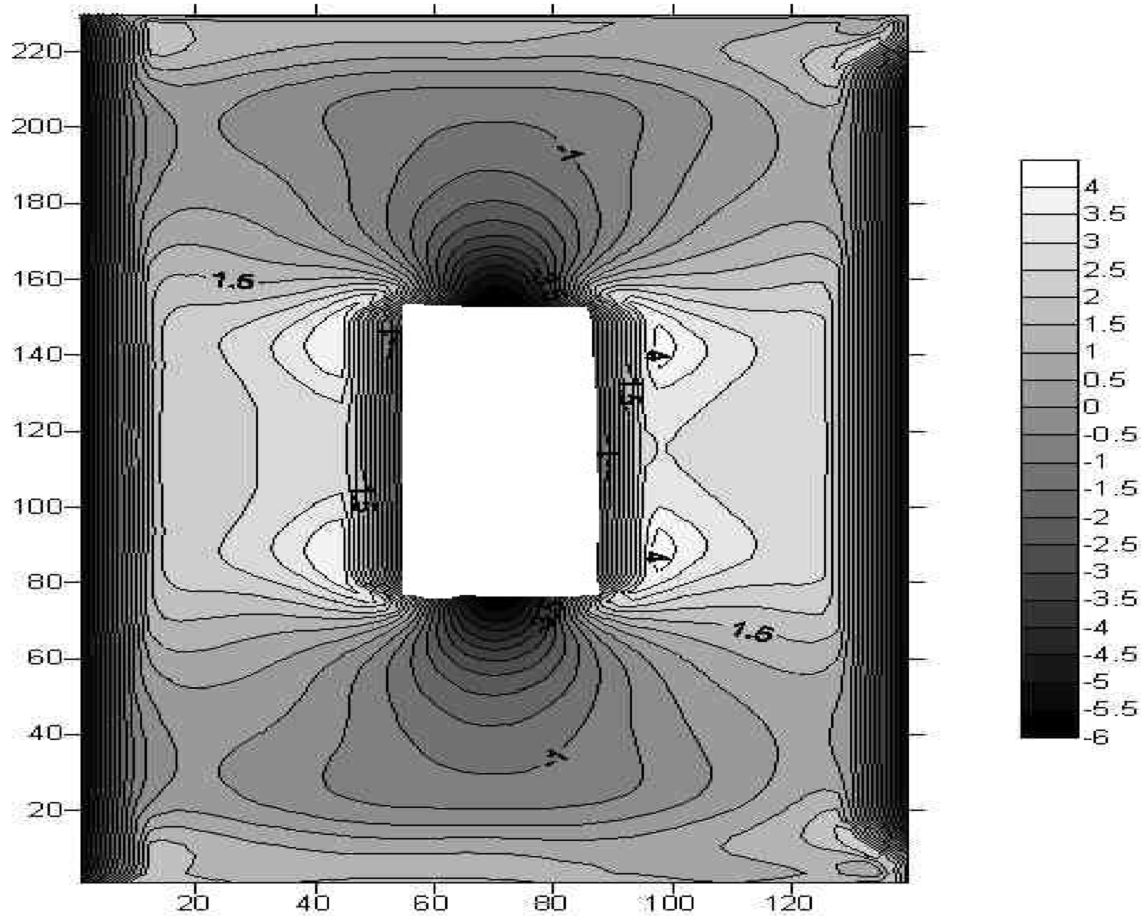

Figure 11. Simulated local scour around the main bridge pier of Dagu bridge

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