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MODELLING SURFACE WATER WAVES USING A HIGH-LEVEL GLN THEORY

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Abstract: This paper concerns the description of surface water waves using the theory of fluid sheets first proposed by Green-Laws & Naghdi (1974). Unlike most traditional wave theories this model is not based upon a velocity potential or stream function representation satisfying Laplace's equation, but instead adopts a shape function representation that approximates the vertical structure of the velocity field. Subject to this approximation, the governing equations are given by the depth-integrated form of the Euler equations, with both the bottom boundary condition and the nonlinear free surface boundary conditions being satisfied exactly.

The principle advantage of the method described herein lies in its versatility. It can be applied in any water depth, to regular and irregular wave fields, and can incorporate varying bottom topography. Furthermore, there are no restrictions placed on the rotationality of the flow field. As a result, it is particularly suited to the study of wave-current interactions where the wave motion itself may become rotational and cannot therefore be modelled by existing irrotational wave solutions.

Key words: Wave modelling, GLN theory, Wave-current interactions, Vorticity

1. INTRODUCTION

Provided one is not specifically concerned with the description of flows immediately adjacent to solid boundaries, the motion associated with the propagation of a wave field may be considered inviscid and solutions conveniently based upon the Euler equations. In this case, numerical integration methods such as finite difference, finite element and finite volume are capable of solving many steady-state problems. However, if the motion is unsteady, involving the evolution of a random wave field over tens or perhaps hundreds of wave cycles, the required computational effort quickly becomes excessive. As a result further simplification is required, with most models assuming the flow is irrotational thereby allowing the application of potential wave solutions. Although there are numerous examples where this is entirely appropriate, wave conditions relevant to engineering design typically arise under storm conditions, where large wind velocities also give rise to co-existing currents. These are not necessarily irrotational and, as a result, potential flow models may be inappropriate to describe the combined wave-current motions. Accordingly, alternative methods are needed to solve the unsteady Euler equations in a computationally efficient way.

The present work addresses this difficulty by applying the theory of fluid sheets first outlined in the work of Green, Laws & Naghdi (1974) and hereafter referred to as GLN theory. Recent examples where the theory has been successfully applied include Green & Naghdi (1986 & 1987), Webster & Shields (1988 & 1991), Ertekin et al. (1984 & 1986), Webster & Kim (1990), Swan et al. (2003) and Chan & Swan (2003a & 2003b). With the exception of the last two contributions these solutions are based on relatively low level approximations or are confined to deep-water conditions. Our present studies have extended these solutions to include wave fields in finite depths using a level III GLN theory.

2. THEORY

2.1 PROBLEM STATEMENT AND SOLUTION DOMAIN

The solution domain is a stationary, two-dimensional Cartesian co-ordinate system, (x, z), where x is measured in the direction of wave propagation and z is orientated vertically upwards from the still-water level. The fluid is assumed to be inviscid, homogeneous and incompressible. The upper boundary is formed by a smooth, time-varying free surface, $z = \zeta(x,t)$, and the velocity vector defining the fluid motion is given by $\mathbf{u} = (u+U, v)$, where U is the current and (u, v) are respectively the horizontal and vertical components of the wave-induced velocity. The current field is assumed to be steady and mass conserving. Applying mass conservation to the wave motion yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \tag{1}$$

The Euler equations describing the combined wave-current motion are given by

$$\begin{gathered}\frac{\partial u}{\partial t} + (u+U)\frac{\partial (u+U)}{\partial x} + v\frac{\partial (u+U)}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + S_x \\ \frac{\partial v}{\partial t} + (u+U)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}\end{gathered} \tag{2}$$

where ρ is the density, P the total pressure and S_x the external force necessary to sustain any horizontal gradient in the steady current, dU/dx. The total pressure consists of the hydrostatic and dynamic components, given by

$$P(x,z,t) = (\zeta - z)\rho g + p \tag{3}$$

where g is the gravitational acceleration and p the dynamic pressure.

On the upper boundary, corresponding to the water-air interface, the kinematic free-surface boundary condition yields

$$\frac{\partial \zeta}{\partial t} + (u+U)\frac{\partial \zeta}{\partial x} = v \qquad \text{on} \qquad z = \zeta(x,t) \tag{4a}$$

This effectively ensures that the water surface is a streamline. Neglecting the effects of surface tension and assuming that the pressure is constant and equal to atmospheric pressure along the water surface, the dynamic free surface boundary condition yields

$$P = 0 \qquad \text{on} \qquad z = \zeta(x,t) \tag{4b}$$

In finite water depths the lower boundary, corresponding to the sea bed, is assumed to be impermeable so that the kinematic bottom-boundary condition requires

$$(u+U)\frac{\partial h}{\partial x} = v \qquad \text{on} \qquad z = -h(x) \tag{4c}$$

The problem statement is completed by the prescription of the boundary conditions acting on the vertical surfaces that close the domain. These are typically problem dependent, but usually involve the specification of the input wave conditions at the upstream end (x=0) together the implementation of effective wave absorption at the downstream end.

2.2 GOVERNING EQUATIONS

Following GLN theory, the velocity field is defined as

$$u(x,z,t) = \sum_{n=1}^{K} u_n(x,t)\lambda_n(z) \tag{5a}$$

$$v(x,z,t)=\sum_{n=1}^{K+1} v_n(x,t)\lambda_n(z) \quad \textit{for finite depth} \tag{5b}$$

$$v(x,z,t)=\sum_{n=1}^{K} v_n(x,t)\lambda_n(z) \quad \textit{for deep water} \tag{5c}$$

where as noted previously $\mathbf{u}=(u(x,z,t)+U(x,z), v(x,z,t))$. The coefficients u_n and v_n represent the unknown variables, λ_n are the prescribed shape functions that define the vertical structure of the velocity field and K determines the level of G-N theory that approximates the complexity of the velocity field.

Substituting equations (5) into (2) and depth integrating using a weighting functions, W, that is assumed to be identical to the shape functions parameter λ, the depth-integrated Euler equations are given by

$$\int_{z=-h}^{z=\zeta}\left[\sum_{n=1}^{K}\frac{\partial u_n}{\partial t}\lambda_n+U\sum_{n=1}^{K}\frac{\partial u_n}{\partial x}\lambda_n+\sum_{m=1}^{K}\sum_{n=1}^{K}u_m\frac{\partial u_n}{\partial x}\lambda_m\lambda_n+\frac{\partial U}{\partial x}\sum_{n=1}^{K}u_n\lambda_n\right.$$
$$\left.+\sum_{m=1}^{K}\sum_{n=1}^{K}u_n v_m\frac{\partial \lambda_n}{\partial z}\lambda_m+\frac{\partial U}{\partial z}\sum_{n=1}^{K}v_n\lambda_n\right]W_r dz=\int_{z=-h}^{z=\zeta}\left[-\frac{1}{\rho}\frac{\partial P}{\partial x}\right]W_r dz \tag{6a}$$

$$\int_{z=-h}^{z=\zeta}\left[\sum_{n=1}^{K}\frac{\partial v_n}{\partial t}\lambda_n+U\sum_{n=1}^{K}\frac{\partial v_n}{\partial x}\lambda_n+\sum_{m=1}^{K}\sum_{n=1}^{K}u_m\frac{\partial v_n}{\partial x}\lambda_m\lambda_n\right.$$
$$\left.+\sum_{m=1}^{K}\sum_{n=1}^{K}v_m v_n\frac{\partial \lambda_n}{\partial z}\lambda_m\right]W_r dz=\int_{z=-h}^{z=\zeta}\left[-\frac{1}{\rho}\frac{\partial P}{\partial z}\right]W_r dz \tag{6b}$$

for $r=1,\ K$

The right-hand side of (6b) can be integrated by parts to yield

$$\int_{z=-h}^{z=\zeta}\left[-\frac{1}{\rho}\frac{\partial p}{\partial z}\right]W_r dz=-\frac{1}{\rho}\left[PW_r\right]_{z=-h}^{z=\zeta}+\frac{1}{\rho}\int_{z=-h}^{z=\zeta}P\frac{\partial W_r}{\partial z}dz \tag{7a}$$

and (6a) can be expanded using Leibnitz's rule to give

$$\int_{z=-h}^{z=\zeta}\left[-\frac{1}{\rho}\frac{\partial P}{\partial x}\right]W_r dz=-\frac{1}{\rho}\frac{\partial}{\partial x}\int_{z=-h}^{z=\zeta}PW_r dz+\frac{1}{\rho}\left[PW_r\frac{\partial z}{\partial x}\right]_{z=-h}^{z=\zeta} \tag{7b}$$

Adopting the expressions noted above it is clear that the pressure terms are compatible and can be eliminated if the shape functions (and hence the weighting functions) are restricted accordingly

$$\frac{\partial W_r}{\partial z}=\sum_{n=1}^{K}a_{rn}\lambda_n \tag{8}$$

where a_{rn} are arbitrary constants. In this manner the continuity equation (1) can be re-written as

$$\frac{\partial u_j}{\partial x}+\sum_{n=1}^{K}a_{nj}v_n=0 \tag{9}$$

where j is again a simple counter that varies from 1 to K. Following this manipulation it should be noted that equation (9) represents an exact expression for the conservation of mass, subject to the prescribed velocity distributions given in equations 5(a-c).

2.3 MODEL CHARACTERISTICS

Following extensive investigations it has been established that a level III GLN model provides an optimal solution in terms of the accuracy achieved and the computational cost or effort. In the present calculations the adopted shape functions are of polynomial form

$$u(x,z,t) = \sum_{n=1}^{K} u_n(x,t)\lambda_n^{n-1}(z)$$
$$v(x,z,t) = \sum_{n=1}^{K+1} v_n(x,t)\lambda_n^{n-1}(z) \quad \textit{for finite depth} \tag{10}$$
$$v(x,z,t) = \sum_{n=1}^{K} v_n(x,t)\lambda_n^{n-1}(z) \quad \textit{for deep water}$$

In finite water depth the parameter λ is set as $(z-h)/(\zeta-h)$, where h is the water depth. In contrast, in deep water $\lambda = ze^{(\alpha z)/(n-1)}$ where α should be close to the corresponding wave number. A full discussion of this latter point is given in Swan *et al*. (2003).

3. MODELLING OF SURFACE WATER WAVES

3.1 REGULAR WAVES

To examine the success of the model in relation to regular wave propagation, a large number of calculations have been undertaken and, depending on the water depth, comparisons made against appropriate analytical wave models including: (a) High-order Stokes solutions; (b) Stream function theory; (c) Cnoidal wave theory; and (d) Solitary wave theory. In all cases the model reproduced the water surface elevations exactly and provided a good description of the underlying water particle kinematics. For example, Figure 1 concerns a Cnoidal wave propagating up a slope. This case corresponds to the experiment conducted by Hansen & Svendson (1979) and is also reported by Webster & Shields (1991). The wave height is 0.04m, the wave frequency is 0.3Hz, the water depth is 0.36m and the slope is 1 in 34.26. Fig. 1(a) describes the time-history of the water surface elevation, $\zeta(t)$, recorded at different locations along the slope, while Fig. 1(b) describes the variation in the wave height in different water depths along the slope and contrasts the predicted results with the laboratory observations.

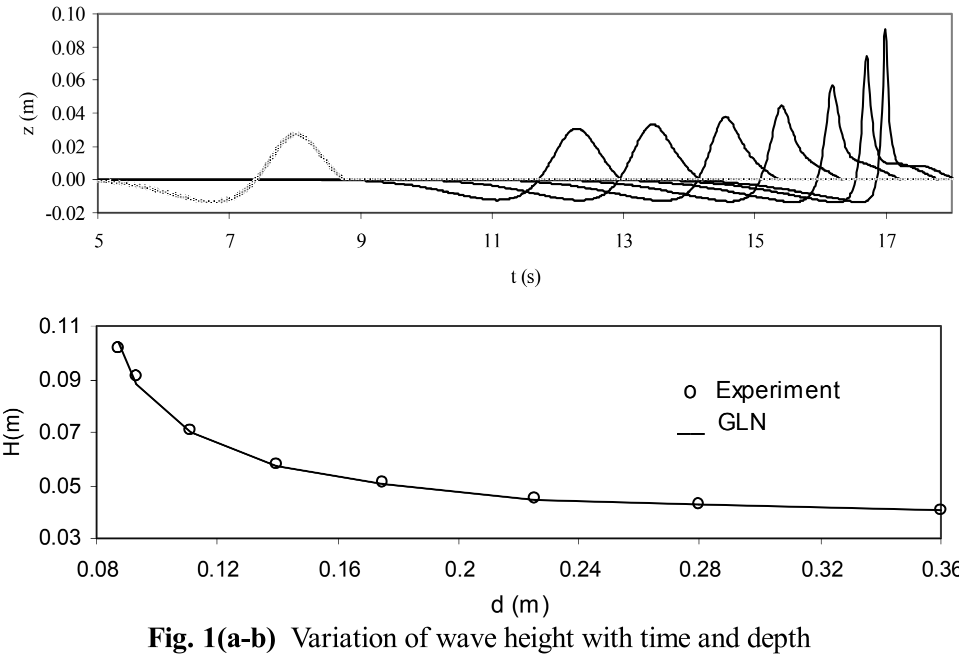

Fig. 1(a-b) Variation of wave height with time and depth

Fig. 2 concerns the case of a linear regular wave propagating in water of depth h=0.3m, past a step of ± 0.05 m located at x_s =30m. The wavelength of the incident wave was 1.325m and the wave height 0.02m. The reflection coefficient K_r and transmission coefficient K_t were determined from the numerical generated water surface elevations. The K_r values were computed to be 0.035 and -0.025 respectively. These correspond closely to the linearly predicted values (Ippen, 1966) of 0.036 and –0.026. Similarly, the transmission coefficients K_t were determined to be of the order of $1+K_r$.

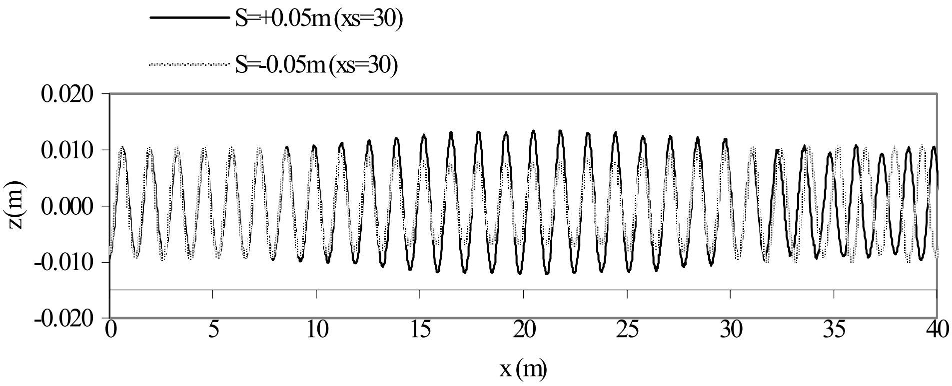

Fig. 2 Spatial profile of regular waves propagating past a step

Fig. 3 shows a series of regular waves propagating onto a steady current. In the region x<5 no current is present. Between 5<x<7 the surface current is increased from 0 to 0.28c, where c is the phase velocity of the wave in the absence of the current. For x>7 the surface current remains constant so that $U_{z=\zeta}$=0.28c. Two cases are presented. The first corresponds to a depth-uniform current, while the second corresponds to a linearly sheared current. In the first case the current stretches the wavelength, with a consequent reduction in the wave amplitude. This result is shown to be in good agreement with linear predictions relating to a depth-uniform current. In the second case the presence of vorticity limits the reduction in the wavelength and therefore also the reduction in the wave amplitude. This is consistent with analytical wave-current models such as Swan & James (2001).

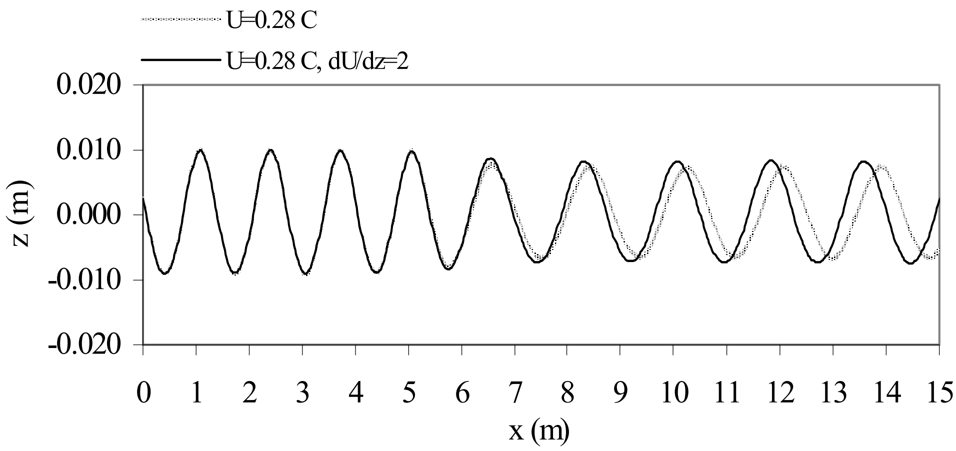

Fig. 3 Spatial profile of regular waves progagating onto current

3.2 Random waves

One of the key advantages of GLN theory is its ability to model the propagation of irregular or random waves. To demonstrate this Fig. 4 concerns a highly nonlinear transient wave group produced by the focusing of freely propagating wave components of differing frequency. In total 3 deep water wave forms are presented. The first concerns the waves generated in the absence of a current (U=0). This case corresponds directly to one of the laboratory test cases reported by Baldock et al. (1996). In this example the linear sum of the amplitudes of the wave components generated at the upstream end is A=55mm. However, due to the nonlinear interactions between the components the largest wave crest corresponds to $\eta_{\max}$=92mm. This value, together with the importance of the nonlinear interactions is clearly reproduced by the GLN theory.

In the remaining two cases the same focussed wave case is generated in the presence of currents, the first depth-uniform and the second exponentially sheared. In the depth-uniform case, U=0.2m/s, the influence of the current is as expected in the sense that the individual waves are stretched, their steepness reduced and the nonlinear interactions less severe. As a result, the corresponding value of $\eta_{\max}$ reduces. In contrast, the case involving an exponentially varying current shows that despite the fact that the surface current and hence the Doppler shift is identical to that in the depth-uniform case, the presence of vorticity acts to increase the nonlinear interactions leading to increased rather than reduced maximum crest elevations. Such results are clearly relevant to offshore engineering design and at present can only be calculated using the GLN theory.

Finally, Fig. 5 provides the first evidence that a GLN theory can be successfully applied to the description of a directionally spread wave field. This case corresponds to a deep water condition in which linear wave components, involving a spread of wave energy in both frequency and direction, propagate and are brought into focus at one point in space and time. This leads to an unsteady wave form that is short-crested. Further comparisons between this formulation of the GLN theory and existing potential wave models capable of incorporating directionality (Bateman et al., 2001) are presently underway.

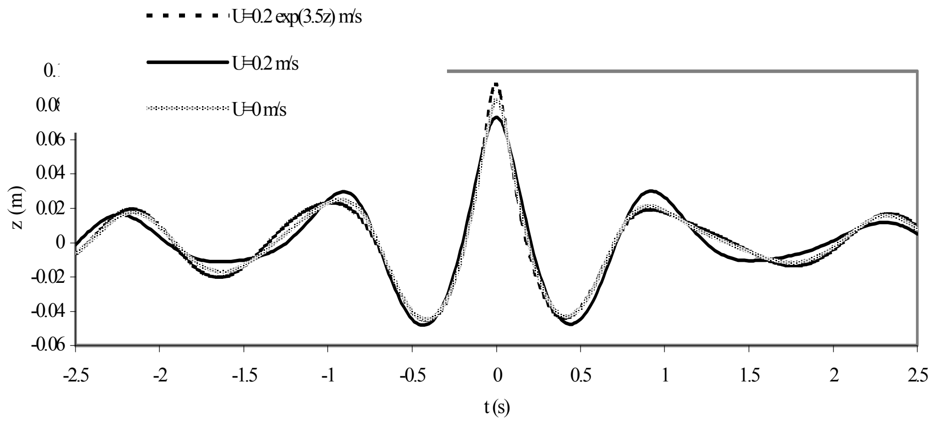

Fig. 4 Temporal variation of free-surface elevation for a uni-directional wave group

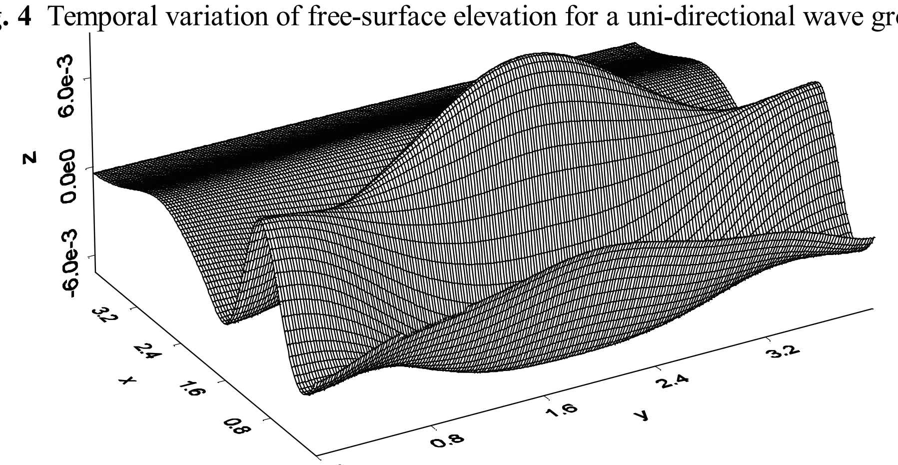

Fig. 5 Spatial profile of maximum crest elevation for a directionally spread wave group

4. CONCLUSIONS

This paper has presented the results of GLN theory applied to a wide range of wave problems. Indeed, the most important feature of the proposed model lies in its versatility. In particular, it can be applied across a full range of water depths, it is appropriate to both steady and unsteady waves including random and focussed wave events, it can deal with variations in bottom topography (both gradual and rapid), it is able to incorporate wave-current interactions including vertically sheared currents with variable vorticity leading to rotational waves and, in its most recent form, is also able to calculate directionally spread wave cases. Computed results demonstrated that the GLN level III model is accurate, robust, computationally efficient and less restricted when compared to many traditional wave models.

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