# NP-completeness for two generalized domination problems*

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**Abstract** We study the complexity of two classes of generalized domination problems: $k$-step domination problem and $k$-distance domination problem. We prove that the decision version of $k$-step domination problem is NP-complete when instances are restricted to chordal graphs or planar bipartite graphs. As corollaries to the results, we obtain new proofs of the NP-completeness of $k$-distance domination problem for chordal graphs and bipartite graphs, and also prove that this problem remains NP-complete even when restricted to planar bipartite graphs.

**Keywords** $k$-step domination, $k$-distance domination, NP-completeness, chordal graphs, planar bipartite graphs

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# 两类广义控制问题的 NP- 完全性

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**摘要** 研究两类广义控制问题的复杂性： $k$- 步长控制问题和 $k$- 距离控制问题，证明了 $k$- 步长控制问题在弦图和平面二部图上都是 NP- 完全的. 作为上述结果的推论，给出了 $k$- 距离控制问题在弦图和二部图上 NP- 完全性的新的证明，并进一步证明了 $k$- 距离控制问题在平面二部图上也是 NP- 完全的.

**关键词** $k$- 步长控制， $k$- 距离控制， NP- 完全性，弦图，平面二部图

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# 0 Introduction

In this paper we in general follow [1] for notation and graph theory terminology. Specifically, let $G = (V, E)$ be a simple graph with vertex set $V$ and edge set $E$, and let $v$ be a vertex in $V$. The (open) neighborhood $N(v)$ of $v$ is defined as the set of vertices adjacent to $v$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The distance $d_G(x, y)$ between two vertices $x$ and $y$ is the length of a shortest $xy$-path in $G$. Let $k$ be a positive integer. For every vertex $v \in V$, the (open) $k$-distance neighborhood $N_{\leqslant k}(v)$ of $v$ is defined as $N_{\leqslant k}(v) = \{u|\ d_G(v, u) \leqslant k\}$. The closed $k$-distance neighborhood $N_{\leqslant k}[v]$ of $v$ is defined as $N_{\leqslant k}[v] = N_{\leqslant k}(v) \cup \{v\}$. The (open) $k$-step neighborhood $N_k(v)$ of $v$ is defined as $N_k(v) = \{u|\ d_G(v, u) = k\}$. The closed $k$-step neighborhood $N_k[v]$ of $v$ is defined as $N_k[v] = N_k(v) \cup \{v\}$.

Given a graph $G = (V, E)$, a vertex $v \in V$ is said to dominate all vertices in its closed neighborhood $N[v]$. A subset $D \subseteq V$ is called a dominating set of $G$ if any vertex in $G$ is dominated by a vertex in $D$. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set of $G$. The domination problem is to find a minimum dominating set of $G$.

Now two types of generalizations of the concept of dominating set are given as follows. Given a graph $G = (V, E)$, a vertex $v \in V$ is said to $k$-distance dominate all vertices in its closed $k$-distance neighborhood $N_{\leqslant k}[v]$. A subset $D \subseteq V$ is called a $k$-distance dominating set of $G$ if any vertex in $G$ is $k$-distance dominated by a vertex in $D$. The $k$-distance domination number $\gamma_{\leqslant k}(G)$ of $G$ is the minimum cardinality of a $k$-distance dominating set of $G$. A $k$-distance dominating set with cardinality $\gamma_{\leqslant k}(G)$ is also called a $\gamma_{\leqslant k}(G)$-set. The $k$-distance domination problem is to find a minimum $k$-distance dominating set of $G$. From the definition above, we can find that a dominating set is a 1-distance dominating set, and thus $\gamma(G) = \gamma_{\leqslant 1}(G)$.

Another generalization is as follows. Given a graph $G = (V, E)$, a vertex $v \in V$ is said to $k$-step dominate all vertices in its closed $k$-step neighborhood $N_k[v]$. A subset $D \subseteq V$ is called a $k$-step dominating set of $G$ if every vertex in $G$ is $k$-step dominated by a vertex in $D$. The $k$-step domination number $\gamma_k(G)$ of $G$ is the minimum cardinality of a $k$-step dominating set of $G$. A $k$-step dominating set with cardinality $\gamma_k(G)$ is also called a $\gamma_k(G)$-set. The $k$-step domination problem is to find a minimum $k$-step dominating set of $G$. From the definition above, we can find that a dominating set is a 1-step dominating set, and thus $\gamma(G) = \gamma_1(G)$.

There are many applications of the above generalizations, and an interpretation in terms of communication networks is presented by [2-3] as follows. If $V$ represents a collection of cities and an edge represents a communication link, then one may be interested in selecting a minimum number of cities as sites for transmitting stations so that every city either contains a transmitter or can receive messages from at least one of the transmitting stations through the links. If only direct transmissions are acceptable, then one wishes to find a minimum 1-dominating set. If communication over paths of $k$ links (but not of $k + 1$ links) is adequate in quality and rapidity, the problem becomes that of determining a minimum

$k$-distance dominating set, i.e. a $k$-distance dominating set with the fewest possible vertices. If we further require that every city not selected is at distance $k$ from at least one of the transmitting stations in the above communication networks, then the problem becomes that of determining a minimum $k$-step dominating set, i.e. a $k$-step dominating set with the fewest possible vertices.

The concept of $k$-distance domination number was introduced by [4], and has been well studied by various authors such as [5-9]. The concept of $k$-step domination number can be seen in some early papers such as [8,10]. Both parameters were summarized in [1,11]. In [5], it was proved that $k$-distance domination problem is NP-complete for chordal graphs and for bipartite graphs. In the present paper, we prove that $k$-step domination problem is NP-complete for chordal graphs and planar bipartite graphs. As a consequence, we obtain new proofs of the NP-completeness of $k$-distance domination problem for chordal graphs and bipartite graphs, and prove that this problem remains NP-complete even when restricted to planar bipartite graphs.

# 1 NP-completeness results

Our proof in this section uses reductions from the following NP-complete problem.

**Exact Cover By 3-Sets**[12]

**Instance**: A finite set $X = \{x_1, x_2, \ldots, x_{3q}\}$ of cardinality $3q$, for some positive integer $q$, and a set $C = \{C_1, C_2, \ldots, C_m\}$ of 3-element subsets of $X$.

**Question**: Does $C$ contain an exact cover for $X$, that is, a subset $\widetilde{C} \subseteq C$ such that every element of $X$ occurs in exactly one 3-element subset of $\widetilde{C}$.

The decision versions for $k$-distance domination and $k$-step domination are stated as follows.

**$k$-Distance Domination**

**Instance**: A graph $G = (V, E)$ and a positive integer $K \leqslant |V|$.

**Question**: Does $G$ have a $k$-distance dominating set of size $\leqslant K$?

**$k$-Step Domination**

**Instance**: A graph $G = (V, E)$ and a positive integer $K \leqslant |V|$.

**Question**: Does $G$ have a $k$-step dominating set of size $\leqslant K$?

**Theorem 1** *For any fixed positive integer $k$, $k$-step domination is* NP*-complete when instances are restricted to chordal graphs or bipartite graphs.*

**Proof** First, $k$-step domination obviously belongs to NP. Secondly, given an instance of exact cover by 3-sets, we construct graph $G(C) = (V, E)$ as follows. The construct is similar to that in [13]. For each element $x_i \in X$, we create a vertex $x_i$ in $V$. For each 3-element subset $C_j$ in $C$, we create a path on $2k+1$ vertices, labelled $c_{j,1}, c_{j,2}, \ldots, c_{j,2k+1}$. We then add edges so that the set of vertices $U = \{c_{1,1}, c_{2,1}, \ldots, c_{m,1}\}$ induces a complete graph (we omit the edges in $G[U]$ in Fig.1). We also add $k$-length paths between the vertices labelled $c_{j,1}$ and the three vertices corresponding to the 3-element subset $C_j$ (these $k$-length paths are represented by dash lines in Fig.1). Note that the graph $G(C)$ is a chordal graph as the vertices in $U$ form a complete subgraph, and note that obviously the graph $G(C)$ can be constructed in polynomial time. We shall show that $C$ contains an exact cover for $X$ (for example, the set $\{C_4, C_5\}$ in Fig. 1) if and only if the chordal graph $G(C)$ has a $k$-step dominating set of size $K = km + q$ (for example, the set of $K = 5k + 2$ shaded vertices in Fig.1).

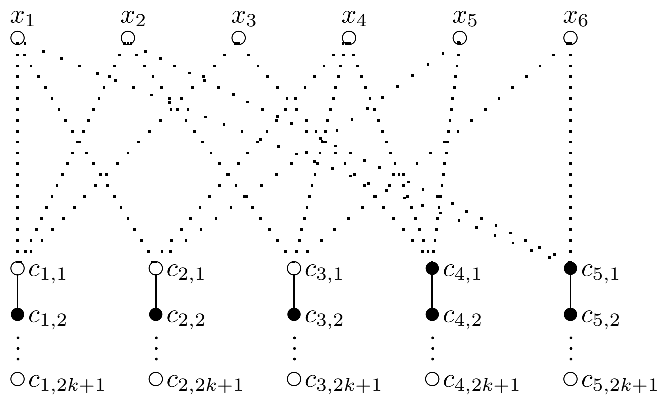


Figure 1: NP-completeness for chordal graphs and bipartite graphs

Suppose first that $C$ contains an exact cover for $X$, that is, a subset $\widetilde{C} \subseteq C$ such that every element of $X$ occurs in exactly one 3-element subset of $\widetilde{C}$. Construct a set $D \subseteq V(G)$ as follows: let

$$D = \{c_{j,1} |\ C_j \in \widetilde{C}, j \in \{1, 2, \cdots, m\}\} \bigcup_{j=1}^{m} \{c_{j,2}, c_{j,3}, \cdots, c_{j,k+1}\}.$$

The set $D$ is clearly a $k$-step dominating set of $G(C)$, and $|D| = km + q$.

Conversely, suppose that the chordal graph $G(C)$ has a $k$-step dominating set $D$ of size $km + q$. We shall show that $C$ contain an exact cover for $X$. For convenience, we take $C_1 = \{x_1, x_2, x_3\}$ for example (see Fig.1, suppose the vertices between the vertices $x_i$ and $c_{1,1}$ are $v_{i,1}, v_{i,2}, \cdots, v_{i,k-1}$ for each $i \in \{1, 2, 3\}$). Note first that, to $k$-step dominate the vertices in $P_{1i} = \{v_{i,1}, v_{i,2}, \cdots, v_{i,k-1}\} \cup \{c_{1,1}, c_{1,2}, \cdots, c_{1,2k+1}\}$ for each $i \in \{1, 2, 3\}$, $D$ must contain at least $k$ vertices of $P_{1i}$. Also note that, if $D$ contains exact $k$ vertices of $P_{1i}$ for some $i \in \{1, 2, 3\}$, then $D$ contains no vertex of $\{x_1, x_2, x_3\}$. Similar conclusions hold for $C_2, \cdots, C_m$. Let $D'$ denote the set of vertices in both $D$ and all $P_{ji}$ for $i \in \{1, 2, \cdots, n\}, j \in \{1, 2, \cdots, m\}$. Thus $|D'| \geqslant km$.

Further, it is easy to see that if $|D'| = km$ then $D'$ must contain exactly $k$ vertices in each set $Q_j = \{c_{j,2}, \cdots, c_{j,2k+1}\}$ for each $j \in \{1, 2, \cdots, m\}$, and thus $D'$ cannot $k$-step dominate any vertex of $X$ in $G(C)$, and therefore there must exist at least $q$ vertices of $U$ contained in $D$ to $k$-step dominate all vertices of $X$ in $G(C)$. In fact, since $|D| = km + q$, $D$

must contain exactly $k$ vertices in each set $Q_j$, and contain exactly $q$ vertices of $U$ to $k$-step dominate all vertices of $X$ in $G(C)$. This means that $C$ contains an exact cover for $X$.

Now, if we let $U$ be a stable set in the construction of $G(C)$, then $G(C)$ is a bipartite graph. By similar arguments to those above we can prove that $C$ contains an exact cover for $X$ if and only if the bipartite graph $G(C)$ has a $k$-step dominating set of size $K = km + q$. This completes the proof of Theorem 1.

Now let us further study the properties of the bipartite graph $G(C)$ constructed in Theorem 1. For convenience, we call every element of $C$ a clause. Note that if the clauses in $C$ are pairwise distinct, then no copy of $K_{3,3}$ appears as a subgraph of $G(C)$. If a clause is duplicated, it is easy to see that combining all copies of this clause into a single vertex of $G(C)$ does not change the result in Theorem 1. Note also that no subgraph of $G(C)$ is isomorphic to $K_5$. Then, by Kuratowski's theorem, we have the following further result.

**Theorem 2** *For any fixed positive integer $k$, $k$-step domination is* NP*-complete even when instances are restricted to planar bipartite graphs.*

Chang and Nemhauserthey[5] proved that $k$-distance domination problem is NP-complete for chordal graphs and for bipartite graphs. Two reductions in [5] were used: the reduction of the 1-domination problem on a general graph to $k$-distance domination problem on a bipartite graph; the reduction of the 1-domination problem on general graphs to the $k$-distance domination problem on chordal graphs.

It is noticeable that, by using the same construction as in Theorem 1 we can obtain new proofs of NP-completeness of $k$-distance domination problem on chordal graphs and bipartite graphs. Moreover, we shall prove that $k$-distance domination problem is NP-complete for planar bipartite graphs, a proper subclass of both planar graphs and bipartite graphs.

**Corollary 1** [5] *For any fixed positive integer $k$, $k$-distance domination is* NP*-complete for chordal graphs or bipartite graphs.*

**Proof** First, we construct the same chordal (bipartite) graph $G(C)$ as that in Theorem 1. Second, by a similar argument to that in Theorem 1, we can prove that $C$ contains an exact cover for $X$ (for example, the set $\{C_4, C_5\}$ in Fig. 1) if and only if the chordal (bipartite) graph $G(C)$ has a $k$-distance dominating set of size $K = m + q$ (for example, the set $\{c_{1,k+1}, c_{2,k+1}, c_{3,k+1}, c_{4,k+1}, c_{5,k+1}, c_{4,1}, c_{5,1}\}$ in Fig. 1).

For the same reason as in the proof of Theorem 2, we get the following strengthened result.

**Theorem 3** *For any fixed positive integer $k$, $k$-distance domination is* NP*-complete even when restricted to planar bipartite graphs.*

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