A Model for a Linked System of Mulit-Purpose Reservoirs with Stochastic Inflows and Demands

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A MODEL FOR A LINKED SYSTEM OF MULTI-PURPOSE
RESERVOIRS WITH STOCHASTIC INFLOWS AND DEMANDS

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FOREWARD

The primary objective of this research project is to develop techniques for the optimal operation of a linked system of multi-purpose reservoirs. Linkage of the system may be through normal river reaches, canals, or through pumping in pipelines. In this report a model is developed which utilizes stochastic inflows with the total system subject to certain constraints. This model will be utilized later in an operational study of an existing system.

ABSTRACT

In Chapter I of this report a model of a single multi-purpose
reservoir with stochastic inflows is addressed. The objective of
the model is the development of an optimal operating policy for
given time sequence of minimum and maximum reservoir levels. The
unregulated inflow into the reservoir is assumed to be stochastic
with known distribution for each time period.

The significant difference between this model and those of
previous investigators is that no linear decision rule is utilized.
Instead, the approach is based on the distribution of the sum of
the inflows over successive time periods. The resultant reservoir
release variables are no longer stochastic values as they were in
previous models.

The resultant constraint set forms a linear system of equations.
Stochastic demands as well as inflows also are considered in the
paper. Example problems are presented to illustrate the models.

In Chapter II, a single multi-purpose reservoir model with
stochastic inflows is extended to a connected system of such reser-
voirs. The reservoirs are considered to be linked by a system of
pumping canals and normal river reaches. The objective of the
model is the optimal operation of the total system subject to certain
restrictions on reservoir operations.

The linked system model is a natural extension of the single
reservoir model proposed in Chapter I. The resulting constraints
for the problem are linear and the decision variables are deter-
ministic rather than random variables. Thus, linear, quadratic
or even general convex objective functions can be handled readily.

ACKNOWLEDGMENTS

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CHAPTER I

A STOCHASTIC MODEL FOR A SINGLE MULTI-PURPOSE RESERVOIR

Operations research methodology and systems analysis has in the last decade facilitated the water resource planner in the development of reservoir management techniques. The Planner must integrate the many functions of a reservoir in order to obtain decision policies.

In a recent paper ReVelle et al. (1969) proposed a linear decision rule for a single reservoir design and operation. The linear decision rule permitted the structure of a chance-constrained linear programming model for determining the reservoir capacity required to maintain a range of storage volumes and releases during specified time periods.

ReVelle's linear decision rule, as applied to a reservoir, has the simple form

$$x = s - b,$$

where x is the release during a period of reservoir operation; s is the storage at the end of the previous period; and b is a decision parameter chosen to optimize some criterion function. The linear decision rule was applied in two contexts: (1) the stochastic contexts where the magnitudes of reservoir inputs are treated as random variables unknown in advance and (2) the deterministic contexts where the magnitude of each input in a sequence is specified in advance.

This article is an extension of the work of ReVelle for reservoir modeling. The emphasis will be mainly on stochastic systems since the deterministic case is merely a special case of the

stochastic model. The linear decision rule discussed by both ReVelle and Loucks (1970) is not utilized in this model. By not restricting the formulation to linear decision rules, several advantages arise. Paramount among these are the ability to include the release quantities x_t in the objective function, the extension to a linked multiple reservoir system is readily obtained, and the inclusion of stochastic as opposed to deterministic demands adds no conceptional difficulties. This approach is applied initially to a single multi-purpose reservoir. The important case of systems of linked reservoirs will be taken up in a subsequent chapter.

Single Multi-purpose Reservoir

In this section a single multi-purpose reservoir with chance-constraints is modeled based on a formulated continuity or material balance equation. The formulation provides decisions that specify the release during different time periods of reservoir operation. The decisions for the entire time horizon are determined by solving a linear programming problem. The linear programming problem is the deterministic equivalent of the original stochastic system. The continuity equation consists of the reservoir inventory for the previous period, random inflow, deterministic demands, and scheduled releases.

Chance-constraints for each time period are established. Chance-constrained means that the specified constraints may not be satisfied all the time, but will be satisfied at least some pre-specified amount. The purpose of utilizing the chance-constrained formulation is the convenience in which the random variables can be handled in the constraints. The stochastic constraints selected involve maintaining (1) a specified maximum capacity minus a time requirement variable for upper storage space and (2) a time dependent minimum pool level.

The chance-constraints contain random inflow for each time period. The random inflows are assumed to be additive and essentially independent from one time period to the next. By making this assumption a density function for the sum of the independent random variables is obtained by convolution. The independence assumption is not necessary. It does, however, simplify the presentation of this approach and hence will be adheared to in further discussion. By using the convoluted random variables for each constraint, a deterministic set of equivalent linear constraints is generated.

Objective functions are then appended to this mathematical formulation for analysis of various decision policies. Both linear and quadratic objective function forms subsequently will be discussed. More general convex objective functions with linear constraints also can be handled readily (Rosen, 1961 and Goldfarb, 1969). However, the size of the general problem which can be solved routinely is much smaller than the more specialized linear and quadratic forms.

Continuity Equation

The continuity equation is based on the reservoir model shown in Figure 1.
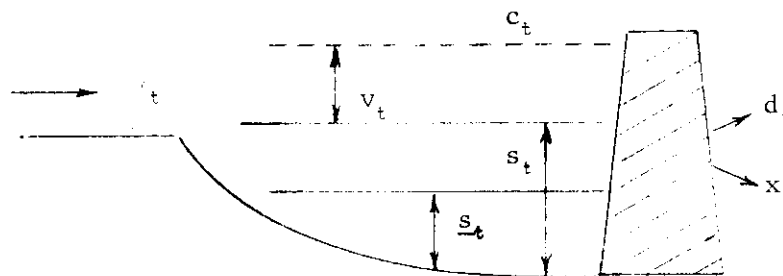
Figure 1: Single Multi-purpose Reservoir

The total unregulated flow γ_t enters the reservoir in time period t. The inflow is randomly distributed with a probability density function (p.d.f.) $f_t(\gamma_t)$. Therefore, the inflow in a particular period is known only with some probability. The inflow plus the storage volume s_{t-1} at end of the previous time period is available for downstream release x_t, and extracted demands d_t.

The current ending storage volume or inventory level s_t is then expressed as

$$(1) \qquad s_t = e_t\, s_{t-1} + \gamma_t - d_t - x_t,$$

where e_t is the fraction of water remaining contingent upon losses due to evaporation.

Chance-Constraints with Stochastic Inflows

The chance-constraint for the probability of not exceeding the specified maximum capacity of the reservoir is given by

$$(2) \qquad P\left\{ s_t \leq c_t - v_t \right\} \geq \alpha_1,$$

where c_t is the total capacity of the reservoir below the maximum water level, v_t is the upper storage space which may be required in time period t (this space might be reserved for flood control or surcharge storage), and α_1 is a specified value between zero and one. The value of α_1 is normally chosen to be reasonably close to one. The complementary probability $(1 - \alpha_1)$ represents the allowable risk that the random variable γ_t will take on values such that

$$s_t > c_t - v_t.$$

The probability at the end of time period t that storage s_t exceeds the minimum pool level $\underline{s}_t$ is written as

$$(3) \qquad\qquad P\left\{\, s_t \geq \underline{s}_t \,\right\} \geq \alpha_2 \,,$$

where α_2 is a preselected minimum allowable probability that must be maintained. The complementary probability $(1 - \alpha_2)$ represents the allowable risk that the random variable γ_t will take on values such that

$$s_t < \underline{s}_t \,.$$

The downstream release x_t must satisfy minimum $\underline{x}_t$ and maximum $\bar{x}_t$ reservoir release constraints

$$(4) \qquad\qquad \underline{x}_t < x_t \leq \bar{x}_t \,,$$

which might be established by water requirements for water control dilution, or recreational purposes.

Convolution Formulation. The chance-constraints can be converted to an equivalent set of linear constraint by convolution. The operation of obtaining the density function of the sum of two independent random variables is called convolution (Feller, 1966). The resulting p.d.f. is usually indicated as $g_o = f_1 * f_2$.

For the case of two continuous random variables x and y with a joint p.d.f. f(x,y), where $-\infty < x < \infty$ and $-\infty < y < \infty$, it is required to determine the p.d.f. of $s = x + y$. To accomplish this it is necessary to consider the cumulative density function. (C.D.F.) of s,

$$G(s) = P\lceil x + y \leq s \rceil \text{ or}$$

thus,

$$G(s) = \int \int f(x,y)\, dx dy$$

$$\text{in which } -\infty < x < \infty, \ -\infty < y < \infty,$$

and integrated over the range $x + y \leq s$.

Changes in the limits of integration are then made to the C.D.F. and

$$G(s) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{s-x} f(x,y)dy \right] dx.$$

Now, $G(s)$ is differentiated with respect to s and the p.d.f. of s is obtained

$$g(s) = \int_{-\infty}^{+\infty} f(x, s-x) dx.$$

When x and y are independent, then

$$f(x,y) = f_1(x) f_2(y)$$

and the resulting p.d.f. of s denoted by $g_e(s)$, is

$$g_e(s) = \int_{-\infty}^{+\infty} f_1(x) f_2(s-x) dx.$$

For the case where it is required to obtain the convolution of three or more random variables the formulas are applied recursively until the density function of the total sum is obtained. As an example, let

$$s = x_1 + x_2 + \ldots + x_n,$$

where x_1, x_2, ..., x_n are independent random variables. Then to obtain the p.d.f. of s, start first by obtaining the p.d.f. of $s_2 = x_1 + x_2$. Next with regard to $s_3 = s_2 + x_3$ and continue in this manner until $s_n = s_{n-1} + x_n$.

The convolution operation also can be extended to the discrete case, that is,

$$p_o(s) = \sum_{all\ x} p_1(x)\ p_2(s-x).$$

As an example of the discrete case, consider the density function of the random inflow Y_t into a reservoir over a certain time period t given by

Y_t	0	1	2
$p(Y_t)$	.2	.3	.5

,

where Y_t is the number of units of inflow (usually expressed in day-second-feet or acre-feet) into the reservoir for the time period. Assuming that this distribution is the same for each time period and that each time period is statistically independent, find the distribution of random inflow for two time periods. By the application of the convolution formula, the discrete case yields,

$$p(0) = p_1(0)\ p_2(0) = .2 \times .2 = .04$$

$$p(1) = p_1(0)\ p_2(1) + p_1(1)\ p_2(0)$$
$$= .2 \times .3 + .3 \times .2 = .12$$

$$p(2) = p_1(0)\ p_2(2) + p_1(1)\ p_2(1) + p_1(2)\ p_2(0)$$
$$= .2 \times .5 + .3 \times .3 + .5 \times .2 = .29$$

$$p(3) = p_1(1)\ p_2(2) + p_1(2)\ p_2(1)$$
$$= .3 \times .5 + .5 \times .3 = .30$$

$$p(4) = p_1(2)\ p_2(2) = .5 \times .5 = .25$$

The distribution of the random inflow for the two time periods is

$Y_1 + Y_2$	0	1	2	3	4
$p(Y_1 + Y_2)$	.04	.12	.29	.30	.25

Deterministic Equivalent. Chance-constraint (2) can now be converted to an equivalent linear constraint. For time period $t = 1$, substitute the continuity equation (1) into constraint (2) to yield

$$P \{ e_1 \, s_0 + Y_1 - d_1 - x_1 \leq c_1 - v_1 \} \geq \alpha_1 .$$

The random variable Y_1 is taken to the right-hand side of the constraint and the inequality sign reversed to give

$$P \{ c_1 - v_1 - e_1 \, s_0 + d_1 + x_1 \geq Y_1 \} \geq \alpha_1 .$$

Since Y_1 has a known p.d.f., the C.D.F. F_{Y_1} evaluated at the argument

$$[c_1 - v_1 - e_1 \, s_0 + d_1 + x_1]$$

must be greater than or equal to α_1. Thus

$$F_{Y_1} [c_1 - v_1 - e_1 \, s_0 + d_1 + x_1] \geq \alpha_1 .$$

For specified α_1, the chance-constraint becomes

$$c_1 - v_1 - e_1 \, s_0 + d_1 + x_1 \geq (R_1)^{\alpha_1} ,$$

where $(R_1)^{\alpha_1}$ is the value of γ_1 from the cumulative distribution F_{γ_1} such that only 100 $(1-\alpha_1)$ per cent of the random values of γ_1 are greater than the argument.

For $t = 2$, constraint (2) is

$$P\left\{ e_2\ s_1 + \gamma_2 - d_2 - x_2 \le c_2 - v_2 \right\} > \alpha_1 .$$

Substituting for s_1 from the continuity equation (1),

$$P\left\{ e_2\ e_1\ s_0 + e_2\ \gamma_1 - e_2\ d_1 - e_2\ x_1 + \gamma_2 - d_2 - x_2 \le c_2 - v_2 \right\} \ge \alpha_1 .$$

Grouping

$$P\left\{ e_2\ e_1\ s_0 - (e_2\ d_1 + d_2) - (e_2\ x_1 + x_2) - c_2 + v_2 \le \right.$$

$$\left. - (e_2\ \gamma_1 + \gamma_2) \right\} \ge \alpha_1$$

and reversing the inequality sign the constraint becomes

$$P\left\{ c_2 - v_2 - e_2\ e_1\ s_0 + (e_2\ x_1 + x_2) + e_2\ d_1 + d_2) \ge \right.$$

$$\left. (e_2\ \gamma_1 + \gamma_2) \right\} \ge \alpha_1 .$$

The C.D.F.

$$F_{e_2\gamma_1 + \gamma_2}\left[(c_2 - v_2 - e_2\ e_1\ s_0) + (e_2\ x_1 + x_2) \right.$$

$$\left. + (e_2\ d_1 + d_2) \right],$$

which is obtained by convoluting the p.d.f.'s of $e_2\ \gamma_1$ and γ_2, evaluated at the argument must be greater than or equal to α_1.

Again for specified α_1, the chance-constraint becomes

$$(c_2 - v_2 - e_2\, e_1\, s_0) + (e_2\, x_1 + x_2) + (e_2\, d_1 + d_2) >$$

$$(e_2\, R_1 * R_2)^{\alpha_1},$$

which is linear in x_1 and x_2. The expression $(e_2\, R_1 * R_2)^{\alpha_1}$ represents a value on the convoluted cumulative distribution of the random variables γ_1 and γ_2 evaluated at the point α_1.

Then for the general nth time period, defining $e_{n+1} = 1$,

$$(5) \quad P\left\{ s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) (\gamma_t - d_t - x_t) \le c_n - v_n \right\} \ge \alpha_1.$$

The sum of the random variables is taken to the right-hand side of the constraint to give

$$P\left\{ s_0 \prod_{t=1}^{n} e_t - \sum_{t=1}^{n} \left[\left(\prod_{k=t+1}^{n} e_k \right) (d_t + x_t) \right] - c_n + v_n \right.$$

$$\left. \le - \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) \gamma_t \right\} \ge \alpha_1$$

Rearranging,

$$P\left\{ c_n - v_n - s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left[\left(\prod_{k=t+1}^{n} e_k \right) (d_t + x_t) \right] \right.$$

$$\left. \ge \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) \gamma_t \right\} \ge \alpha_1.$$

The C.D.F. for $\xi_n = \sum_{t=1}^{n} (\prod_{k=t+1}^{n} e_k) \gamma_t$ evaluated at the argument is

$$F_{\xi_n} \left\{ c_n - v_n - s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left[(\prod_{k=n+1}^{n} e_k) (d_t + x_t) \right] \right\},$$

which must be greater than or equal α_1. By specifying α_1 the chance-constraint becomes

$$(c_n - v_n - s_0 \prod_{t=1}^{n} e_t) + \sum_{t=1}^{n} \left[(\prod_{k=n+1}^{n} e_k) (d_t + x_t) \right] \geq$$

$$\sum_{t=1}^{n} \left\{ \prod_{k=t+1}^{n} e_k \right\} \gamma_t$$

or

$$(6) \quad (c_n - v_n - s_0 \prod_{t=1}^{n} e_t) + \sum_{t=1}^{n} \left[(\prod_{k=t+1}^{n} e_k) (d_k + x_t) \right]$$

$$\geq (R_{*_n})^{\alpha_1},$$

where $(R_{*_n})^{\alpha_1}$ is the value at the α_1 point on the cumulative of the convoluted distribution.

Constraint (6) is the deterministic equivalent of (2), where the release quantities x_t are the decision variables. All other variables are state variables selected by the water resource planner.

Chance-constraint (3) is next converted to an equivalent linear constraint for two time periods followed by the general constraint for the nth time period.

Constraint (3) is

$$P\{ s_t \geq \underline{s}_t \} \geq \alpha_2,$$

and for $t = 1$

$$P\{ s_1 \geq \underline{s}_1 \} \geq \alpha_2.$$

Then, substitution of (1) into (3) yields

$$P\{ e_1 s_0 + \gamma_1 - d_1 - x_1 \geq \underline{s}_1 \} \geq \alpha_2 .$$

Taking the random variable to the right-hand side

$$P\{ e_1 s_0 - d_1 - x_1 - \underline{s}_1 \geq -\gamma_1 \} \geq \alpha_2$$

and reversing the inequality yields

$$P\{ (\underline{s}_1 - e_1 s_0) + d_1 + x_1 \leq \gamma_1 \} \geq \alpha_2.$$

The deterministic equivalent for this equation, one minus the C.D.F., F_{γ_1}, evaluated at the argument

$$[(\underline{s}_1 - e_1 s_0) + d_1 + x_1]$$

must be greater than or equal to α_2. Thus,

$$1 - F_{\gamma_1} [(\underline{s}_1 - e_1 s_0) + d_1 + x_1] \geq \alpha_2$$

or

$$F_{\gamma_1} \cdot [\, (\, \underline{s_1} - e_1 \, s_0 \,) + d_1 + x_1 \,] \leqslant 1 - \alpha_2 \; .$$

For specified α_2 the chance-constraint becomes

$$(\, \underline{s_1} - e_1 \, s_0 \,) + d_1 + x_1 \leqslant (R_1)^{\, 1 - \alpha_2} \; ,$$

where $(R_1)^{\, 1 - \alpha_2}$ is the value of $(1 - \alpha_2)$ from the cumulative distribution F_{γ_1} .

For $t = 2$, constraint (3) is

$$P \, \{ \, s_2 \geq \underline{s_2} \, \} \geq \alpha_2 \; .$$

Substitution of continuity equation (1) twice yields

$$P \, \{ \, -e_2 \, x_1 + e_2 \, \gamma_1 - e_2 \, d_1 + e_2 \, e_1 \, s_0 + \gamma_2 - d_2 - x_2$$

$$\geq \underline{s_2} \, \} \geq \alpha_2 \; .$$

Regrouping and reversing the inequality yields

$$P \, \{ \, (\, \underline{s_2} - e_2 \, e_1 \, s_0 \,) + (e_2 \, d_1 + d_2 \,) + (\, e_2 \, x_1 + x_2 \,)$$

$$\leqslant (\, e_2 \, \gamma_1 + \gamma_2 \,) \, \} \geq \alpha_2 \; .$$

As before, to obtain the deterministic equivalent for this equation, one minus the C.D.F. evaluated at the argument must be greater than or equal to α_2,

$$1 - F_{e_2 \, \gamma_1 \, + \, \gamma_2} \, [\, (\, \underline{s}_2 - e_2 \, e_1 \, s_0 \,) + (\, e_2 \, d_1 + d_2 \,)$$

$$+ (\, e_2 \, x_1 + x_2 \,) \,] \geq \alpha_2 \, .$$

Letting $(\, e_2 \, R_1 \, \ast \, R_2 \,)^{1 - \alpha_2}$ represent the $(1 - \alpha_2)$ value on the cumulative distribution $F_{e_2 \, \gamma_1 \, + \, \gamma_2}$, the deterministic constraint is

$$(\, \underline{s}_2 - e_2 \, e_1 \, s_0 \,) + (\, e_2 \, d_1 + d_2 \,) + (\, e_2 \, x_1 + x_2 \,) \leq (\, e_2 \, R_1 \, \ast \, R_2 \,)^{1 - \alpha_2}.$$

The generalized formulation of equation (3) follows that of equation (2) and the convolution of the sum of the random variables ξ_n is in general:

$$\underline{s}_n - s_0 \, \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\, \prod_{k=t+1}^{n} e_k \right) (\, d_t + x_t \,)$$

$$\leq \left\{ \, \mathop{\ast}_{t=1}^{n} \left(\, \prod_{k=t+1}^{n} e_k \right) R_t \, \right\}^{1 - \alpha_2}$$

or

$$(7) \quad (\underline{s}_n - s_0 \, \prod_{t=1}^{n} e_t \,) + \sum_{t=1}^{n} \left(\, \prod_{k=t+1}^{n} e_k \, \right) (\, d_t + x_t \,) \leq (\, R_{\ast_n})^{1 - \alpha_2} \, .$$

Constraint (7) is the linear deterministic equivalent of (3) with the release x_t ($t = 1, 2, \ldots, n$) being the decision variables necessary to insure that the storage s_n exceeds the minimum pool level $\underline{s}_n$ with a probability α_2.

Stochastic Inflows and Demands

The development for both stochastic inflows and stochastic demands is similar to that for stochastic inflows. The main difference is that the convolution of the inflows minus the demands must now be obtained.

Consider the general stochastic-constraint on reservoir capacity (5),

$$P\left\{ s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) (\gamma_t - d_t - x_t) \leq c_n - v_n \right\} \geq \alpha_1 .$$

The sum of the random variables γ_t and d_t is again taken to the right-hand side of the constraint and, upon rearranging,

$$(8) \quad P\left\{ c_n - v_n - s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) x_t \geq \right.$$

$$\left. \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) (\gamma_t - d_t) \right\} \geq \alpha_1 .$$

The distribution of the random variable

$$\xi_n = \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) (\gamma_t - d_t)$$

must now be obtained. This can be accomplished by convolution or by application of Fourier transforms (Parzen, 1960). Letting F_{ξ_n} represent the distribution function (C.D.F.) of ξ_n, equation (8) becomes

$$F_{\xi_n} \left[c_n - v_n - s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) x_t \right] \geq \alpha_1 .$$

Thus,

$$(9) \quad c_n - v_n - s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) x_t \geq R^{\alpha_1}_{(\gamma^*_d)_n} \quad ,$$

where $R^{\alpha_1}_{(\gamma^*_d)_n}$ represents the value of the random variable ξ_n for which $100\alpha_1$ per cent of the area of the distribution is to the left of ξ_n, or equivalently the α_1 point on the C.D.F. F_{ξ_n}.

Constraint (9) is the deterministic equivalent of equation (2) when both the demands and inflows are stochastic. Similarly, equation (3) becomes,

$$(10) \quad \underline{s}_n - s_0 \prod_{t=1}^{n} e_t + \sum_{t=1}^{n} \left(\prod_{k=t+1}^{n} e_k \right) x_t \leq R^{1-\alpha_2}_{(\gamma^*_d)_n} \quad .$$

The minimum and maximum constraints on releases remain as before. Hence, for stochastic inflows and demands, the constraints on the system are equations (9), (10), and (4).

Reservoir Models and Solutions

In this section three example problems are solved to illustrate the chance-constraint formulation. The first two example problems are solved by linear programming. One of the problems assumes stochastic inflow and the other example both stochastic inflow and demand. The third example problem appends a quadratic objective function to the original model, which is then solved by quadratic programming.

Linear Objective Function with Stochastic Inflow

In the single multi-purpose reservoir example, it is assumed
that the objective is to maximize the profit of releasing x_t units
of water for two time periods. The releases are subject to the
chance-constraints

$$P \{ s_t < c_t - v_t \} \geq \alpha_1$$

and

$$P \{ s_t \geq \underline{s}_t \} \geq \alpha_2$$

from which the equivalent deterministic constraints (6) and (7)
were derived.

For two time periods, constraints (6) and (7) are, respectively,

$$t=1 \begin{cases} - s_0 \, e_1 + d_1 + x_1 + c_1 - v_1 > R_1^{\alpha_1} \\ \\ - s_0 \, e_1 + d_1 + x_1 + \underline{s}_1 < R_1^{1-\alpha_2} \end{cases}$$

and

$$t=2 \begin{cases} - s_0 \, e_1 \, e_2 + e_2 \, x_1 + x_2 + e_2 \, d_1 + d_2 + c_2 - v_2 > (e_2 \, R_1 * R_2)^{\alpha_1} \\ \\ - s_0 \, e_1 \, e_2 + e_2 \, x_1 + x_2 + e_2 \, d_1 + d_2 + \underline{s}_2 < (e_2 \, R_1 * R_2)^{1-\alpha_2} \end{cases}$$

The maximum and minimum release constraints also must be satisfied:

$$t=1 \quad \begin{cases} x_1 \geq \underline{x_1} \\ x_1 \leq \bar{x_1} \end{cases}$$

and

$$t=2 \quad \begin{cases} x_2 \geq \underline{x_2} \\ x_2 \leq \bar{x_2} \end{cases}$$

By assuming the values in Table 1 for the state variables, the problem to be solved is

minimize $\quad x_0 = x_1 + x_2$

subject to:

$$\begin{aligned}
-x_1 &\leq 2 \\
x_1 &\leq 5 \\
x_1 &\leq 7 \\
-x_1 &\leq -1 \\
-.95\,x_1 - x_2 &\leq 11.1 \\
.95\,x_1 + x_2 &\leq 5.9 \\
x_2 &\leq 8.0 \\
-x_2 &\leq -3.0
\end{aligned}$$

Application of linear programming reveals the critical values of release x_1 and x_2 as 1 and 3 units, respectively, and a minimum cost x_0 of 4 units.

Table 1

Stochastic Inflow Data

Variable		
Time	1	2
d_t	6.0	8.0
$c_t - v_t$	15.0	25.0
$\bar{x}_t$	7.0	8.0
$\underline{x}_t$	1.0	3.0
e_t	1.0	0.95
$\underline{s}_t$	3.0	3.0
s_0	8.0	
$R_1^{\alpha_1}$	11.0	
$R_1^{(1-\alpha_1)}$	6.0	
$(e_2 \, R_1 * R_2)^{\alpha_1}$		20.0
$(e_2 \, R_1 * R_2)^{1-\alpha_2}$		15.0

Linear Objective Function with Stochastic Inflow and Demand

In the previous example problem, only stochastic inflows were assumed. The distribution of inflows was known and the input to the program was listed in Table 1. In this example problem, the same system is assumed. Now, however, the demands and inflows are assumed to be normally distributed with means and variances listed in Table 2. The corresponding convoluted inflows and demands to be used in the problem formulation are also listed. The minimum reservoir level for time period two was adjusted to a unit value. This adjustment was necessary, since the previous problem with these inflows and demands is infeasible.

The solution to this problem is again obtained by the method of linear programming. The critical values of releases in periods one and two, x_1 and x_2, are 1.3474 and 3.0, respectively, with a maximum profit x_0 of 4.347 units.

Quadratic Objective Function

Consider a quadratic objective function for the same system as example 1. Let the objective be:

$$\text{minimize } x_0 = x_1 + x_2 + 3 (x_1 - 3)^2 + 5 (x_2 - 5)^2 + 3 x_1 x_2 .$$

The solution to this problem is readily obtained by quadratic programming techniques (Frank and Wolf, 1956, and Wolfe, 1959). The critical values of releases x_1 and x_2 are 1.0 and 4.6, respectively, with a minimum cost x_0 of 32.2 units.

Table 2

Stochastic Inflow and Demand Data

Variable		
Time	1	2
$c_t - v_t$	15.0	25.0
x_t	7.0	8.0
$\underline{x}_t$	1.0	3.0
e_t	1.0	.95
$\underline{s}_t$	3.0	1.00
s_0	8.0	
$E\{d_t\}$	6.0	8.0
$\sigma^2\{d_t\}$	1.0	1.0
$E\{v_t\}$	8.0	7.0
$\sigma^2\{v_t\}$	1.0	1.0
$R_1^{\alpha_1}$	4.336	
$R_1^{1-\alpha_2}$	-.336	
$(e_2 R_1 * R_2)^{\alpha_1}$		4.12
$(e_2 R_1 * R_2)^{1-\alpha_2}$		-2.32

Conclusions

In this paper an extension of the single multiple purpose stochastic constrained reservoir model was presented. The linear decision rules utilized by ReVelle et al. (1969) and Loucks (1970) are omitted in the model. The purpose of using linear decision rules is to disconnect the release in the nth period from the ending inventory level in period n-1. The advantage of a linear decision rule is that only the random inflow for the current period need be considered. However, the actual quantity to be released in the nth period is not known until the random inflows in periods 1 through n-1 are observed. Thus, for planning purposes where operation of the reservoir is important or when the release variables are represented in the objective function, this formulation is unsatisfactory since releases are actually random variables and not exactly determined by the reservoir planner.

The formulation proposed here requires that the distributions of the sums of the random inflows for all time periods be obtained. This is a relatively simple task for models with a large number of time periods. Since by the central limit theorem (Parzen, 1960), the distribution of the sums derived from the sampling of the parent distribution tends to become normal as the sample size increases.

By not using any form of decision rule, the constraints on upper and lower release quantities become deterministic and need not be represented by chance-constrained formulations. Also, quadratic or even general convex objective functions of the release quantities can be considered. The main advantage of this model is that it can be expanded readily to encompass systems of linked reservoirs with stochastic constraints.

CHAPTER II

STOCHASTIC MODEL FOR CONNECTED

MULTI-PURPOSE RESERVOIRS

In the previous chapter a chance - constrained model was proposed
for a single multi-purpose reservoir. The model was based on a
material balance equation and made extensive use of distributions
of the sum of random variables. The natural extension of the single
multi-purpose reservoir model is to link a system of such reservoirs.

Linked System of Multi-Purpose Reservoirs

For the development of a linked system of reservoirs, it is
assumed that there are two general linkage types. These linkages
consist of the normal channel flow for reservoir releases, and
pipe lines or pumping canals. The model is completely general in
the sense that any connecting system can be modeled. Thus, each
reservoir could be connected to every other reservoir and could
receive releases from any or all other reservoirs as dictated by the
particular system under consideration.

For the purposes of this discussion, each reservoir in each time
period is assumed to receive random unregulated inflow, regulated
inflow from reservoir releases, and inflow from pumping. The reservoir
level is depleted by means of scheduled releases, deterministic
demands, evaporation and seepage losses, and pumping to other
reservoirs. Stochastic demands can be handled by a simple extension
to the model presented. The method of making this adjustment was
indicated in Chapter I.

The system of multi-purpose reservoirs with chance-constraints
is modeled based on material balance equations for reservoir inventory
levels. The formualtion provides decisions that specify the release

and pumping quantities during different time periods of system
operation. The release and pumping decisions for the entire planning
horizon are determined by solving a linear programming problem. The
linear problem is the deterministic equivalent of the original chance-
constrained system.

The chance-constraints for each reservoir and each time period
are identical to those developed for a single multi-purpose reservoir.
The random inflows for each reservoir are assumed to be independent
from one time period to the next. However, this assumption, while
simplifying, is not necessary to the model development.

Continuity Equation

The kth reservoir in the linked system is based on the model
shown in Figure 2 and the following notation:

m — the number of reservoirs,

γ_t^k — random unregulated inflow into reservoir k in time period t,

c_t^k — capacity of reservoir k in period t,

v_t^k — design maximum capacity in period t,

s_t^k — ending reservoir inventory level for period t,

$\underline{s}_t^k$ — minimum specified inventory level,

e_t^k — fraction of inventory remaining after evaporation and
  seepage losses,

x_t^k — scheduled downstream release from reservoir,

$p_{,t}^k$ — scheduled pumping quantity from reservoir k to reservoir j,

$\bar{p}_{jt}^k$ — maximum pumping capacity from reservoir k to reservoir j,

x_t^{-k} - maximum downstream release,

$\underline{x}_t^{k}$ - minimum downstream release, and

d_t^{k} - deterministic extracted demand.
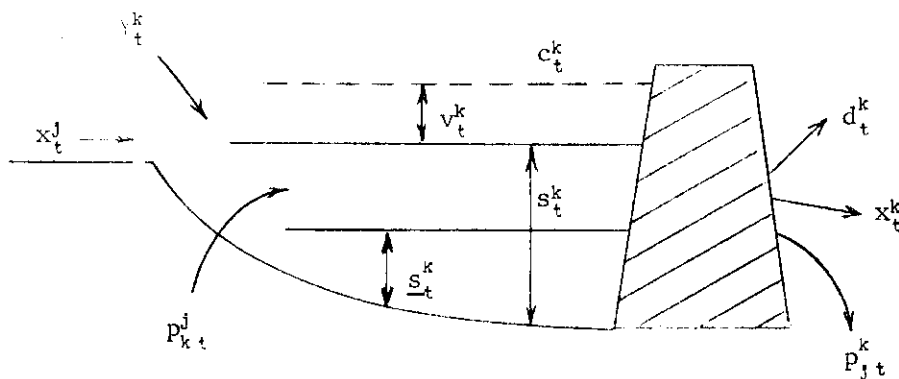
Figure 2. Linked Multi-purpose Reservoir

The total unregulated random inflow v_t^k enters the kth reservoir in time period t. The inflow is randomly distributed with a p.d.f. $f_t(v_t^k)$. The regulated inflow x_t^j is the release from the jth reservoir into reservoir k. The regulated pumped inflow p_{kt}^j is the water pumped from reservoir j into reservoir k. The pumping and release into reservoir k can be from several reservoirs, thus j can vary over all reservoir numbers.

The releases from the kth reservoir in time period t are (1) the deterministic extracted demands d_t^k, (2) the decision variable for downstream release x_t^k, and (3) the decision variable p_{jt}^k for pumping water from reservoir k to reservoir j; again, reservoir k could pump to several different reservoirs. The releases plus the inflows and

previous storage volume constitute the current inventory level s_t^k. The continuity equation for reservoir k in time period t is

$$(11) \quad s_t^k = e_t^k s_{t-1}^k + v_t^k - d_t^k - x_t^k + \sum_{\substack{j=1 \\ j \neq k}}^{m} (I_j^k x_t^j + O_k^j p_{kt}^j - O_j^k p_{jt}^k),$$

where

$$I_j^k \quad - \quad \begin{cases} 1 \text{ if reservoir j release flows into reservoir k} \\ 0 \text{ otherwise, and} \end{cases}$$

$$O_j^k \quad - \quad \begin{cases} 1 \text{ if reservoir k pumps to reservoir j} \\ 0 \text{ otherwise.} \end{cases}$$

Chance-Constraints with Stochastic Inflows

The chance-constraint for the probability of not exceeding the maximum capacity of the kth reservoir is

$$(12) \quad P\{ s_t^k < c_t^k - v_t^k \} \geq \alpha_1^k, \quad k = 1,2, \ldots, m, \text{ and}$$

$$t = 1,2, \ldots, T,$$

where c_t^k is the design maximum capacity of the kth reservoir, v_t^k is the upper storage space required in time period t of the kth reservoir, and α_1^k are the specified constants between zero and one.

The probability α_2^k, at the end of time period t, for storage s_t^k to exceed the minimum pool level is

$$(13) \quad P\{ s_t^k \geq \underline{s}_t^k \} \geq \alpha_2^k, \quad k = 1,2, \ldots, m,$$

where $\underline{s}_t^k$ is the minimum storage that must be maintained.

The downstream release x_t^k must satisfy the minimum $\underline{x}_t^k$ and the maximum x_t^{-k} reservoir releases constraints and maximum pumping capacity constraints, i.e.,

(14)
$$\underline{x}_t^k < x_t^k < x_t^{-k} \quad \forall \ k, \ t, \ \text{and}$$

$$0 \le p_{jt}^k \le p_{jt}^{-k} \quad \forall \ k, \ t, \ j.$$

The chance-constraints (12) and (13) are converted to their equivalent linear deterministic constraints in a similar manner as the chance-constraints of the single multi-purpose reservoir model. The general equivalent linear deterministic result for chance-constraint (12) with t=n is

(15)
$$c_n^k - v_n^k - s_0^k \prod_{t=1}^{n} e_t^k + \sum_{t=1}^{n} \left\{ \left(\prod_{L=t+1}^{n} e_L \right) \cdot \left[d_t^k + x_t^k - \sum_{\substack{j=1 \\ j \ne k}}^{m} (I_j^k x_k^j \right. \right.$$
$$\left. \left. + O_k^j p_{kt}^j - O_j^k p_{jt}^k) \right] \right\} \ge R_{k,*n}^{\alpha_1} \ ,$$

where $R_{k,*n}^{\alpha_1}$ are the values at the α_1^k points on the convoluted distributions. Chance-constraint (13) becomes

(16)
$$\underline{s}_n^k - s_0^k \prod_{t=1}^{n} e_t^k + \sum_{t=1}^{n} \left(\prod_{L=t+1}^{n} e_L \right) \left[d_t^k + x_t^k - \sum_{\substack{j=1 \\ j \ne k}}^{m} (I_j^k x_t^j \right.$$
$$\left. + O_k^j p_{kt}^j - O_j^k p_{jt}^k) \right] \sim R_{k,*n}^{(1-\alpha_2)} \ ,$$
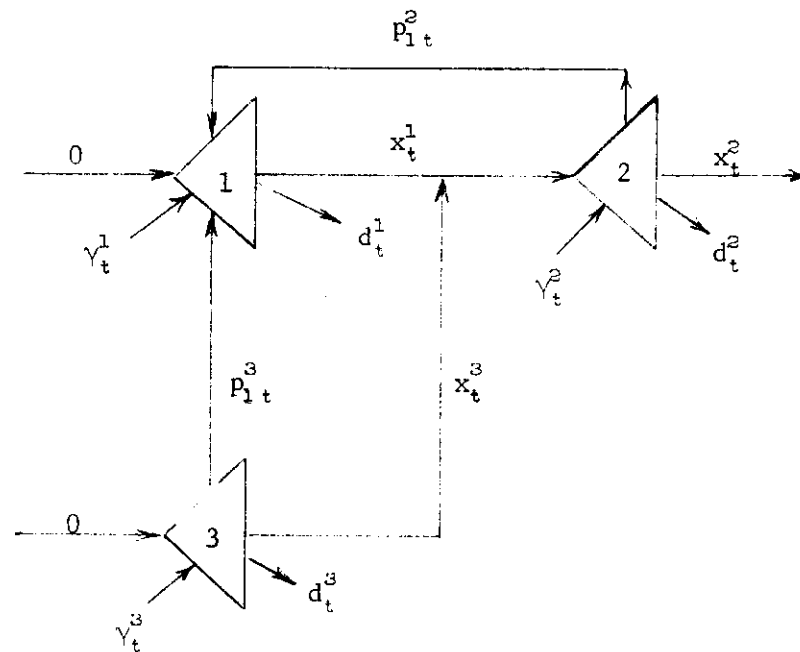
Figure 3: System of Connected Multi-purpose Reservoirs

The structure of the problem can now be put in the form

$$\text{minimize } Z = \underline{h} \underline{y}$$

$$\text{subject to} \qquad (\underline{A}, \underline{I})\underline{y} = \underline{b},$$

$$\underline{\ell} < \underline{y} < \underline{u}$$

where

$$\underline{u} < \underline{\ell} < \underline{0} ,$$

and where $\underline{y}$ is the decision vector consisting of release, pumping, slack, and artificial variables.

Table 3

Data Used in Example Problem for Linked System of Reservoirs

Time	1			2		
Reservoir	1	2	3	1	2	3
d_t^k	6.0	5.0	10.0	8.0	7.0	7.0
$c_t^k - v_t^k$	10.0	20.0	15.0	10.0	19.0	16.0
$\bar{x}_t^k$	7.0	15.0	20.0	8.0	12.0	20.0
$\underline{x}_t^k$	1.0	2.0	1.0	3.0	3.0	1.0
e_t^k	1.0	1.0	1.0	0.95	0.97	0.98
$\underline{s}_t^k$	3.0	4.0	3.0	3.0	2.0	4.0
s_0^k	8.0	20.0	6.0			
$R_1^{\alpha_1}$	11.0	10.0	12.0			
$R_1^{(1-\alpha_2)}$	6.0	9.0	8.0			
$(e_2 R_1 * R_2)^{\alpha_1}$				20.0	15.0	20.0
$(e_2 R_1 * R_2)^{1-\alpha_2}$				15.0	14.0	17.0
$\bar{p}_{j,t}^k$		10.0	5.0		10.0	5.0

The values chosen for the cost coefficient of the objective function are

$$\underline{h} = [1.0, -2.0, 0.0, -.75, .65, 1.0, -2.1, 0.0, -.80, .70]$$

and the linear constraints are given by specifying the matrix $\underline{A}$ and the vectors $\underline{b}$, $\underline{\ell}$, $\underline{u}$ as

$$\underline{A} = \begin{bmatrix}
-1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-.95 & 0 & 0 & .95 & .95 & -1 & 0 & 0 & 1 & 1 \\
.95 & 0 & 0 & -.95 & -.95 & 1 & 0 & 0 & -1 & -1 \\
.97 & -.97 & .97 & -.97 & 0 & 1 & -1 & 1 & -1 & 0 \\
-.97 & .97 & -.97 & .97 & 0 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & -.98 & 0 & -.98 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & .98 & 0 & .98 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

$$
\underline{b} = \begin{bmatrix} -3 \\ 5 \\ -5 \\ 20 \\ 7 \\ 1 \\ -3.9 \\ 5.9 \\ 6.45 \\ 19.55 \\ 6.92 \\ 2.08 \end{bmatrix}
\qquad
\underline{\ell} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\qquad
\underline{u} = \begin{bmatrix} 7 \\ 15 \\ 20 \\ 10 \\ 5 \\ 8 \\ 12 \\ 20 \\ 10 \\ 5 \end{bmatrix}
$$

Application of the revised simplex method in conjunction with the bounded variable technique reveals the critical values of $\underline{y}$:

$$
\underline{y} = \begin{bmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \\ p_{11}^2 \\ p_{11}^3 \\ x_2^1 \\ x_2^2 \\ x_2^3 \\ p_{12}^2 \\ p_{12}^3 \end{bmatrix}
=
\begin{bmatrix} 7 \\ 9 \\ 1 \\ 4 \\ 0 \\ 8 \\ 3 \\ 1 \\ 4.85 \\ .1 \end{bmatrix}
$$

where $R_{k,*n}^{(1-\alpha_2)}$ are the values at the $1-\alpha_2^k$ points on the convoluted distributions. Equations (15) and (16) are linear in the decision variables x_t^k and p_{jt}^k. The releases and pumping units during different time periods of reservoir operation are, therefore, determined by solving a mathematical programming problem with linear constraints.

Linear Objective Function

The example presented is a system of multi-purpose reservoirs. Figure 3 is the model formulated for illustration. The linked reservoir system is composed of three reservoirs, two of which have pumping capabilities. Random inflows and predetermined demands are assumed for each reservoir. Table 3 describes the state variables assumed for each reservoir and time period. The objective is to minimize the operating cost of the system for two time periods.

The decision variables are to be determined for each time period. They are: the units of water released from reservoirs one, two, and three; and the number of units of water pumped into reservoir one from reservoirs two and three. The total number of variables to be determined is the product of the number of time periods with the sum of the number of reservoirs and pumping variables.

The decision variables must satisfy the equivalent deterministic constraints (15), (16), and the upper and lower limits on release (14). By taking advantage of the fact that the decision variables are bounded from above and below, the number of constraints can be reduced considerably for linear programming formulation of the problem. Using the bounded variable techniques discussed in Taha (1970), the resulting number of constraints is the product of twice the number of reservoirs multiplied by the number of time periods.

and an optimum cost of operating the system as -16.11 or a profit of 16.11 units.

Conclusions

The development of a mathematical model for a linked system of multi-purpose reservoirs with stochastic unregulated inflows is obtained as a straight forward generalization of the single reservoir model developed in Chapter I. The chance-constrained formulation for reservoir capacities and minimum inventory levels converts to a linear system of constraints. Linear, quadratic, or even general convex objective functions can be appended to this system and the solution obtained with facility.

The seemingly difficult, or at least time consumming, task of obtaining the distributions for the sums of the random inflows for models with many time periods is actually simple. As the number of time periods increase, the sum of random variables, independent of the basic distribution, are approximately normally distributed. This result is due to the well-known central limit theorem (Parzen, 1960).

If linear objective functions are assumed, which could be operational or of a capacity nature, very large problems can be solved. Since the cummulative inflows will be nearly normally distributed for these problems, their formulations and solutions are thus a matter of course. However, the problem of capacity expansion is generally not well modeled as a continuous linear problem. Capacity expansions are usually limited to certain periods and have nonlinear costs as a function of size.

CHAPTER III

PROGRAM DOCUMENTATION

The linear programming system which is used to solve the
multiple reservoir chanced-constrained problems is based on the
revised simplex method with bounded variables. The bounded variables
procedure is a method by which constraints that are merely upper or
lower bounds on individual variables are handled implicitly rather
than explicitly in the program. Since the work involved in solving
linear programs is mainly a function of the number of constraints,
speed and accuracy can be improved significantly by utilizing the
bounded variables procedure. This is particularly important for
linked multiple reservoir models since at least one-half of the
constraints are bounding constraints. A detailed discussion of the
revised simplex procedure with bounded variables is given by Taha
(1970, Chapter 8).

Although a large number of excellent codes are available, the
authors developed the code which is used in the model. The program
is FORTRAN based which offers the greatest flexibility for interfacing
with the remaining subroutines and for conversion to other computers.
The program was developed to run under the WATFOR, WATFIV or OS 360
FORTRAN systems on the IBM 360/65.

Program Structure

The basic structure of the computer program consists of a
master program which reads the input data, develops the linear
programming formulation, submits the problem to the linear program-
ming subroutine LPSIM, and interprets the results for printout. The
general problem solved by the linear program (l.p) subroutine is

$$\text{maximize} \qquad \underline{c}\ \underline{y}$$

$$\text{subject to} \qquad (\underline{A},\underline{I})\underline{y} = \underline{b} > \underline{o}$$

$$\underline{\lambda} < \underline{y} < \underline{u} \quad ,$$

where $\underline{y}$ is the decision vector and consists of release, pumping, slack and artificial variables. The variables are stored into $\underline{y}$ in the following manner (the notation is that given in the previous chapter):

$$
\underline{y} = \overbrace{[x_1^1, x_1^2, \ldots, x_1^m,}^{\text{all releases}} \overbrace{p_{11}^2, p_{11}^3, \ldots, p_{21}^1, \ldots;}^{\text{all pumping variables}} \quad \text{time 1}
$$

$$
x_2^1, x_2^2, \ldots, x_2^m, \quad p_{12}^2, \quad \ldots, p_{22}^1, \ldots; \quad \text{time 2}
$$

$$
x_T^1, x_T^2, \ldots, x_T^m, \quad p_{1T}^2, \ldots \qquad] \quad \text{time T}
$$

In narrative form the sequence of variables is (1) each release variable x_t^k for each reservoir for time period one, followed by all first period pumping variables; (2) the pumping variables are ordered by all pumping into reservoir one, then reservoir two, etc.; (3) this sequence is repeated for each of the T time periods; and (4) the slack and artificial variables occur next, one each for every constraint.

Program Input Formats

The notation which is used in FORTRAN to designate whether a variable is an integer or a floating point variable is I or F,

respectively. For I type variables, the data must be read in right adjusted in the field. For floating point variables, F designation, the decimal point must be included and the data inserted anywhere within the specified field. A field type of A is used for title information and can be any alpha or numeric character.

CARD	FIELD	TYPE	DESCRIPTION
1	1	I5	NT - number of time periods,
	2	I5	NR - number of reservoirs.

The remaining cards are read in sets. One set for each reservoir. All of the pertinent data for each reservoir is included within the set. Each reservoir in the system must be given a designation number; these numbers must be sequencial starting with one. A title card also is included to identify the reservoir.

SET CARD	FIELD	TYPE	DESCRIPTION
1	1	A80	reservoir title card, can be any alpha-numeric characters, a maximum of 80 columns is available.
2	1	F10	reservoir extracted demand for period one,
	2	F10	demand for period two,
	.		.
	.		.
	.		.
	NT	F10	demand for period NT (If more than 8 periods are to be studied, continue data on successive cards. A maximum of 8 periods per card until NT reached.).
3	1	I5	NP-number of other reservoirs that this reservoir can pump to (If this number is zero, place a one in column 5 of the card.),

SET CARD	FIELD	TYPE	DESCRIPTION
	2	I5	the number of one of the reservoirs which is pumped to,
	3	I5	the number of one of the reservoirs which is pumped to,

.

.

.

(Repeat reservoir numbers until all reservoirs that are pumped to from this reservoir have been included. The maximum number on each card is 9 reservoirs on the first card. This is followed successively by 10 reservoirs per card. The limit of 9 on the first card is because NP takes up one field on this card.).

(The following set cards 4 and 5 are repeated for each reservoir pumped to from this reservoir or NP times. However, if no pumping is allowed from the reservoir omit set cards 4 and 5.)

SET CARD	FIELD	TYPE	DESCRIPTION
4	1	F10	pumping canal maximum capacity for first time period,
	.		
	.		
	.		
	NT	F10	pumping canal maximum capacity for time period NT.
5	1	F10	profit per unit pumped through canal in first time period (costs are considered as negative profit),
	.		
	.		
	.		
	NT	F10	profit per unit pumped through canal in period NT.

SET CARD	FIELD	TYPE	DESCRIPTION
6	1	I5	NP-number of other reservoirs for which the normal channel release flows into this reservoir (If this number is zero, place a one in column 5.),
	2	I5	the number of one of the reservoirs which releases into this reservoir,
	. . .		. . .
	NP	I5	the number of one of the reservoirs which releases into this reservoir (Maximum number per card is 9 on first and 10 each on successive cards. Reasons are same as that for set card 3.).
7	1	F10	maximum inflow into reservoir in period one (This is the α_1 point from distribution inflow and was designated by $(R_{*_1})^{\alpha_1}$ in previous discussions.),
	2	F10	maximum inflow into reservoir in period two, $(R_{*_2})^{\alpha_1}$,
	. . .		
	NT	F10	maximum inflow into reservoir in last time period
8	1	F10	minimum inflow into reservoir in period one, $(R_{*_1})^{1-\alpha_2}$,
	. . .		
	NT	F10	minimum reservoir inflow in period NT, $(R_{*_{NT}})^{1-\alpha_2}$.
9	1	F10	maximum reservoir capacity minus surcharge $(c_1^k - v_1^k)$ for time period one,
	. . .		

SET CARD	FIELD	TYPE	DESCRIPTION
	NT	F10	maximum reservoir capacity minus surcharge $(c^k_{NT} - v^k_{NT})$ for time period NT.
10	1	F10	minimum reservoir pool level $\underline{s}^k_1$ for time period one,
	.		
	.		
	.		
	NT	F10	minimum reservoir pool level $\underline{s}^k_{NT}$
11	1	F10	maximum normal channel release $\overline{x}^k_1$ from reservoir in period one,
	.		
	.		
	.		
	NT		maximum normal channel release $\overline{x}^k_{NT}$ from reservoir in period NT.
12	1	F10	minimum normal channel release $\underline{x}^k_1$ from reservoir in period 1,
	.		
	.		
	.		
	NT	F10	minimum normal channel release $\underline{x}^k_{NT}$ from reservoir in period NT.
13	1	F10	profit per unit for releasing from reservoir in period 1 (cost considered as negative profit),
	.		
	.		
	.		
	NT	F10	profit per unit for releasing from reservoir in period NT.
14	1	F10	evaporation factor, e^k_1, for time period (This is the fraction of previous period ending reservoir inventory quantity which is not lost due to evaporation or leakage.),
	.		
	.		
	.		

SET CARD	FIELD	TYPE	DESCRIPTION
	NT	F10	evaporation factor, e^k_{NT}, for time period NT.
15	1	F10	starting reservoir water quantity.

A list, of the input cards for the linked multiple reservoir example, Figure 3 discussed in Chapter II and using the data in Table 3, is given below.

Program Output

The program output is self explanatory. Therefore, a computer run for data listed above is included without further discussion.

EXAMPLE DATA LIST

```
COLUMNS
1    5     1     1     2     2     3
           0     5     0     5     0
------------------------------------

      2     3
 RESERVOIR ONE
6.          8.
     1     0
     1     0
11.         20.
6.          15.
 10.0        10.0
3.          3.
7.          8.
1.          3.
1.0         1.0
1.          .95
8.
 RESERVOIR TWO
5.          7.
     1     1
10.         10.
-.75        -.80
     2     1     3
10.         15.
9.          14.0
20.0         19.0
4.          2.
15.          12.
2.           3.
-2.          -2.1
1.           .97
20.
 RESERVOIR THREE
   10.0        7.0
     1     1
5.          5.
.65          .70
     1     0
   12.0      20.0
    8.0        17.0
15.0          16.0
    3.0         4.0
    20.0         20.0
    1.0         1.0
0.
    1.0         .98
    6.0
```

```
NUMBER OF TIME PERIODS=      2
NUMBER OF RESERVOIRS  =      3


TIME PERIOD          1          2


RESERVOIR  1                RESERVOIR ONE
        DEMANDS      6.000      8.000
PUMP TO  0,CAP.      0.000      0.000
PUMPING PROFIT       0.000      0.000
RELEASE FROM         0
        INFLOWS     11.000     20.000
      L INFLOWS      6.000     15.000
  CAP.-FREEBD.      10.000     10.000
          SMIN       3.000      3.000
  FLOW LIMIT U       7.000      8.000
  FLOW LIMIT L       1.000      3.000
RELEASE PROFIT       1.000      1.000
   EVAPORATION       1.000      0.950
   STARTING RESERVOIR QUANTITY      8.000


RESERVOIR  2                RESERVOIR TWO
        DEMANDS      5.000      7.000
PUMP TO  1,CAP.     10.000     10.000
PUMPING PROFIT      -0.750     -0.800
RELEASE FROM         1          3
        INFLOWS     10.000     15.000
      L INFLOWS      9.000     14.000
  CAP.-FREEBD.      20.000     19.000
          SMIN       4.000      2.000
  FLOW LIMIT U      15.000     12.000
  FLOW LIMIT L       2.000      3.000
RELEASE PROFIT      -2.000     -2.100
   EVAPORATION       1.000      0.970
   STARTING RESERVOIR QUANTITY     20.000


RESERVOIR  3                RESERVOIR THREE
        DEMANDS     10.000      7.000
PUMP TO  1,CAP.      5.000      5.000
PUMPING PROFIT       0.650      0.700
RELEASE FROM         0
        INFLOWS     12.000     20.000
      L INFLOWS      8.000     17.000
  CAP.-FREEBD.      15.000     16.000
          SMIN       3.000      4.000
  FLOW LIMIT U      20.000     20.000
  FLOW LIMIT L       1.000      1.000
RELEASE PROFIT       0.000      0.000
   EVAPORATION       1.000      0.980
   STARTING RESERVOIR QUANTITY      6.000
```

```
L.P. PROBLEM SIZE NUMBER OF VARIABLES    10        NUMBER OF CONSTANTS   12
TAB  1 -1.00  0.00  0.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00 -3.00

TAB  2  1.00  0.00  0.00 -1.00 -1.00  0.00  0.00  0.00  0.00  0.00  5.00

TAB  3  1.00 -1.00  1.00 -1.00  0.00  0.00  0.00  0.00  0.00  0.00 -5.00

TAB  4 -1.00  1.00 -1.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00 20.00

TAB  5  0.00  0.00 -1.00  0.00 -1.00  0.00  0.00  0.00  0.00  0.00  7.00

TAB  6  0.00  0.00  1.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00  1.00

TAB  7 -0.95  0.00  0.00  0.95  0.95 -1.00  0.00  0.00  1.00  1.00 -3.90

TAB  8  0.95  0.00  0.00 -0.95 -0.95  1.00  0.00  0.00 -1.00 -1.00  5.90

TAB  9  0.97 -0.97  0.97 -0.97  0.00  1.00 -1.00  1.00 -1.00  0.00 -3.55

TAB 10 -0.97  0.97 -0.97  0.97  0.00 -1.00  1.00 -1.00  1.00  0.00 19.55

TAB 11  0.00  0.00 -0.98  0.00 -0.98  0.00  0.00 -1.00  0.00 -1.00  6.92

TAB 12  0.00  0.00  0.98  0.00  0.98  0.00  0.00  1.00  0.00  1.00  2.08
```

VAR	LOWER BOUND	UPPER BOUND
1	1.000	7.000
2	2.000	15.000
3	1.000	20.000
4	0.000	10.000
5	0.000	5.000
6	3.000	8.000
7	3.000	12.000
8	1.000	20.000
9	0.000	10.000
10	0.000	5.000

```
INFEASIBLE INITIAL RHS FOR CONSTRAINT    1   -2.0000
INFEASIBLE INITIAL RHS FOR CONSTRAINT    3   -5.0000
INFEASIBLE INITIAL RHS FOR CONSTRAINT    9   -4.5500
```

OPTIMAL SOLUTION

```
X(  1) =      7.000        0.1000000E 01
X(  2) =      9.000       -0.2000000E 01
X(  3) =      1.000        0.0000000E 00
X(  4) =      4.000       -0.7500000E 00
X(  6) =      8.000        0.1000000E 01
X(  7) =      3.000       -0.2100000E 01
X(  8) =      1.000        0.0000000E 00
X(  9) =      4.850       -0.8000000E 00
X( 10) =      0.100        0.7000000E 00
X( 13) =      2.000        0.0000000E 00
X( 17) =     15.000        0.0000000E 00
X( 19) =      8.000        0.0000000E 00
X( 23) =      2.000        0.0000000E 00
X( 27) =      0.150        0.0000000E 00
X( 29) =     15.850        0.0000000E 00
X( 31) =      9.000        0.0000000E 00
OBJECTIVE FUNCTION VALUE IS        -0.1611002E 02
```

```
TIME PERIOD    1
     RELEASE FROM RESERVOIR    1 INTO CHANNEL BED IS       7.0
     RELEASE FROM RESERVOIR    2 INTO CHANNEL BED IS       9.0
     PUMPING FROM RESERVOIR    2 TO RESERVOIR    1 IS      1.0
     RELEASE FROM RESERVOIR    3 INTO CHANNEL BED IS       4.0
     PUMPING FROM RESERVOIR    3 TO RESERVOIR    1 IS      0.0


TIME PERIOD    2
     RELEASE FROM RESERVOIR    1 INTO CHANNEL BED IS       8.0
     RELEASE FROM RESERVOIR    2 INTO CHANNEL BED IS       3.0
     PUMPING FROM RESERVOIR    2 TO RESERVOIR    1 IS      1.0
     RELEASE FROM RESERVOIR    3 INTO CHANNEL BED IS       4.9
     PUMPING FROM RESERVOIR    3 TO RESERVOIR    1 IS      C.1
```

One type of output which isn't displayed above is that for an infeasible problem. In this case, all of the above information is given along with an analysis of the infeasibilities which is similar to:

```
TIME    1 RESERVOIR    1 CAPACITY CONSTRAINT VIOLATED BY       8.0
```

Program Listing

The reservoir model program is listed below. A complete input system for running the program is depicted as:

```
                              /*
                         INPUT DATA
                    //$DATA
                 INVER
              LPSIM PROG.
           PROB. GEN. ROUT.
        WHATV CONTROL
             CARD
```
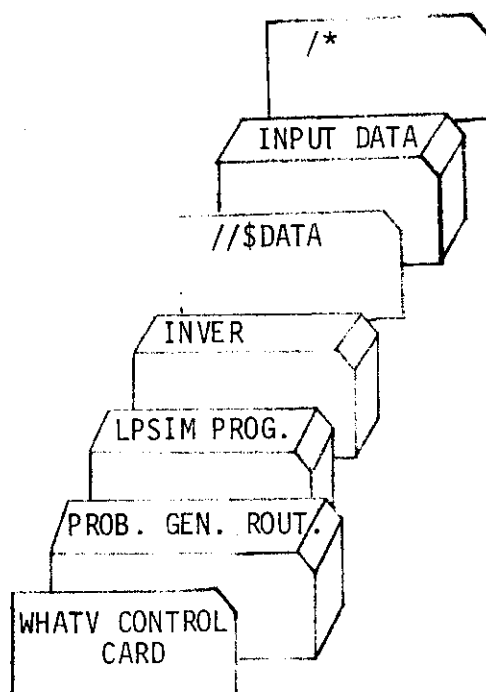
```
      REAL MSIO
      INTEGER T
      DATA ART /-1.E9/
      COMMON NXB(25),CB(25)
      DIMENSION XUB(99), XLB(99), PA(25), P(25), C(99) ,XP(9
     *-  9)
      DIMENSION          NPV(25),IO(25,25),D(25,25),
     1 RIF(20,25),RIFL(25,25), NPVSX(25)
      DIMENSION  RCAP(25,25),RFUL(25,25),RFLL(25,25)
     1 ,SMIN(25,25),E(25,25),SO(25),NRF(25),NSRF(25,25)
     2,MSIO(25, 5,25),TAB(25,99) , NPTR(25), NVPTR(25,25)
     3, ITIB(20), PRRES(25,25),PROPUM(25, 5,25)
3     FORMAT(8F10.3)
4     FORMAT('1')
5     WRITE(6,4)
      READ(5,6,END=500) NT,NR
6     FORMAT(10I5)
    8 FORMAT(20A4)
      WRITE(6,10)NT,NR
10    FORMAT( 5X,'NUMBER OF TIME PERIODS= ',I5/5X,
     1 'NUMBER OF RESERVOIRS  = ' I5/)
      WRITE(6,11) (J,J=1,NT)
   11 FORMAT('-      TIME PERIOD'10(I7,3X)
     1, 10(/13X, 10I10))
      NPVV=0
      DO 40 I=1,NR
      READ(5,8) ITIB
      WRITE(6,19) I, ITIB
   19 FORMAT('-    RESERVOIR' I3,10X, 20A4)
      READ(5,3) (D(I,J),J=1,NT)
      WRITE(6,21)   ( D(I,J),J=1,NT)
   21 FORMAT(8X' DEMANDS' 10F10.3,10(/16X 10F10.3))
      DO 920 T=1,NT
      PROPUM(I,1,T) =0.0
  920 MSIO(I,1,T) = 0.0
      READ(5,6) NP,( IO(I,J),J=1,NP)
      IF( IO(I,1) .EQ. 0) GO TO 922
C     READ PUMPING CAP. AND COSTS
      DO 921 J=1,NP
      READ(5,3) ( MSIO(I,J,T),T=1,NT)
  921 READ(5,3)(PROPUM(I,J,T),T=1,NT)
  922 DO 23 J=1,NP
      WRITE(6,22)    IO(I,J),  ( MSIO(I,J,T),T=1,NT)
   22 FORMAT(' PUMP TO'I3',CAP.' 10F10.3,
     1 10(/16X, 10F10.3))
   23 WRITE(6,24) ( PROPUM(I,J,T),T=1,NT)
   24 FORMAT('  PUMPING PROFIT' 10F10.3,
     1 10(/16X, 10F10.3))
      NPV(I)=NP
      IF(IO(I,1).EQ.0) NPV(I)=0
```

```
            READ(5,6) NP,( NSRF(I,J),J=1,NP)
            NRF(I) = NP
            IF( NSRF(I,1) .EQ.0) NRF(I) = 0
            WRITE(6,25)   (NSRF(I,J),J=1,NP)
     25     FORMAT(4X,'RELEASE FROM' 10I10,10(16X/10I10))
            READ(5,3) (RIF(I,J),J=1,NT)
            WRITE(6,28) (RIF(I,J),J=1,NT)
     28     FORMAT(9X,'INFLOWS',10F10.3,10(/16X,F10.3))
            READ(5,3) (RIFL(I,J),J=1,NT)
            WRITE(6,29) (RIFL(I,J),J=1,NT)
     29     FORMAT(7X,'L INFLOWS',10F10.3,10(/16X,F10.3))
            READ(5,3) (RCAP(I,J),J=1,NT)
            WRITE(6,30) (RCAP(I,J),J=1,NT)
     30     FORMAT(4X,'CAP.-FREEBD.'10F10.3,10(/16X,F10.3))
            READ (5,3) (SMIN(I,J),J=1,NT)
            WRITE(6,31)(SMIN(I,J),J=1,NT)
     31     FORMAT(12X'SMIN'   10F10.3,10(/16X,10F10.3))
            READ(5,3) (RFUL(I,J),J=1,NT)
            WRITE(6,32) (RFUL(I,J),J=1,NT)
     32     FORMAT(4X,'FLOW LIMIT U'   10F10.3,10(/16X,10F10.3))
            READ(5,3) (RFLL(I,J),J=1,NT)
            WRITE(6,34) (RFLL(I,J),J=1,NT)
     34     FORMAT(4X,'FLOW LIMIT L'   10F10.3,10(/16X,10F10.3))
     C      PROFIT FROM RELEASE BY TIME PEROID
            READ(5,3) ( PRRES(I,J),J=1,NT)
            WRITE(6,35)(PRRES(I,J),J=1,NT)
     35  FORMAT(2X'RELEASE PROFIT' 10F10.3,10(/16X,10F10.3))
            READ (5,3)     (E(I,J),J=1,NT)
            WRITE(6,36)    (E(I,J),J=1,NT)
     36     FORMAT(5X,'EVAPORATION'  ,10F10.3,10(/16X,10F10.3))
            READ(5,3)     SO(I)
            WRITE(6,38)   SO(I)
     38     FORMAT(5X,'STARTING RESERVOIR QUANTITY 'F10.3)
     40     CONTINUE
     C       COMPUTE THE L.P. SIZE
            NV=NR*NT
            NP=0
            NPVSX(1)=0
            DO 45 I=1,NR
            NP=NPV(I)+NP
            NPTR(I) = 0
     45     NPVSX(I+1)=NP
     C
     C       SETUP INTO PUMPING VAR. ARRAY
     C       NPTR(K) - NO. INFLOWS TO RES. K
     C       NVPTR(K,J)-NO. OF VAR. PUMPED INTO RES. K
            DO 48 K=1,NR
            J= NPV(K)
            IF(J.LE.0) GO TO 48
            N= NPVSX(K)+ NR
```

```
          DO 47 I=1,J
C         RES. PUMPED INTO
          L= IO(K,I)
          K1= NPTR(L) + 1
          N= N+1
          NVPTR(L,K1) = N
   47 NPTR(L) = K1
   48 CONTINUE
          NV=NV+NP*NT
          NC= 2*NR*NT
          WRITE(6,57) NV,NC
   57     FORMAT('1L.P. PROBLEM SIZE NUMBER OF VARIABLES '
      1,I5,5X ' NUMBER OF CONSTANTS' I5)
          NVT=2*NC+ NV
          DO   59 I=1,NVT
          C(I) = 0.0
          XLB(I) = 0.0
          XP(I) = 0.0
   59 XUB(I) = 1.0E30
          NRHS=NV+1
          NOBJ=NC+1
          DO 60 I=1,NOBJ
          DO 60 J=1,NRHS
   60     TAB(I,J)=0.0
          DO 90 N=1,NT
          ICN= 2*NR*(N-1)
          DO 90 K=1,NR
          ICN=ICN+1
          PE=SO(K)
          DO 65 T=1,N
   65     PE=PE*E(K,T)
          TAB(ICN,NRHS)=-RIF(K,N)-PE+RCAP(K,N)
          TAB(ICN+1,NRHS)=RIFL(K,N)+PE-SMIN(K,N)
          DO 85 T=1,N
          INVR=(NP+NR)*(T-1)
          PE=1.0
          IF(T.EQ.N) GO TO 73
          K1=T+1
          DO 70 L=K1,N
   70     PE=PE*E(K,L)
   73     TAB(ICN,NRHS)=TAB(ICN,NRHS)+PE*D(K,T)
          TAB(ICN+1,NRHS)=TAB(ICN+1,NRHS)-PE*D(K,T)
C         RELEASE VARIABLE RES. K
          ISUB= INVR+K
          TAB(ICN,ISUB) = -PE
          TAB(ICN+1,ISUB ) = PE
          IF(NPV(K).EQ.0) GO TO 77
          IXV=INVR+NR+NPVSX(K)
          K1=NPV(K)
          DO 75 J=1,K1
```

```
      IXV=IXV+1
C      PUMPING  FROM RES. K VAR.
      TAB(ICN+1,IXV)=PE
75    TAB(ICN,IXV)=-PE
77    K1= NRF(K)
      IF(K1.EQ. 0) GC TO 80
      DO 79 L=1,K1
      J= NSRF(K,L)
C      VAR. INTO RES. K THAT ARE RELEASES FROM OTHER RES.
      TAB(ICN,INVR+J) = PE                     .
79    TAB(ICN+1,INVR+J) = -PE
C      VAR. FOR PUMPING INTO RES. K FROM OTHER RES.
   80 K1= NPTR(K)
      IF( K1.EQ.0) GO TO 85
      DO 82 L=1,K1
      J= NVPTR(K,L)
      TAB(ICN,INVR+J) = PE
   82 TAB(ICN+1,INVR+J) = -PE
   85 CONTINUE
C      RELEASE VAR. CAP. BY TIME PEROID
      XUB(INVR+K) = RFUL(K,N)
      XLB(INVR+K) = RFLL(K,N)
      C(INVR+K) = PRRES(K,N)
      ICN=ICN+1
      K1=NPV(K)
      IF(K1.EQ.0) GO TO 90
      IXV=INVR+NR+NPVSX(K)
      DO 89  L=1,K1
      IXV=IXV+1
C      PUMPING VAR. CAP. BY TIME PEROID
      XUB(IXV) = MSIO(K,L,N)
      C(IXV) = PROPUM(K,L,N)
89     CONTINUE
   90 CONTINUE
      DO 110 I=1,NC
  110 WRITE(6,115) I,( TAB(I,J),J=1,NRHS)
  115 FORMAT(' TAB 'I3,11F6.2,9(/ 8X,11F6.2))
      IXV= NV
      NV= 2*NC+ NV
      DO 135 I=1,NC
      PA(I) = 0.0
  135 P(I) = TAB(I,NRHS)
      DO 137 I=NRHS, NV
      DO 137 J=1,NC
  137 TAB(J,I) = 0.0
      DO 140 I=1,NC,2
      IXV= IXV+ 1
      TAB(I,IXV) = 1.0
C      STARTING BASIS VARIABLE IN LP SUB.
      NXB(I  )=    IXV
```

```
         CB(I) = C(IXV)
         IXV= IXV+1
C      THIS IS THE ARTIFICIAL VARIABLE
         TAB(I,IXV) = -1.0
         C(IXV)=ART
         IXV= IXV+1
         TAB(I+1,IXV) = 1.0
C      STARTING BASIS VARIABLE IN LP SUB.
         NXB(I+1)=   IXV
         CB(I+1)=C(IXV)
         IXV=IXV+1
C      THIS IS THE ARTIFICIAL VARIABLE
         TAB(I+1,IXV) = -1.0
  140 C(IXV)=ART
         CALL LPSIM(NV,NC,0,0,TAB,P,PA,C,XUB,XLB,XZERO,XP)
         WRITE(6,300) XZERO,( XP(I),I=1,NV)
  300 FORMAT('0    SOLUTION  XO,X(I) '  1F10.4,
     1 10( /22X 5F10.4 ))
         IX= 0
         WRITE(6,4)
         DO 330 T=1,NT
         WRITE(6,320) T
  320 FORMAT('-' 14X'TIME PERIOD '13)
         DO 330 K=1,NR
         IX= IX+1
         WRITE(6,325) K, XP(IX)
  325 FORMAT(20X'RELEASE FROM RESERVOIR'I4,
     1 ' INTO CHANNEL BED IS' F9.1)
         NP= NPV(K)
         IF( NP.EQ.0) GO TO 330
         DO 329 J=1,NP
         IX= IX+1
         WRITE(6,327) K, IO(K,J),XP(IX)
  327 FORMAT(20X'PUMPING FROM RESERVOIR'I4,
     1 ' TO RESERVOIR'I4,' IS' F9.1)
  329 CONTINUE
  330 CONTINUE
         WRITE(6,4)
         DO 350 T=1,NT
         DO 350 K=1,NR
         IX=IX+2
         IF( XP(IX) .EQ. 0.0) GO TO 340
         WRITE(6,335) T,K,XP(IX)
  335 FORMAT(15X'TIME' I4,' RESERVOIR'I4,' CAPACITY '
     1 'CONSTRAINT VIOLATED BY'F10.1)
  340 IX= IX+2
         IF( XP(IX) .EQ. C.0) GO TO 350
         WRITE(6,345) T,K, XP(IX)
  345 FORMAT(15X'TIME' I4,' RESERVOIR'I4,' MIN POOL '
     1 'CONSTRAINT VIOLATED BY'F10.1)
```

```
  350 CONTINUE
      GO TO 5
  500 STOP
      END
      SUBROUTINE
     1        LPSIM(NV,NC,K,L,A,P,PA,C,XUB,XLB,XZERO,XP)
C     PROGRAM TO DO REVISED SIMPLEX AND PARAMETRIC
C     RHS RANGING, NO ATTEMPT FOR PROGRAMMING
C     EFFICIENCY HAS BEEN MADE, THE PROBLEM FORM IS
C                              8-20-71   GUY CURRY
C     MAX  CX
C     S.T. (A,I)X = P GE 0, X GE 0
C
C     ALSO   P + Y*PA FOR RHS PARAMETRIC ANALYSIS
C
C     NV - NUMBER OF VARIABLES, INCLUDES ALL SLACKS, ETC.
C     NC - NUMBER OF CONSTRAINTS
C     NPA - ZERO FOR REV. SIMPLEX
C     NPA - ONE FOR PARA. RHS
C     NPA - TWO FOR STOCHASTIC SOL.
C     NARTV - NUMBER OF ARTIFICIAL VARIABLES
C     THEY ARE NOT USED IN THE PARA OR STOCH ANALYSIS
C
C     PROGRAM EXTENDED TO DO BOUNDED VARIABLES 12-8-71
C     LOWER BOUNDS ARE SUBSTITUTED OUT IN THIS MODEL
C
C     IXP(I)=0   MEANS ORIGINAL VARIABLES IN PROBLEM
C     IXP(I)=1   MEANS COMPLEMENT OF VARIABLE IN PROBLEM
C
      DATA  TOL/0.0001/
      COMMON NXB,CB
      DOUBLE PRECISION BI,ALPHA ,ZCB, ZMC
      DIMENSION A(25,99),BI(25,25),XP(99),
     1 PCTH(25,25), P(25), PA(25), C(99), CB(25), XB(25),
     2 NXB(25), ZCB(25), ALPHA(25), ZMC(99), STH(25)
     3,OBJV(25),XLB(99),XUB(99),IXP(99)
C
C
C     WRITE(6,11) NV,NC
C  11 FORMAT('1' 15X 'REVISED SIMPLEX      NUMBER OF VARIABLE
     *-  S IS'
C    1 I3,5X'NUMBER OF CONSTRAINTS IS' I3 )
C     WRITE(6,12)
C 12  FORMAT('-(A,I),P,PA ' )
      DO 15 I=1,NV
   15 IXP(I) = 0
C     DO 20 I=1,NC
C     WRITE(6,21)( A(I,J),J=1,NV),P(I),PA(I)
C  20 CONTINUE
C  21 FORMAT( 8F10.3 )
```

```
      DO 30 I=1,NC
      DO 25 J=1,NC
  25  BI(I,J) = 0.0
  30  BI(I,I) = 1.0
      WRITE(6,935)
 935  FORMAT('        VAR    LOWER BOUND      UPPER BOUND' )
C     WRITE(6,35)( C(I),I=1,NV)
C  35 FORMAT('0 C  ' 10E10.3,10(/5X10E10.3))
      DO 937 I=1,NV
      IF( XUB(I).GT. 1.0E29 .AND. XLB(I).LE.0.0) GO TO 937
      WRITE(6,938) I, XLB(I),XUB(I)
      IF( XLB(I) .LE. 0.0) GO TO 937
      XUB(I) = XUB(I) -XLB(I)
      DO 936 J=1,NC
 936  P(J) = P(J) - A(J,I)*XLB(I)
 937  CONTINUE
 938  FORMAT(I5, F13.3, F16.3 )
      DO 953 I=1,NC
      IF( P(I) .GE. 0.0) GO TO 953
      WRITE(6,941) I, P(I)
 941  FORMAT('  INFEASIBLE INITIAL RHS FOR CONSTRAINT'
     1 I5,F10.4)
C     CONVERT TO POSITIVE RHS CONSTRAINTS
      P(I) = -P(I)
      DO 945 J=1,NV
 945  A(I,J) = -A(I,J)
      J= NXB(I)   +1
      NXB(I) = J
      CB(I) = C(J)
 953  XB(I) = P(I)
C     WRITE(6,105)( XB(I),I=1,NC)
C        Z(J) - C(J) = CB*BI*PJ - CJ
  39  ZMIN = 0.0
      DO 40 I=1,NC
      ZCB(I) = 0.0
      DO 40 J=1,NC
  40  ZCB(I) = ZCB(I) + CB(J) *BI(J,I)
C     WRITE(6,45) ( ZCB(I),I=1,NC )
C  45 FORMAT( '0 CB*BI ' 10F10.5  )
      DO 60 I=1,NV
      ZMC(I) =   -C(I)
      DO 50 J=1,NC
  50  ZMC(I) = ZMC(I) + ZCB(J) * A(J,I)
      IF( ZMC(I) .GT. ZMIN ) GO TO 60
      ZMIN = ZMC(I)
      IMIN = I
  60  CONTINUE
C     WRITE(6,65) ( ZMC(I),I=1,NV)
C  65 FORMAT('0ZJ-CJ ' 9E12.5 )
C        IF CJ-CJ GE 0 FOR ALL J, OPT. SOL. GO TO 200
```

```
      IF (ZMIN .GE. -0.00100) GO TO 200
      DO 70 I=1,NC
      ALPHA(I)= 0.0
      DO 70 J=1,NC
   70 ALPHA(I) = ALPHA(I) + BI(I,J) * A(J,IMIN)
C     WRITE(6,75) IMIN,( ALPHA(I),I=1,NC)
C  75 FORMAT('- ENTERING VARIABLE IS'I3,' ALPHA IS'( 7E12.5,
C    1/34X))
C
C        COMPUTE LEAVING VARIABLE BY MIN (XB(I)/ ALPHA(I),
C                                  FOR ALPHA(I) LT 0 )
C
      ICODE=0
      VALUE=XUB(IMIN)
      ILEAVE= 0
      DO 90 I=1,NC
      IF (ALPHA(I).LE. -TOL ) GO TO 80
      IF (ALPHA(I).LE.  TOL ) GO TO 90
C        POSITIVE ALPHA
      RATIO =XB(I)/ALPHA(I)
      IF (RATIO .GE. VALUE ) GO TO 90
      ILEAVE=I
      VALUE =RATIO
      ICODE =1
      GO TO 90
C        NEGATIVE ALPHA
   80 J=NXB(I)
      RATIO= (-XUB(J) + XB(I)) / ALPHA(I)
      IF (RATIO .GE. VALUE ) GO TO 90
      ILEAVE =I
      VALUE =RATIO
      ICODE =-1
   90 CONTINUE
      IF( VALUE .LT. 1.0E28) GO TO 91
      IF(ILEAVE) 110,110,91
   91 IF(ICODE .LE. 0 ) GO TO 125
   92 J=NXB(ILEAVE)
C     WRITE(6,95) J
C  95 FORMAT('0 LEAVING VARIABLE IS' I3 )
      CB(ILEAVE) = C(IMIN)
      NXB(ILEAVE)= IMIN
      CALL NEWBI( NC, BI , ALPHA, ILEAVE )
   97 DO 100 I=1,NC
      XB(I) = 0.0
      DO 100 J=1,NC
  100 XB(I) = XB(I) + BI(I,J) * P(J)
C     WRITE(6,105)( XB(I),I=1,NC)
C 105 FORMAT(' XB ' 10F10.5)
      GO TO 39
  110 WRITE(6,115)
```

```
  115 FORMAT('- UNBOUNDED SOLUTION ' )
      RETURN
  125 IF( ICODE.NE.0) GO TO 150
C     NONBASIC VARIABLE CAN NOT ENTER BECAUSE IT HAS
C     ENCOUNTERED ITS UPPER BOUND, REPLACE IT BY ITS
C     COMPLEMENT AND CONTINUE
      J=IMIN
      IF( IXP(J) .EQ. 0) GO TO 127
      IXP(J) = 0
      GO TO 130
  127 IXP(J) = 1
  130 DO 132 I=1,NC
      A(I,J) = -A(I,J)
  132 P(I) = P(I) + A(I,J)*XUB(J)
      C(J) = -C(J)
C     WRITE(6,135) J,XUB(J)
C 135 FORMAT('- VARIABLE'I5,5X'WAS SUBSTITUED AT ITS'
C    1' UPPER BOUND'F10.3 )
      GO TO 97
C         ALPHA IS NEGATIVE
  150 J=NXB(ILEAVE)
C     WRITE(6,155) J
C 155 FORMAT( '0 LEAVING VARIABLE IS ' I3, ' THEN '
C    1,'REPLACE BY ITS COMPLEMENT')
      CB(ILEAVE) = C(IMIN)
      NXB(ILEAVE)=IMIN
      CALL NEWBI (NC,BI,ALPHA,ILEAVE)
      IXP(J) =1
      DO 160 I=1,NC
      A(I,J) =-A(I,J)
  160 P(I)=P(I) +A(I,J)*XUB(J)
      C(J) = -C(J)
      GO TO 97
C
C     OPTIMAL SOLUTION HAS BEEN OBTAINED
  200 XZERO = 0.0
      WRITE(6,205)
  205 FORMAT('1 OPTIMAL SOLUTION'/)
      DO 206 I=1,NV
      XP(I) = XLB(I)
      IF( IXP(I) .EQ.0) GO TO 206
      C(I) = -C(I)
      XP(I) =XP(I) +XUB(I)
  206 CONTINUE
      DO 210 I=1,NC
      J= NXB(I)
      IF( IXP(J) .EQ.0) GO TO 209
      XP(J)=XP(J)-XB(I)
      GO TO 210
  209 XP(J)=XP(J)+XB(I)
```

```
210     CONTINUE
        DO 212 I=1,NV
        IF( XP(I).EQ.0.0) GO TO 212
        XZERO=XZERO+C(I)* XP(I)
        WRITE(6,211)I, XP(I),C(I)
211     FORMAT(' X('I3')=' F10.3,5XE20.7)
212     CONTINUE
        WRITE(6,215) XZERO
   215 FORMAT( ' OBJECTIVE FUNCTION VALUE IS ' E20.7,/'1')
        RETURN
        END
        SUBROUTINE NEWBI (NC, BI, ALPHA, IL )
        DOUBLE PRECISION BI,ALPHA,  BETA, P,DABS
        DIMENSION BI(25,25), ALPHA(25) , BETA(25)
        P = 1.0/ ALPHA(IL)
        DO  5 I=1,NC
     5 BETA(I) = BI(IL,I)
        DO 20 I=1,NC
        IF(I .EQ. IL ) GO TO 10
        DO 9 J=1,NC
        BI(I,J) = BI(I,J) - P*ALPHA(I)*BETA(J)
     9  IF(DABS(BI(I,J)).LE.1.0D-09)BI(I,J)=0.0D00
        GO TO 20
    10 DO 15 J=1,NC
    15 BI(I,J) = BI(I,J)*P
    20 CONTINUE
C       DO 30 I=1,NC
C   30 WRITE(6,35) ( BI(I,J),J=1,NC)
C   35 FORMAT(' BI ' 10F10.5 )
        RETURN
        END
```

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