Testing Linear Dependence of Hyperexponential Elements*

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Abstract

A Wronskian (resp. Casoratian) criterion is useful to test linear dependence of elements in a differential (resp. difference) field over constants. We generalize this criterion for invertible hyperexponential elements in a differential-difference ring extension over a field F. The generalization also enables us to connect similarity and F-linear dependence of invertible hyperexponential elements.

1 Introduction

Hyperexponential functions in several variables are an abstraction of common properties of exponential functions, radical functions and hypergeometric terms. They appear in many applications such as automatic proofs of combinatorial identities ([10]), and factorization of finite-dimensional linear functional systems ([15]). In practice, it is necessary to check linear dependence of hyperexponential elements. For example, to compute hyperexponential solutions of linear functional systems, one often needs to decide whether a solution is linearly dependent on other solutions over the constants, or over the ground field.

There are some well-known criteria for linear dependence of elements of a differential (difference) field F. If F is an ordinary differential (resp. difference) field, then a finite number of elements of F are linearly dependent over the constants if and only if their Wronskian (resp. Casoratian, see [3, page 271]) equals zero. If F is a partial differential field, then a finite number of elements of F are linearly dependent over the constants if and only if some Wronskian-like determinants vanish ([5, page 86]). However, the above criteria are not valid in general for differential-difference rings that contain zero-divisors. The following example is from [4, Example 6.1].

Example 1.1 *Let R be the ring of infinite sequences of the form $(a_1, a_2, \cdots)$ for $a_i \in \mathbb{C}$, where addition and multiplication are defined coordinatewise. Let I be the ideal of all sequences with at most a finite number of nonzero terms*

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and let $\mathcal{S}_{\mathbb{C}} = R/I$. The shift map $\sigma : (a_1, a_2, \cdots) \mapsto (a_2, a_3, \cdots)$ defines an automorphism of $\mathcal{S}_{\mathbb{C}}$. Every element $c \in \mathbb{C}$ is regarded as a sequence $(c, c, \cdots)$. So the set of all constants of $\mathcal{S}_{\mathbb{C}}$ with respect to σ is $\mathbb{C}$. Consider two sequences $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ in $\mathcal{S}_{\mathbb{C}}$, where

$$x_i = \begin{cases} 1 & i \equiv 0 \mod 4 \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad y_i = \begin{cases} 1 & i \equiv 2 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

By a straightforward calculation, we have

$$\text{Casoratian}(x, y) = \begin{vmatrix} x & y \\ \sigma(x) & \sigma(y) \end{vmatrix} = 0.$$

However, the sequences x and y are linearly independent over $\mathbb{C}$. Here both sequences x and y are zero-divisors.

Hyperexponential functions are usually called exponential functions in the differential case and hypergeometric sequences in the difference case. They can be regarded as elements in differential-difference rings. The results of this paper include a criterion for determining whether hyperexponential elements are linearly dependent over the constants, and a method for determining whether they are linearly dependent over the ground field.

The rest of this paper is organized as follows. After introducing the notions of Δ-rings and hyperexponential elements in Section 2, we present in Section 3 a method for determining whether hyperexponential elements (vectors) are linearly dependent over the constants. A method is described in Section 4 for checking their linear dependence over the ground field.

2 Hyperexponential elements

Let R be a commutative ring. A derivation δ on R is an additive map from R to itself satisfying

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \text{for all } a, b \in R.$$

The pair (R, δ) is called an ordinary differential ring. For an automorphism σ of R, the pair (R, σ) is called an ordinary difference ring. If R is a field, then (R, δ) and (R, σ) are called ordinary differential and difference fields, respectively.

Let Δ be a finite set of commuting maps from R to itself. A map in Δ is assumed to be either a derivation or an automorphism. The pair (R, Δ) is called a *differential-difference ring*, or a Δ-*ring* for short. It is a Δ-field when R is a field. Clearly, a Δ-ring is a partial differential (resp. difference) ring if Δ contains only derivations (resp. automorphisms).

An element c of R is called a *constant* with respect to a derivation δ if $\delta(c) = 0$. An element c is called a *constant* with respect to an automorphism σ if $\sigma(c) = c$. An element c of R is called a *constant* if it is a constant with respect to all the maps in Δ. The set of constants of R, denoted by C_R, is a subring, and it is a subfield if R is a field.

In the sequel, let F be a Δ-field. It is sometimes referred as the ground field. A commutative ring R containing F is called a Δ-*extension* of F if every derivation in Δ can be extended to a derivation of R, every automorphism in Δ can be extended to an automorphism of R, and the extended maps commute pairwise.

A nonzero element h in a Δ-extension of F is said to be *hyperexponential over F with respect to a map ϕ in Δ* if $\phi(h) = r_\phi h$ for some $r_\phi \in F$. The element r_ϕ is called the *certificate* of h with respect to ϕ. An element h is said to be *hyperexponential* over F if it is hyperexponential with respect to all the maps in Δ. In particular, every nonzero element of F is hyperexponential.

Example 2.1 *Consider the Δ-ring $(\mathcal{S}_{\mathbb{C}}, \{\sigma\})$ in Example 1.1. It is a Δ-extension of $\mathbb{C}$. Recall that a sequence $(a_n)_{n=1}^{\infty} \in \mathcal{S}_{\mathbb{C}}$ is rational if there exists a rational function $f(x) \in \mathbb{C}(x)$ such that $a_n = f(n)$ for all but finitely many $n \in \mathbb{Z}^+$ (see [10, Definition 8.2.1]). The set $\mathcal{R}_{\mathbb{C}}$ of all rational sequences is a Δ-field, and $\mathcal{S}_{\mathbb{C}}$ is also a Δ-extension of $\mathcal{R}_{\mathbb{C}}$. The sets of constants of $\mathcal{S}_{\mathbb{C}}$ and $\mathcal{R}_{\mathbb{C}}$ are both equal to $\mathbb{C}$.*

One can easily verify that the sequence $(a, a^2, a^3, \ldots)$*, where* $a \in \mathbb{C}$ *and* $a \neq 0$*, is hyperexponential (hypergeometric) over* $\mathbb{C}$*, while the sequence* $(1!, 2!, 3!, \ldots)$ *is hyperexponential (hypergeometric) over* $\mathcal{R}_{\mathbb{C}}$*.*

Let R be a Δ-extension of F. For a first-order system

$$\phi(z) = f_\phi z \quad \text{for all } \phi \in \Delta, \tag{1}$$

where $f_\phi \in F$, a nonzero solution of (1) in R is hyperexponential over F. Conversely, a hyperexponential element h in R over F is a solution of a system in the form (1), where f_ϕ is the certificate of h with respect to ϕ.

Example 2.2 *Consider the* Δ*-field* $(\mathbb{C}(x,k), \{\delta, \sigma\})$*, where* x *and* k *are indeterminates, and* $\delta = \frac{d}{dx}$ *and* σ *is the shift operator sending* k *to* $k+1$*. Every nonzero element of* $\mathbb{C}(x,k)$ *is clearly hyperexponential. The expression* $kx^{\frac{1}{3}}x^k$ *may be understood as a solution of the first-order system*

$$\left\{ \delta(z) = \frac{1+3k}{3x} z, \ \sigma(z) = \frac{(k+1)x}{k} z \right\}.$$

Thus, $kx^{\frac{1}{3}}x^k$ *is in some* Δ*-extension of* $\mathbb{C}(x,k)$ *(see [1, Theorem 1]), and is hyperexponential over* $\mathbb{C}(x,k)$*.*

The product of two hyperexponential elements in a Δ-extension of F is again hyperexponential. If an invertible element of R is hyperexponential, so is its inverse. However, the sum of two hyperexponential elements is not necessarily hyperexponential. Consider the Δ-extension $\mathcal{S}_{\mathbb{C}}$ of $\mathbb{C}$ in Example 2.1. The sequences $(1,1,1,1,\ldots)$ and $(1,-1,1,-1,\ldots)$ are both hyperexponential over $\mathbb{C}$, but their sum $(2,0,2,0,\ldots)$ is not. More alarmingly, this sum is a zero divisor in $\mathcal{S}_{\mathbb{C}}$. Thus, in general, hyperexponential elements cannot live in a field.

Two hyperexponential elements in a Δ-extension of F are said to be *similar* if they are linearly dependent over F. The similarity among hyperexponential elements is an equivalence relation.

Notation. For an integer $m > 1$, an element of R^m is always understood as a *column* vector with m entries. For $r_1, \ldots, r_n \in R$, $\operatorname{diag}(r_1, \ldots, r_n)$ denotes the $n \times n$ matrix whose entries are $r_1, \ldots, r_n$ on the principal diagonal and are zero elsewhere.

Consider a matrix system

$$\phi(\mathbf{z}) = A_\phi \mathbf{z} \quad \text{for all } \phi \in \Delta, \tag{2}$$

where $\mathbf{z}$ is a column vector consisting of m unknowns and the A_ϕ are $m \times m$ matrices over F. A nonzero solution $h\mathbf{v}$ of (2), where h is hyperexponential in some Δ-extension R of F and $\mathbf{v} \in F^m$, corresponds to a one-dimensional submodule of the Laurent-Ore module associated with (2) (see [8, §3]). Such a submodule helps us to factorize (2) over F. This observation motivates us to define that a nonzero vector $\mathbf{h} \in R^m$ is *hyperexponential* over F if its nonzero entries are hyperexponential and similar to each other. We choose to exclude zero vector in our definition since it corresponds to a trivial solution. Clearly, $\mathbf{h} \in R^m$ is hyperexponential if and only if it can be written as a product of a hyperexponential element h of R and a nonzero vector $\mathbf{v}$ in F^m. For a hyperexponential vector $\mathbf{h} = h\mathbf{v}$ as given above, we have $\mathbf{h} = (fh)\left(\frac{1}{f}\mathbf{v}\right)$ for any nonzero element $f \in F$. Moreover, if $\mathbf{h} = g\mathbf{u}$, where $g \in R$ is hyperexponential over F and $\mathbf{u}$ is in F^m, then h and g are similar over F.

3 Linear dependence over the constants

In this section we generalize a well-known classical result on Wronskian (Casoratian) determinants. Recall that our ground field F is a Δ-field. We denote by Θ the (commutative) monoid generated by the maps in Δ under composition. Moreover, Θ_k stands for the subset of Θ consisting of maps that are composed of at most k maps in Δ. In particular, Θ_0 contains only the identity map.

Let R be a Δ-extension of F. For an element θ in Θ and $\mathbf{r}_1 = (r_{11}, \ldots, r_{m1})^T, \ldots, \mathbf{r}_n = (r_{1n}, \ldots, r_{mn})^T$ in R^m, we form an $m \times n$ matrix whose (i,j)-entry equals $\theta(r_{ij})$. For brevity, we denote this matrix by $(\theta(\mathbf{r}_1), \ldots, \theta(\mathbf{r}_n))$. For a finite subset $\Theta' \subset \Theta$, $(\Theta'(\mathbf{r}_1), \ldots, \Theta'(\mathbf{r}_n))$ stands for the stacking of the blocks of the form $(\theta(\mathbf{r}_1), \ldots, \theta(\mathbf{r}_n))$ for all $\theta \in \Theta'$. This $m|\Theta'| \times n$ matrix has entries in R. The order to stack the blocks is insignificant in the sequel.

Example 3.1 *Let F be the Δ-field in Example 2.2 and $R = F[\exp(x), \exp(-x), x^k, x^{-k}]$. We extend the derivation δ to R by letting $\delta(\exp(x)) = \exp(x)$ and $\delta(x^k) = kx^{k-1}$, and extend the automorphism σ by $\sigma(\exp(x)) = \exp(x)$ and $\sigma(x^k) = x^{k+1}$. Then R becomes a Δ-extension of F. Let*

$$\mathbf{h}_1 = \begin{pmatrix} kx^k \\ x^{k+1} \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} k\exp(x) \\ x\exp(x) \end{pmatrix}$$

be two vectors in R^2. It is easy to verify that the two vectors are hyperexponential over F. The matrices $(\mathbf{h}_1, \mathbf{h}_2)$, $(\delta(\mathbf{h}_1), \delta(\mathbf{h}_2))$ and $(\sigma(\mathbf{h}_1), \sigma(\mathbf{h}_2))$ are

$$M_{\mathbf{1}} = \begin{pmatrix} kx^k & k\exp(x) \\ x^{k+1} & x\exp(x) \end{pmatrix}, \quad M_\delta = \begin{pmatrix} k^2x^{k-1} & k\exp(x) \\ (k+1)x^k & (x+1)\exp(x) \end{pmatrix}$$

and

$$M_\sigma = \begin{pmatrix} (k+1)x^{k+1} & (k+1)\exp(x) \\ x^{k+2} & x\exp(x) \end{pmatrix},$$

respectively. Let $\Theta' = \{\mathbf{1}, \delta, \sigma\}$, where $\mathbf{1}$ stands for the identity map. Then $(\Theta'(\mathbf{h}_1), \Theta'(\mathbf{h}_2))$ equals the 6×2 matrix formed by stacking $M_{\mathbf{1}}, M_\delta$ and M_σ.

The reader may find in [2] a general definition of the rank of a matrix over a commutative ring. Let $\mathbf{h}_1, \ldots, \mathbf{h}_n \in R^m$ be hyperexponential. The next lemma enables us to use linear algebra over F to compute the rank of $(\Theta'(\mathbf{h}_1), \ldots, \Theta'(\mathbf{h}_n))$, which is an $m|\Theta'| \times n$ matrix over R.

Lemma 3.2 *Let F be a Δ-field and R be a Δ-extension of F. Let $\mathbf{h}_1, \ldots, \mathbf{h}_n$ be hyperexponential vectors in R^m and Θ' a finite subset of Θ. Then there exist an $m|\Theta'| \times n$ matrix $A_{\Theta'}$ over F and $h_1, \ldots, h_n$ in R, hyperexponential over F, such that*

$$(\Theta'(\mathbf{h}_1), \ldots, \Theta'(\mathbf{h}_n)) = A_{\Theta'} \cdot \operatorname{diag}(h_1, \ldots, h_n). \tag{3}$$

In particular, the rank of $(\Theta'(\mathbf{h}_1), \ldots, \Theta'(\mathbf{h}_n))$ is equal to that of $A_{\Theta'}$ if $h_1, \ldots, h_n$ are invertible in R.

Proof. Write $\mathbf{h}_i = h_i \mathbf{v}_i$, where $h_i \in R$ is hyperexponential and $\mathbf{v}_i \in F^m$ is nonzero for $i = 1, \ldots, n$. Applying an operator $\theta \in \Theta$ to $h_i \mathbf{v}_i$ yields $\theta(\mathbf{h}_i) = h_i \mathbf{v}_{i,\theta}$ for some vectors $\mathbf{v}_{i,\theta} \in F^m$, since h_i is hyperexponential. Let A_θ be the $m \times n$ matrix whose i-th column is composed of the entries of $\mathbf{v}_{i,\theta}$. Then $(\theta(\mathbf{h}_1), \ldots, \theta(\mathbf{h}_n))$ is equal to $A_\theta \cdot \operatorname{diag}(h_1, \ldots, h_n)$. The matrix $A_{\Theta'}$ is constructed by stacking all the A_θ for $\theta \in \Theta'$. □

Note that every hyperexponential vector $\mathbf{h}$ can be written in the form $h\mathbf{v}$ in many ways, so the matrix $A_{\Theta'}$ in Lemma 3.2 depends on the choices of $h_1, \ldots, h_n$.

Example 3.3 *The matrix $(\Theta'(\mathbf{h}_1), \Theta'(\mathbf{h}_2))$ in Example 3.1 may be decomposed as*

$$\begin{pmatrix} k & k \\ x & x \\ \frac{k^2}{x} & k \\ k+1 & x+1 \\ (k+1)x & k+1 \\ x^2 & x \end{pmatrix} \cdot \begin{pmatrix} x^k & 0 \\ 0 & \exp(x) \end{pmatrix}.$$

A generalized Wronskian (Casoratian) criterion for hyperexponential vectors is given below.

Proposition 3.4 *Let F be a Δ-field, R be a Δ-extension of F, and $\mathbf{h}_1, \ldots, \mathbf{h}_n$ be hyperexponential vectors in R^m. Denote by W_i the $m|\Theta_{i-1}| \times i$ matrix $(\Theta_{i-1}(\mathbf{h}_1), \ldots, \Theta_{i-1}(\mathbf{h}_i))$ for $i = 1, \ldots, n$. Then* *(i) If $\mathbf{h}_1,\ldots,\mathbf{h}_n$ are linearly dependent over C_R, then W_n has rank less than n.*

(ii) Suppose that there exist invertible and hyperexponential elements $h_1, \ldots, h_n$ in R such that

$$\mathbf{h}_1 = h_1\mathbf{v}_1, \ldots, \mathbf{h}_n = h_n\mathbf{v}_n \quad \textit{for some } \mathbf{v}_1,\ldots,\mathbf{v}_n \textit{ in } F^m.$$

If W_n has rank less than n, then $\mathbf{h}_1,\ldots,\mathbf{h}_n$ are linearly dependent over C_R.

Proof. Suppose that $\mathbf{h}_1,\ldots,\mathbf{h}_n$ are linearly dependent over C_R. Then there exist $c_1,\ldots,c_n \in C_R$, not all zero, such that $\sum_{i=1}^n c_i\mathbf{h}_i = 0$. For any $\theta \in \Theta$, we have $\sum_{i=1}^n c_i\theta(\mathbf{h}_i) = 0$. In particular,

$$W_n \cdot (c_1, c_2, \ldots, c_n)^T = 0,$$

thus the rank of W_n is less than n by Theorem 5.3 in [2].

We prove the second assertion by induction on n. The assertion clearly holds for $n = 1$. Assume that it holds for lower values of n, and that, furthermore, any $n-1$ elements among $\mathbf{h}_1,\ldots,\mathbf{h}_n$ are linearly independent over C_R. From Lemma 3.2 there exists an $m|\Theta_{n-1}| \times n$ matrix A_n with entries in F such that $W_n = A_n \cdot \mathrm{diag}(h_1,\ldots,h_n)$. The invertibility of $h_1, \ldots, h_n$ implies that the rank of A_n equals that of W_n, and hence A_n has rank less than n. There exist $f_1,\ldots,f_n \in F$, not all zero, such that $A_n \cdot (f_1,\ldots,f_n)^T = 0$. Setting $x_i = h_i^{-1} f_i$, $i = 1,\ldots,n$, yields $W_n \cdot (x_1,\ldots,x_n)^T = 0$. Without loss of generality, suppose that x_n is nonzero. Then x_n can be set to be 1 since it is invertible. Thus

$$\sum_{i=1}^{n-1} x_i\theta(\mathbf{h}_i) + \theta(\mathbf{h}_n) = 0 \quad \text{for all } \theta \in \Theta_{n-1}. \tag{4}$$

In particular,

$$\sum_{i=1}^{n-1} x_i\theta(\mathbf{h}_i) + \theta(\mathbf{h}_n) = 0 \quad \text{for all } \theta \in \Theta_{n-2}. \tag{5}$$

Applying a derivation $\delta \in \Delta$ to (5) yields

$$\sum_{i=1}^{n-1} \delta(x_i)\theta(\mathbf{h}_i) + \sum_{i=1}^{n-1} x_i\delta\circ\theta(\mathbf{h}_i) + \delta\circ\theta(\mathbf{h}_n) = 0 \quad \text{for all } \theta \in \Theta_{n-2},$$

which, together with (4) for $\delta\circ\theta$, implies that $\sum_{i=1}^{n-1} \delta(x_i)\theta(\mathbf{h}_i) = 0$ for all $\theta \in \Theta_{n-2}$.

If $\delta(x_i)$ is nonzero for some i with $1 \le i \le n-1$, then the matrix W_{n-1} has rank less than $n-1$ by Theorem 5.3 in [2]. Accordingly, $\mathbf{h}_1,\ldots,\mathbf{h}_{n-1}$ would be linearly dependent over C_R by the induction hypothesis, a contradiction. Therefore, $\delta(x_i) = 0$ for $i = 1,\ldots,n-1$ and for every derivation $\delta \in \Delta$.

Likewise, applying an automorphism $\sigma \in \Delta$ to (5) yields

$$\sum_{i=1}^{n-1} \sigma(x_i)\sigma\circ\theta(\mathbf{h}_i) + \sigma\circ\theta(\mathbf{h}_n) = 0 \quad \text{for all } \theta \in \Theta_{n-2},$$

which, together with (4) for $\sigma\circ\theta$, implies

$$\sum_{i=1}^{n-1} (\sigma(x_i) - x_i)\,\sigma\circ\theta(\mathbf{h}_i) = 0 \quad \text{for all } \theta \in \Theta_{n-2},$$

therefore $\sum_{i=1}^{n-1}(x_i - \sigma^{-1}(x_i))\theta(\mathbf{h}_i) = 0$ for all $\theta \in \Theta_{n-2}$. A similar argument shows that $\sigma(x_i) = x_i$ for $i = 1,\ldots,n-1$ and for every automorphism $\sigma \in \Delta$. So $x_1, \ldots, x_{n-1}$ are all constants, and $\mathbf{h}_1, \ldots, \mathbf{h}_n$ are linearly dependent over C_R by (4). □

The invertibility of hyperexponential elements $h_1,\ldots,h_n$ is essential for the proof of the second assertion of Proposition 3.4. So far we have failed to weaken the assumption on invertibility.

An algebraic ideal I in a Δ-ring is said to be *invariant* if $\phi(I) \subset I$ for all $\phi \in \Delta$. A Δ-ring R is said to be *simple* if its invariant ideals are (0) and R. Let F be a Δ-field contained in R. For every hyperexponential element $h \in R$ over F, the ideal (h) is invariant. Thus, h is invertible if R is simple. Examples for simple Δ-rings are Picard-Vessiot extensions for finite-dimensional linear functional systems (see [11, 12] for the ordinary case, and [1] for the partial case).

The difference ring $\mathcal{S}_{\mathbb{C}}$ in Example 2.1 is not simple (see [11, page 5]), while hyperexponential (hypergeometric) sequences in $\mathcal{S}_{\mathbb{C}}$ over $\mathcal{R}_{\mathbb{C}}$ are all invertible.

The matrix $(\Theta'(\mathbf{h}_1), \Theta'(\mathbf{h}_2))$ in Example 3.3 has rank two, therefore by Proposition 3.4(i) the vectors $\mathbf{h}_1, \mathbf{h}_2 \in R^2$ in Example 3.1 are linearly independent over C_R.

Example 3.5 *Let the Δ-field F be the same as in Example 2.2. Consider the system*

$$\frac{dz}{dx} = z \quad \text{and} \quad \sigma(z) = kz. \tag{6}$$

The Picard-Vessiot extension R of (6) is a simple Δ-ring with $C_R = \mathbb{C}$ (see [1, Theorem 2])). We denote a solution of (6) in R by y, which is hyperexponential over F. Let us decide if the three vectors

$$\mathbf{h}_1 = (x+k)y \begin{pmatrix} \frac{1}{k(1-k)} \\ \frac{1}{x(x+1)} \end{pmatrix}, \quad \mathbf{h}_2 = \frac{y}{k} \begin{pmatrix} \frac{x+1}{k-1} \\ \frac{-k}{x} \end{pmatrix}, \quad \mathbf{h}_3 = y \begin{pmatrix} \frac{1}{k} \\ \frac{1-k}{x(x+1)} \end{pmatrix}$$

are $\mathbb{C}$-linearly dependent. Let $\Theta_2 = \{1, \delta, \delta^2, \sigma, \sigma^2, \sigma\delta\}$. By Lemma 3.2

$$(\Theta_2(\mathbf{h}_1), \Theta_2(\mathbf{h}_2), \Theta_2(\mathbf{h}_3)) = A \cdot \operatorname{diag}\left((x+k)y, \frac{y}{k}, y\right)$$

for some 12×3 matrix A over F. A direct calculation shows that the rank of A is less than 3. Since R is simple and y is invertible in R, then $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ are linearly dependent over $\mathbb{C}$ by the second assertion of Proposition 3.4.

The next corollary is a "scalar" version of Proposition 3.4.

Corollary 3.6 *Let F be a Δ-field, R be a Δ-extension of F and $h_1, \ldots, h_n$ be invertible and hyperexponential elements of R. Then $h_1, \ldots, h_n$ are linearly dependent over C_R if and only if the matrix $(\Theta_{n-1}(h_1), \ldots, \Theta_{n-1}(h_n))$ has rank less than n.*

Proof. Setting $m = 1$ and $\mathbf{v}_i = 1$ for $i = 1, \ldots, n$ in the proof of Proposition 3.4 yields the corollary. □

Assume that $R = F$, which is a Δ-field. Proposition 3.4 is applicable to test linear dependence of vectors in F^m over C_F, because every nonzero element of F is invertible and hyperexponential over F. Proposition 3.4 specializes to the well-known Wronskian (resp. Casoratian) when $m = 1$ and Δ contains only a derivation (resp. an automorphism). It corresponds to Theorem 1 in [5, page 86] if Δ contains only derivations, and to the difference analogue of that theorem if Δ contains only automorphisms.

4 Linear dependence over the ground field

Let F be a Δ-field and R a Δ-extension of F. We study how to determine whether a finite number of hyperexponential elements (resp. vectors) in R (resp. in R^m) are linearly dependent over F. First, we present a generalization of Proposition 3.3 in [7], which connects the notion of similarity with linear dependence of hyperexponential elements over F.

Proposition 4.1 *Let F be a Δ-field and R a Δ-extension of F. Assume that $C_R = C_F$. Let $h_1, \ldots, h_n \in R$ be invertible and hyperexponential over F. Then $h_1, \ldots, h_n$ are pairwise dissimilar if and only if they are linearly independent over F.*

Proof. If $h_1,\ldots,h_n \in R$ are linearly independent over F, so are h_i and h_j with $i \neq j$. Thus $h_1,\ldots,h_n$ are pairwise dissimilar by definition.

We prove the converse by induction on n. The proposition clearly holds for $n=2$. Assume that it holds for lower values of n. Suppose that $h_1,\ldots,h_n$ are pairwise dissimilar but are linearly dependent over F. A possible rearrangement of indices leads to

$$h_n = f_1h_1 + f_2h_2 + \ldots + f_{n-1}h_{n-1} \quad \text{for some } f_1,\ldots,f_{n-1} \in F. \tag{7}$$

By the induction hypothesis, $h_1,\ldots,h_{n-1}$ are linearly independent over F. The assumption $C_R = C_F$ then implies that $f_1h_1,\ldots,f_{n-1}h_{n-1}$ are linearly independent over C_R. From Corollary 3.6, there exist $\theta_1,\theta_2\ldots,\theta_{n-1}$ in Θ such that $D = \det(\theta_i(f_jh_j))_{1\le i\le n-1, 1\le j\le n-1}$ is nonzero. Since $h_1,\ldots,h_{n-1},h_n$ are hyperexponential, there exist r_{ij} in F such that $\theta_i(f_jh_j) = r_{ij}h_j$ for all i,j with $1 \le i \le n-1$ and $1 \le j \le n-1$, and, moreover, r_{in} in F such that $\theta_i(h_n) = r_{in}h_n$ for all i with $1 \le i \le n-1$. Applying $\theta_1,\ldots,\theta_{n-1}$ to (7) then yields a linear system

$$\begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1,n-1} \\ r_{21} & r_{22} & \ldots & r_{2,n-1} \\ \vdots & \vdots & \ldots & \vdots \\ r_{n-1,1} & r_{n-1,2} & \ldots & r_{n-1,n-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{pmatrix} = \begin{pmatrix} r_{1n}h_n \\ r_{2n}h_n \\ \vdots \\ r_{n-1,n}h_n \end{pmatrix}.$$

Since $D = \prod_{i=1}^{n-1} h_i \det(r_{ij})$ is nonzero, the coefficient matrix (r_{ij}) is of full rank. By Cramer's rule, $h_i = q_ih_n$ for some $q_i \in F$. So h_i and h_n are similar for $i = 1,\ldots,n-1$, a contradiction. □

The assumption $C_R = C_F$ in Proposition 4.1 is indispensable. For example, let $F = \mathbb{Q}$, $R = \mathbb{Q}(\pi)$ and δ be the derivation on R that maps everything to zero. Then $C_F = \mathbb{Q}$ and $C_R = \mathbb{Q}(\pi)$. The elements $\pi+2$, $\pi+3$ and $\pi+4$ are invertible in R and hyperexponential over F. They are linearly dependent over $\mathbb{Q}$ but dissimilar to each other.

If the ground field F has characteristic zero and C_F is algebraically closed, then Picard-Vessiot extensions of F contain no new constants (see [1, Theorem 5]). The fields of constants of $\mathcal{S}_{\mathbb{C}}$ and of $\mathcal{R}_{\mathbb{C}}$ in Example 2.1 are both equal to $\mathbb{C}$. So Proposition 4.1 is applicable to hyperexponential elements in these two Δ-extensions.

An immediate application of Proposition 4.1 is

Corollary 4.2 *Let F be a Δ-field and R a Δ-extension of F. Assume that $C_R = C_F$. Let $h_1,\ldots,h_n \in R$ be invertible and hyperexponential over F. Then $h_1,\ldots,h_n$ are algebraically dependent over F if and only if there exist integers $e_1,\ldots,e_n$ such that $h_1^{e_1}\cdots h_n^{e_n} \in F$.*

Proof. If $h_1^{e_1}\cdots h_n^{e_n} \in F$ for some $e_1,\ldots,e_n \in \mathbb{Z}$, then $h_1,\ldots,h_n$ satisfy a nonzero n-variate polynomial over F which has two nonzero terms. Thus $h_1,\ldots,h_n$ are algebraically dependent over F.

Conversely, suppose that $(h_1,\ldots,h_n)$ is a solution of some nonzero n-variate polynomial over F. Then the power products of $h_1,\ldots,h_n$ are linearly dependent over F. Proposition 4.1 implies that there exist two such power products similar to each other over F, since all the power products of $h_1, \ldots, h_n$ are invertible and hyperexponential. It follows that there exist $e_1,\ldots,e_n \in \mathbb{Z}$ such that $h_1^{e_1}\cdots h_n^{e_n} \in F$. □

Corollary 4.2 appears to be known at least in the ordinary differential and difference cases (see [13, 14]).

Example 4.3 *Let $\mathcal{S}_{\mathbb{C}}$ and $\mathcal{R}_{\mathbb{C}}$ be the same as those in Example 2.1. Assume that $c \in \mathbb{C}$ is neither zero nor a root of unity. We show that the hypergeometric sequence $S = (c, c^2, c^3, \ldots) \in \mathcal{S}_{\mathbb{C}}$ is transcendental over $\mathcal{R}_{\mathbb{C}}$. Suppose the contrary, then $S^e \in \mathcal{R}_{\mathbb{C}}$ for some $e \in \mathbb{Z}^+$ by Corollary 4.2. Thus, there exist a rational function $f(x) \in \mathbb{C}(x)$ and an integer $m \in \mathbb{Z}^+$ such that $c^{ne} = f(n)$ for all integers n with $n \ge m$. It follows that $c^e = \frac{f(n+1)}{f(n)}$ for all integers $n \ge m$. Note that the rational function $\frac{f(x+1)}{f(x)}$ can be written as $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are monic polynomials in $\mathbb{C}[x]$. Since $q(n)c^e = p(n)$ for all integer $n \ge m$, we have $q(x)c^e = p(x)$ in $\mathbb{C}[x]$. Thus, $c^e = 1$, a contradiction.*

A few words need to be said about similarity. Let F be a Δ-field and R be a Δ-extension of F with $C_R = C_F$. Assume that $h_1, h_2 \in R$ are invertible and hyperexponential over F. Then one can decide whether h_1 and h_2 are similar by their certificates. Note that $\frac{h_1}{h_2}$ is hyperexponential over F, so we let $f_\phi \in F$ be the certificate of $\frac{h_1}{h_2}$ with respect to ϕ for $\phi \in \Delta$. Consider the first-order system:

$$\phi(z) = f_\phi z \quad \text{for every } \phi \in \Delta. \tag{8}$$

Note that $\frac{h_1}{h_2}$ is a solution of (8) in R. If h_1 and h_2 are similar, then $\frac{h_1}{h_2} \in F$ by definition, and so (8) has a nonzero solution in F. Conversely, if (8) has a nonzero solution $r \in F$, then there exists $c \in C_R$ such that $\frac{h_1}{h_2} = cr$. Since $C_R = C_F$, $\frac{h_1}{h_2}$ is in F, i.e. h_1 and h_2 are similar. Therefore, under the assumption given above, h_1 and h_2 are similar if and only if (8) has a solution in F. One may find in [9, Proposition 6.3](see also [6, §6]) a method for determining whether (8) has a nonzero rational solution, when F is the field of rational functions and Δ consists of usual partial differential and shift operators.

At last, we present how to test linear dependence of hyperexponential vectors over the ground field F. Recall that R is a Δ-extension of F with $C_R = C_F$. Let the vectors $\mathbf{h}_1, \ldots, \mathbf{h}_n \in R^m$ be hyperexponential over F. Write $\mathbf{h}_i = h_i \mathbf{v}_i$ where $\mathbf{v}_i \in F^m$ and $h_i \in R$ for $i = 1, \ldots, n$. In addition, the h_i are invertible and hyperexponential. A procedure for testing the F-linear dependence of $\mathbf{h}_1, \ldots, \mathbf{h}_n$ are outlined below.

1. Partition $h_1, \ldots, h_n$ into equivalence classes $H_1, \ldots, H_k$ with respect to similarity. If $k = n$, then $\mathbf{h}_1, \ldots, \mathbf{h}_n$ are linearly independent over F.

2. Suppose $k < n$. Partition the set $\{\mathbf{h}_1, \ldots, \mathbf{h}_n\}$ into k subsets:

$$G_i = \{g_i \mathbf{u}_{i,1}, \ldots, g_i \mathbf{u}_{i,s_i}\}, \quad i = 1, \ldots, k.$$

   where g_i is in H_i and the $\mathbf{u}_{i,j}$ are in F^m.

3. $\mathbf{h}_1, \ldots, \mathbf{h}_n$ are linearly dependent over F if and only if there exists an integer l with $1 \le l \le k$ such that $\mathbf{u}_{l,1}, \ldots, \mathbf{u}_{l,s_l}$ are linearly dependent over F.

The conclusion made in the first step is immediate from Proposition 4.1. We now prove the conclusion in the last step. Assume that $\mathbf{h}_1, \ldots, \mathbf{h}_n$ are linearly dependent over F. Using the notation introduced in the second step, we have

$$\sum_{i=1}^{k} g_i \underbrace{\left(\sum_{j=1}^{s_i} f_{ij} \mathbf{u}_{i,j} \right)}_{\mathbf{w}_i} = 0 \tag{9}$$

where the f_{ij} are in F, not all zero. If $\mathbf{w}_i$ is a nonzero vector for some i with $1 \le i \le k$, then (9) implies that $g_1, \ldots, g_k$ are linearly dependent over F, a contradiction to Proposition 4.1. Thus all the $\mathbf{w}_i$ are zero vectors. Since the f_{ij} are not all zero, there exists an integer l with $1 \le l \le k$ such that $\mathbf{u}_{l,1}, \ldots, \mathbf{u}_{l,s_l}$ are linearly dependent over F.

When the above procedure is applicable, the C_F-linear dependence of hyperexponential vectors can also be tested by first investigating their F-linear dependence, and then checking C_F-linear dependence among several groups of vectors with entries in F. However, Proposition 3.4 is applicable to determine whether hyperexponential vectors are C_R-linearly dependent even in the case $C_R \neq C_F$.

5 Concluding remarks

In this paper, we present a criterion for determining whether invertible hyperexponential elements are linearly dependent over the constants. The criterion allows us to check linear dependence of a finite number of (invertible) hyperexponential elements over the ground field, provided that we can determine the similarity of these elements.

As part of the future work, observe that the sums of hyperexponential elements in a Δ-ring R form a Δ-subring. It would be interesting to test the C_R-linear dependence of elements in that subring.

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References

[1] M. Bronstein, Z. Li, and M. Wu. Picard–Vessiot extensions for linear functional systems. In M. Kauers, ed., *Proc. International Symposium on Symbolic and Algebraic Computation*, 68–75. ACM Press, 2005.

[2] W.C. Brown. *Matrices over Commutative Rings*, Pure and Applied Math **169**. Marcel Dekker, Inc., New York, 1993.

[3] R.M. Cohn. *Difference Algebra*. Intersci. Publishers, New York, 1965.

[4] P.A. Hendriks and M.F. Singer. Solving difference equations in finite terms. *J. Symb. Comput*, **27**:239–260, 1999.

[5] E.R. Kolchin. *Differential Algebra and Algebraic Groups, Pure and Applied Math* **54**. Academic Press, New York-London, 1973.

[6] G. Labhan and Z. Li. Hyperexponential solutions of finite-rank ideals in othorgonal Ore algebras. In J. Gutierrz, ed., *Proc. International Symposium on Symbolic and Algebraic Computation*, 213–220. ACM Press, 2004.

[7] Z. Li, F. Schwarz, and S. Tsarev. Factoring systems of linear PDE's with finite-dimensional solution spaces. *J. Symb. Comput.*, **36**:443–471, 2003.

[8] Z. Li, M.F. Singer, M. Wu, and D. Zheng. A recursive method for determining the one-dimensional submodules of Laurent-Ore modules. In J. Dumas, ed., *Proc. International Symposium on Symbolic and Algebraic Computation*, 220–227. ACM Press, 2006.

[9] Z. Li and D. Zheng. Determining whether a multivariate hyperexponential function is algebraic. *J. Syst. Sci. and Compl.*, **9**:352-364, 2006.

[10] M. Petkovšek, H. Wilf, and D. Zeilberger. *A=B*. A K Peters, Ltd, 1996.

[11] M. van der Put and M.F. Singer. *Galois Theory of Difference Equations, Lecture Notes in Mathematics* **1666**. Springer, 1997.

[12] M. van der Put and M.F. Singer. *Galois Theory of Linear Differential Equations, Grundlehren der Mathematischen Wissenschaften* **328**. Springer, Heidelberg, 2003.

[13] M. Rosenlicht. On Liouville's theory of elementary functions. *Pacific Journal of Mathematics*, **65**(2):485-492, 1976.

[14] M.F. Singer. *Private comunication.* 2007.

[15] M. Wu. *On Solutions of Linear Functional Systems and Factorization of Modules over Laurent-Ore Algebras*. PhD thesis, Chinese Academy of Sciences and Université de Nice, 2005.