

# Complete Numerical Isolation of Real Roots in Zero-dimensional Triangular Systems

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## Abstract

We present a complete numerical algorithm of isolating all the real zeros of a zero-dimensional triangular polynomial system  $F_n \subseteq \mathbb{Z}[x_1, \dots, x_n]$ . Our system  $F_n$  is general, with no further assumptions. In particular, our algorithm successfully treat multiple zeros directly in such systems. A key idea is to introduce evaluation bounds and sleeve bounds. We also present a much more efficient algorithm for zero-dimensional triangular systems without multiple roots. We implemented our algorithms and promising experimental results are shown.

*Key words:* Triangular system, real zero isolation, sleeve bound, evaluation bound

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## 1. Introduction

Many problems in the computational sciences and engineering can be reduced to the solving of polynomial equations. There are two basic approaches to solving such polynomial systems – numerically or algebraically. Usually, the numerical methods have no global guarantees of correctness. Algebraic methods for solving polynomial systems include Gröbner bases, characteristic sets, CAD, and resultants (3; 4; 5; 7; 17; 19; 21; 26; 27). One general idea in polynomial equation solving is to reduce the original system into a triangular system. Zero-dimensional polynomial systems are among the most important cases to solve. This paper considers zero-dimensional triangular systems only.

A zero-dimensional triangular system has the form  $F_n = \{f_1, \dots, f_n\}$ , where each  $f_i \in \mathbb{Z}[x_1, \dots, x_i]$  ( $i = 1, \dots, n$ ) and  $x_i$  is a variable of  $f_i$ . We are interested in real zeros of  $F_n$ . A real zero of  $F_n$  is  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  such that

$$f_1(\xi_1) = f_2(\xi_1, \xi_2) = \dots = f_n(\xi_1, \dots, \xi_n) = 0.$$

The standard idea here is to first solve for  $f_1(x_1) = 0$ , and for each solution  $x_1 = \xi_1$  of  $f_1$ , we find the solutions of  $x_2 = \xi_2$  of  $f_2(\xi_1, x_2) = 0$ , etc. The problem is reduced to solving univariate polynomials of the form  $f_i(\xi_1, \dots, \xi_{i-1}, x_i) = 0$ . Such polynomials have algebraic number coefficients. We could isolate roots of such polynomials by using standard root isolation algorithms such as the Sturm sequence method, but using algebraic number arithmetic. But even for very small  $n$ , such algorithms are too slow. The numerical approach is to replace the  $\xi_i$ 's by approximations, and thus reduce the problem to isolating roots of such numerical polynomials. The challenge is how to guarantee completeness of such numerical algorithms.

We will provide a numerical algorithm that solves such triangular systems completely in the following precise sense: given an  $n$ -dimensional box  $R = J_1 \times \dots \times J_n \subseteq \mathbb{R}^n$  where  $J_i$  are intervals, and any precision  $\varepsilon > 0$ , it will isolate the real zeros of  $F_n$  in  $R$  to precision  $\varepsilon$ .

Our solution places no restriction on  $F_n$ . The reason why we consider general zero-dimensional triangular systems is that the triangular systems derived in cylinder algebraic decomposition or topology determination (8) are generally with multiple roots and even non regular (for definition see (26)). Our goal is to deal with such triangular systems directly without factorization or gcd computation over algebraic number fields. It is well known that these computations are expensive.

Many algorithms that seek to provide “exact numerical” solution assume computation over the rational numbers  $\mathbb{Q}$ . But this is much less efficient than using dyadic numbers: let  $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$  denote the set of dyadic numbers (or bigfloats)(29). Most current fast algorithms for bigfloats can be derived from Brent’s work (6). In the following, we use the symbol  $\mathbb{F}$  to denote either  $\mathbb{D}$  or with  $\mathbb{Q}$ . We use intervals to isolate real numbers: let  $\square\mathbb{F}$  denote the set of intervals of the form  $[a, b]$  where  $a \leq b \in \mathbb{F}$ .

Given a polynomial  $f \in \mathbb{R}[x]$  and an interval  $I = [a, b] \in \square\mathbb{F}$ , we construct two polynomials  $f^u, f^d \in \mathbb{F}[x]$ , called *sleeve functions*, such that

$$f^u(x) > f(x) > f^d(x), \forall x \in I.$$

We call  $(I, f^u, f^d)$  a **sleeve** of  $f$  over  $I$  and

$$SB_I(f^u, f^d) := \sup\{f^u(x) - f^d(x) : x \in I\} \quad (1)$$

the **sleeve bound**. Note that the coefficients of  $f^u, f^d$  are in  $\mathbb{F}$ , but  $f$  have real coefficients which can be arbitrarily approximated. Based on the sleeve of  $f$ , we have two complete algorithms for general zero-dimensional triangular system and zero-dimensional triangular system without multiple roots, respectively.

The key idea for general triangular system is the introduction of **evaluation bounds**. For a polynomial  $f \in \mathbb{R}[x]$  and a subset  $I \subseteq \mathbb{R}$ , let its **evaluation bound** be

$$EB_I(f) := \min\{|f(z)| : z \in \text{Zero}_I(f') \cup \{a, b\} \setminus \text{Zero}_I(f)\}, \quad (2)$$

If the following **sleeve-evaluation inequality**

$$SB_I(f^u, f^d) < EB_I(f) \quad (3)$$

holds, we show that the isolating intervals of  $f^u f^d$  can be used to define isolating intervals of  $f$ . The use of evaluation bounds appear to be new. It is the ability to compute lower estimates on  $EB_I(f)$  that allows us to detect zeros of even multiplicities.

For triangular systems without multiple roots, we introduce a much more efficient method without computing the evaluation bound. The basic idea is that when  $f$  is square free,  $\frac{\partial f}{\partial x}$  has no root in a neighborhood of each real root of  $f$ . Based on this, we give a criterion to use the roots of the sleeve function to isolate the roots of the triangular system. Experiments show that the algorithm can be used to isolate the roots for large scale triangular systems efficiently.

As a consequence of the above analysis, the real roots isolation for  $f$  is reduced to real roots isolation for the sleeve functions  $f^d$  and  $f^u$ . Univariate root isolation is a well-developed subject in its own right, with many efficient solutions known (1; 10; 12; 14; 16; 22; 24). In our implementation, we use the method of Rouillier and Zimmermann (22).

The idea of using a sleeve to solve equations was used in (23) and (18). In particular, Lu et al (18) proposed an algorithm to isolate the real roots of triangular systems. Their method could solve many problems in practice, but is not complete and cannot handle multiple zeros. Collins et al (11) considered the problem with interval arithmetic methods and Descartes' method using floating point computation. Again, they pointed out that if a real coefficient is implicitly zero, the method will fail. Xia and Yang (28) consider real root isolation of a semi-algebraic set. They ultimately considered the regular and square-free triangular systems. They mentioned that the method is not complete and will fail in some cases. Eigenwillig et al considered root-isolation for real polynomials with bitstream coefficients (13). Their algorithm requires  $f$  to be squarefree. Our evaluation bound is similar to the curve separation bound in (30). It seems difficult to define sleeves for non-triangular systems, because the variables appear simultaneously. Interesting work on general polynomial systems was done by Hong and Stahl (15).

This is an extended version of our ISSAC paper (9). The efficient algorithm for triangular systems without multiple roots in Section 5 is newly added. The key concept of faithful sleeve in (4) is newly defined to make the algorithm simpler and more precise. Also, proofs for Lemmas 1, 3, 7, 11, 14, 15, 16, and 23 omitted in the ISSAC paper are included.

In Section 2, we describe the basic technique of using sleeves and evaluation bounds of  $f$ . In Section 3, we give methods to compute evaluation bounds, to construct sleeves. and to derive a sleeve bound for a triangular system. In Section 4, we present the root isolation algorithm for triangular systems. Experimental results are also presented. In Section 5, we introduce an algorithm for triangular systems without multiple roots. We conclude the paper in Section 6.

## 2. Root Isolation for Real Univariate Polynomials

We give a framework for isolating the real roots of a univariate polynomial equation with real coefficients.

## 2.1. Evaluation and Sleeve Bounds

Let  $\mathbb{Q}$  be the field of rational numbers,  $\mathbb{R}$  the field of real numbers,  $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$  the set of dyadic numbers, and  $\mathbb{F}$  denote either  $\mathbb{D}$  or  $\mathbb{Q}$ . In this section, we fix  $f, f^u, f^d$  to be  $C^1$  functions, and  $I \in \square\mathbb{F}$ .

For any real function  $f$ , let  $\mathbf{Zero}_I(f)$  denote the set of distinct real zeros of  $f$  in the interval  $I$ . If  $I = \mathbb{R}$ , then we simply write  $\mathbf{Zero}(f)$ . If  $\#(\mathbf{Zero}_I(f)) = 1$ , we call  $I$  an **isolating interval** of  $f$ . Sometimes, we need to count the zeros up to the parity (i.e., evenness or oddness) of their multiplicity. Call a zero  $\xi \in \mathbf{Zero}(f)$  an **even zero** if its multiplicity is even, and **odd zero** if its multiplicity is odd. Define the multiset  $\mathbf{ZERO}_I(f)$  whose underlying set is  $\mathbf{Zero}_I(f)$  and where the multiplicity of  $\xi \in \mathbf{ZERO}_I(f)$  is 1 (resp., 2) if  $\xi$  is an odd (resp., even) zero of  $f$ .

To avoid special treatment near the endpoints of an interval, we assume

$$f^u(a)f^d(a) > 0, \quad f^u(b)f^d(b) > 0. \quad (4)$$

We say that the sleeve  $(I, f^u, f^d)$  is **faithful** for  $f$  if (4) and (3) are both satisfied.

Intuitively,  $f$  is nicely behaved when if we restrict  $f$  to a neighborhood of a zero  $\xi$  where  $|f| < EB(f)$ . This is illustrated in Figure 1.

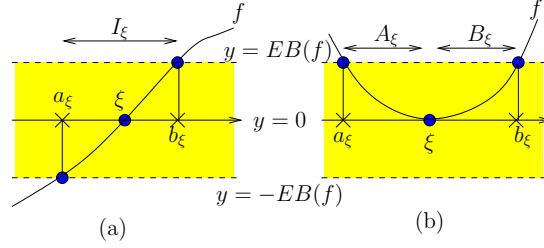


Fig. 1. Neighborhood of  $\xi$ :  $I_\xi = A_\xi \cup \{\xi\} \cup B_\xi$ .

Given  $f$  and  $I$ , define the polynomials

$$\widehat{f}(x) := f(x) - EB_I(f), \quad \overline{f}(x) := f(x) + EB_I(f).$$

If  $\xi \in \mathbf{Zero}_I(f)$ , we define the points  $a_\xi, b_\xi$  as follows:

$$a_\xi := \max\{\{a\} \cup \mathbf{Zero}(\widehat{f} \cdot \overline{f}) \cap (-\infty, \xi)\}, \quad (5)$$

$$b_\xi := \min\{\{b\} \cup \mathbf{Zero}(\widehat{f} \cdot \overline{f}) \cap (\xi, +\infty)\}. \quad (6)$$

Then define the open intervals (see Figure 1):

$$A_\xi := (a_\xi, \xi), \quad B_\xi := (\xi, b_\xi) \text{ and } I_\xi := (a_\xi, b_\xi). \quad (7)$$

Basic properties of these intervals are captured below.

LEMMA 1. Let  $(I, f^u, f^d)$  be a faithful sleeve for  $f$ . For all  $\xi, \zeta \in \mathbf{Zero}_I(f)$ , we have:

- (i) If  $\xi \neq \zeta$  then  $I_\xi$  and  $I_\zeta$  are disjoint.
- (ii)  $\mathbf{Zero}_I(f^u f^d) \subseteq \bigcup_\xi I_\xi$ .
- (iii-a)  $A_\xi \cap \mathbf{Zero}(f^u)$  is empty iff  $A_\xi \cap \mathbf{Zero}(f^d)$  is non-empty.
- (iii-b)  $B_\xi \cap \mathbf{Zero}(f^u)$  is empty iff  $B_\xi \cap \mathbf{Zero}(f^d)$  is non-empty.
- (iv) The derivative  $f'$  has a constant sign in  $A_\xi$  or  $B_\xi$  for any  $\xi \in \mathbf{Zero}_I(f)$ .

**Proof.** (i) Suppose  $\xi < \zeta$  are consecutive zeros of  $\mathbf{Zero}_I(f)$ . Then either  $f$  is positive on  $(\xi, \zeta)$  or  $f$  is negative on  $(\xi, \zeta)$ . Without loss of generality,  $f$  is positive on  $(\xi, \zeta)$ . Then the multiset  $\mathbf{ZERO}_I(\widehat{f}) = \mathbf{ZERO}(f - EB_I(f))$  has at least two zeros (they may have the same value) in  $(\xi, \zeta)$ . This proves  $b_\xi \leq a_\zeta$  and so  $I_\xi$  and  $I_\zeta$  are disjoint.

(ii) Let  $z \in \mathbf{Zero}_I(f^u f^d)$ . Then (3) implies that  $|f(z)| < EB_I(f)$ . By the definition of evaluation bound, this also means that  $f'(z) \neq 0$ . Thus there are two cases: either  $f(z)f'(z) > 0$  or  $f(z)f'(z) < 0$ . First, suppose  $f(z)f'(z) > 0$ . Then there is a unique largest  $\xi \in \mathbf{Zero}(f)$  that is less than  $z$ , and there is a unique smallest  $b_\xi \in \mathbf{Zero}(\widehat{f})$  that is greater than  $z$ . This proves that  $z \in (\xi, b_\xi)$ . Similarly, if  $f(z)f'(z) < 0$ , we will see that  $z \in (a_\xi, \xi)$  for some  $\xi \in \mathbf{Zero}_I(f)$ .

(iii-a) Either  $f(a_\xi) > 0$  or  $f(a_\xi) < 0$ . If  $f(a_\xi) > 0$  then (3) implies  $f^d(a_\xi) > 0$ . But  $f^d(\xi) < 0$ , and hence  $A_\xi \cap \mathbf{Zero}(f^d)$  is non-empty. Now, since  $f^u$  is positive over  $A_\xi$ , we conclude that  $A_\xi \cap \mathbf{Zero}(f^u)$  is empty. The other case,  $f(a_\xi) < 0$  will similarly imply that  $A_\xi \cap \mathbf{Zero}(f^d)$  is empty and  $A_\xi \cap \mathbf{Zero}(f^u)$  is non-empty.

(iii-b) This is similar to (iii-a).

(iv) It is obvious. Otherwise, we assume there exist  $s \in A_\xi$  (for  $B_\xi$ , the proof is similar) such that  $f'(s) = 0$ . We derive a contradiction from the definitions of  $a_\xi$  by (5), where  $A_\xi = (a_\xi, \xi)$ .  $\square$

If  $s, t \in \mathbf{Zero}_I(f^u f^d)$  such that  $s < t$  and  $(s, t) \cap \mathbf{Zero}_I(f^u f^d)$  is empty, then we call  $(s, t)$  a **sleeve interval** of  $(I, f^u, f^d)$ . From Lemma 1(iii), we have

COROLLARY 2. Each zero of  $\mathbf{Zero}_I(f)$  is isolated by some sleeve interval of  $(I, f^u, f^d)$ .

LEMMA 3. Let  $(I, f^u, f^d)$  be a faithful sleeve. For all  $\xi \in \mathbf{Zero}_I(f)$ , the multiset  $\mathbf{ZERO}_{B_\xi}(f^u \cdot f^d)$  has odd size. Similarly, the multiset  $\mathbf{ZERO}_{A_\xi}(f^u \cdot f^d)$  has odd size. As a consequence, the multiset  $\mathbf{ZERO}_{I_\xi}(f^u f^d)$  has even size.

**Proof.** We just prove the result for the multiset  $\mathbf{ZERO}_{B_\xi}(f^u \cdot f^d)$ . Without loss of generality, let  $f(b_\xi) > 0$  (the case  $f(b_\xi) < 0$  is similar). By the sleeve-evaluation inequality,  $f^d(b_\xi) > 0$ . Note that when  $b_\xi = b$ , the inequality is also true since  $(I, f^u, f^d)$  is faithful. But  $f^d(\xi) < 0$ . Hence  $f^d$  has an odd number of zeros (counting multiplicities) in the interval  $B_\xi = (\xi, b_\xi)$ . Moreover,  $f^u > f$  implies  $f^u$  has no zeros in  $B_\xi$ .  $\square$

It follows from the preceding lemma that for each zero  $\xi$  of  $f$ , the multiset  $\mathbf{ZERO}_{I_\xi}(f^u f^d)$  has even size. Hence the multiset  $\mathbf{ZERO}_I(f^u f^d)$  has even size, say  $2m$ . So we may denote the sorted list of zeros of  $\mathbf{ZERO}_I(f^u f^d)$  by

$$(t_0, t_1, \dots, t_{2m-1}) \tag{8}$$

where  $t_0 \leq t_1 \leq \dots \leq t_{2m-1}$ . Intervals of the form  $J_i := [t_{2i}, t_{2i+1}]$  where  $t_{2i} < t_{2i+1}$  are called **candidate intervals** of the sleeve. We immediately obtain:

COROLLARY 4. Each  $\xi \in \mathbf{Zero}_I(f)$  is contained in some candidate interval of a faithful sleeve  $(I, f^u, f^d)$ .

Which of these candidate intervals actually contain zeros of  $f$ ? To do this, we classify a candidate interval  $[t_{2j}, t_{2j+1}]$  in (8) into two types:

$$\left. \begin{array}{l} \text{(Odd): } t_{2j} \in \mathbf{Zero}(f^d) \text{ iff } t_{2j+1} \in \mathbf{Zero}(f^u) \\ \text{(Even): } t_{2j} \in \mathbf{Zero}(f^d) \text{ iff } t_{2j+1} \in \mathbf{Zero}(f^d) \end{array} \right\} \quad (9)$$

We call a candidate interval  $J$  an **odd** or **even candidate interval** if it satisfies (9)(Odd) or (9)(Even). We now treat the easy case of deciding which candidate intervals are isolating intervals of  $f$ :

LEMMA 5 (Odd Zero). Let  $J$  be a candidate interval. The following are equivalent:

- (i)  $J$  is an odd candidate interval.
- (ii)  $J$  contains a unique zero  $\xi$  of  $f$ . Moreover  $\xi$  is an odd zero of  $f$ .

**Proof.** Let  $J = [t, t']$ .

(i) implies (ii): Without loss of generality, let  $f^u(t) = 0$  and  $f^d(t') = 0$ . Thus,  $f(t) < 0$  and  $f(t') > 0$ . Thus  $f$  has an odd zero in  $J$ . By Corollary 2, we know that candidate intervals contain at most one distinct zero.

(ii) implies (i): Since  $\xi$  is an odd zero, we see that  $f$  must be monotone over  $J$ . Without loss of generality, assume  $f$  is increasing. This implies  $f^d(t) < 0$  and hence  $f^u(t) = 0$ . Similarly,  $f^u(t') > 0$  and hence  $f^d(t') = 0$ . Hence  $J$  is an odd candidate.  $\square$

Isolating zeros of even multiplicity is more subtle and will be dealt with in the next section.

## 2.2. Monotonicity Property

We will exploit a special property of  $(I, f^u, f^d)$  for  $f$ :

$$\frac{\partial f^u}{\partial x} \geq \frac{\partial f}{\partial x} \geq \frac{\partial f^d}{\partial x} \quad \text{holds in } I \quad (10)$$

We call this the **monotonicity property**. In this subsection, we assume (10) and the faithfulness of the sleeve. We now strengthen one half of Lemma 3 above.

LEMMA 6. For any  $\xi \in \mathbf{Zero}_I(f)$ , there is a unique zero of odd multiplicity of  $f^u \cdot f^d$  in  $A_\xi = (a_\xi, \xi)$ .

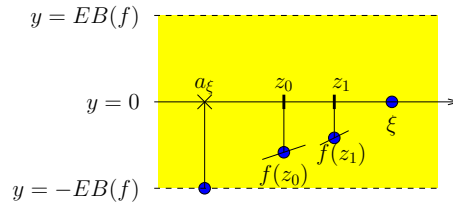


Fig. 2.  $A_\xi$  has a unique zero of  $f^u \cdot f^d$ : CASE of  $f^u(z_0) = f^u(z_1) = 0$ .

**Proof.** Alternatively, this lemma says that the multiset  $\mathbf{ZERO}_{A_\xi}(f^u f^d)$  has size 1.

By way of contradiction, suppose  $z_0 \leq z_1$  are two zeros of  $f^u f^d$  in  $A_\xi = (a_\xi, \xi)$ . Note that we allow the possibility that  $z_0 = z_1$  (in which case  $z_0$  is an even root of  $f^u f^d$ ). From Lemma 1(iii), we know that either  $z_0, z_1 \in \text{ZERO}(f^u)$  or  $z_0, z_1 \in \text{ZERO}(f^d)$  (i.e., it is not possible that one is a zero of  $\text{ZERO}(f^u)$  and the other is a zero of  $\text{ZERO}(f^d)$ ). There are two cases:

(A)  $z_0, z_1$  are roots of  $f^u$ . See Figure 2. By Rolle's theorem, there exists  $z \in [z_0, z_1]$  such that  $\frac{\partial f^u}{\partial x}(z) = 0$ . Therefore, there exist  $z^- < z < z^+$  that are arbitrarily close to  $z$  such that

$$\frac{\partial f^u}{\partial x}(z^-) \cdot \frac{\partial f^u}{\partial x}(z^+) < 0. \quad (11)$$

On the other hand, note that  $f(z_j) < f^u(z_j) = 0$  for  $j = 0, 1$ . Since  $f(\xi) = 0$ , and  $z_j < \xi$ , this means that the interval  $(z_j, \xi)$  contains a point  $z$  with  $f'(z) > 0$ . But  $f'$  has constant sign in  $A_\xi$  from Lemma 1 (iv), and so this sign of  $f'$  is positive. Then by monotonicity (10),

$$\frac{\partial f^u}{\partial x}(z^-) \geq f'(z^-) > 0, \quad \text{and} \quad \frac{\partial f^u}{\partial x}(z^+) \geq f'(z^+) > 0. \quad (12)$$

Now we see that (11) and (12) are contradictory.

(B)  $z_0, z_1$  are roots of  $f^d$ . We similarly derive a contradiction.  $\square$

COROLLARY 7. If  $t_{2j}$  is an even zero of  $f^u f^d$ , then  $[t_{2j}, t_{2j+1}]$  contains no zero of  $f$ .

If  $t_{2j}$  is an even zero we have either  $t_{2j} = t_{2j+1}$  or  $t_{2j} = t_{2j-1}$ . But for the former case,  $(t_{2j}, t_{2j+1})$  clearly has no zeros of  $f$ . The next result is a consequence of monotonicity and faithfulness:

LEMMA 8. The interval  $J_0 = [t_0, t_1]$  is a candidate interval and it isolates a zero of  $f$ .

In Lemma 5, we showed that (9)(Odd) holds iff  $J_j$  isolates an odd zero of  $f$ . The next result shows what condition must be added to (9)(Even) in order to characterize the isolation of even zeros.

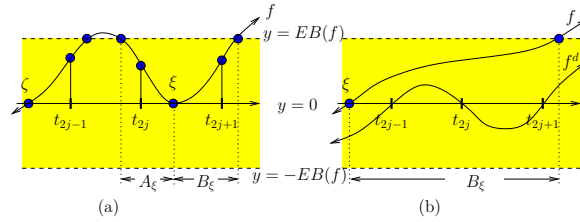


Fig. 3. Detection of even zero when  $t_{2j}, t_{2j+1} \in \text{Zero}_I(f^d)$ : (a) even zero, (b) no zero

LEMMA 9 (Even Zero). Let  $J_j = [t_{2j}, t_{2j+1}]$  ( $j > 0$ ) be an even candidate interval. Then  $J_j$  isolates an even zero  $\xi$  of  $f$  iff

- (i)  $f^d(t_{2j}) = 0$  and  $\frac{\partial f^u}{\partial x}$  has real zero in  $(t_{2j-1}, t_{2j+1})$ , or
- (ii)  $f^u(t_{2j}) = 0$  and  $\frac{\partial f^d}{\partial x}$  has real zero in  $(t_{2j-1}, t_{2j+1})$ .

**Proof.** Note that since  $j > 0$ , then  $t_{2j-1}$  is a zero of  $f^d$  iff  $t_{2j}$  is a zero of  $f^d$ . Let  $t_{2j}$  be a zero of  $f^d$ . So  $f^d(t_{2j+1}) = 0$  and  $t_{2j+1} \in B_\xi$  for a zero  $\xi$  of  $f$ . This means  $\frac{\partial f}{\partial x}$  is

positive in  $(\xi, t_{2j+1})$ . There are two cases: (a)  $t_{2j} < \xi < t_{2j+1}$  or (b)  $\xi < t_{2j} < t_{2j+1}$ . If (a), then  $t_{2j-1} \in B_\zeta$  for some zero  $\zeta$  of  $f$  and  $\zeta \neq \xi$  (see Figure 3(a)). By (3), we have  $0 < f^u(t_{2j-1}) < EB(f)$ ,  $0 < f^u(t_{2j}) < EB(f)$ . Since  $t_{2j-1} \in B_\zeta$ ,  $t_{2j} \in A_\xi$  and  $\zeta \neq \xi$ , there exists a point  $\eta \in (t_{2j-1}, t_{2j})$  such that  $f(\eta) \geq EB(f)$ . So  $f^u(\eta) > EB(f)$ . That means there is an extremum point of  $f^u$  in  $(t_{2j-1}, t_{2j})$ . That is, there exists a zero of  $\frac{\partial f^u}{\partial x}$  in  $(t_{2j-1}, t_{2j}) \subset (t_{2j-1}, t_{2j+1})$ . If (b), then  $\frac{\partial f^u}{\partial x}(x) > 0$  for all  $x \in (t_{2j-1}, t_{2j+1})$  since  $\frac{\partial f}{\partial x}$  has constant sign in  $B_\xi$  (see Figure 3(b)). We finish the proof.  $\square$

### 2.3. Effective Root Isolation of $f$

So far, we have been treating the roots  $t_j$  of  $f^u f^d$  exactly. But in our algorithms, we only have numbers in  $\mathbb{F}$ . We now want to replace  $t_j$  by their isolating intervals  $[a_j, b_j]$ . As usual, we assume that  $(I, f^u, f^d)$  is faithful and satisfies the monotonicity property (10). Let  $\text{ZERO}_I(f^u f^d)$  be the sorted list given in (8), and  $[a_i, b_i]$  an isolating interval of  $t_i$ , where any two distinct intervals  $[a_i, b_i]$  and  $[a_j, b_j]$  are disjoint. Let

$$SL_{f,I} = ([a_0, b_0], [a_1, b_1], \dots, [a_{2m-1}, b_{2m-1}]) \quad (13)$$

be the isolating intervals for roots of  $f^u f^d$  in  $\text{ZERO}_I(f^u f^d)$ . Assume that  $[a_i, b_i] = [a_j, b_j]$  iff  $t_i = t_j$ . Note that  $t_i = t_j$  implies  $|i - j| \leq 1$ . Let  $K_i := [a_{2i}, b_{2i+1}]$ .

By Corollary 7,  $J_i$  is not an isolating interval if  $t_{2i}$  is an even zero. Hence, we call  $K_i$  an **effective candidate** iff  $t_{2i} < t_{2i+1}$  and  $t_{2i}$  is an odd zero. Thus,  $K_i$  contains the candidate interval  $J_i = [t_{2i}, t_{2i+1}]$ . Furthermore,  $K_i$  is called an **effective even candidate** (resp., **effective odd candidate**) if  $J_i$  is an even (resp., odd) candidate interval (cf. (9)).

Our next theorem characterizes when  $K_i$  is an isolating interval of  $f$ . This is the “effective version” of Lemma 5 and Lemma 9. But before this theorem, we provide a useful partial criterion:

LEMMA 10. Let  $K_i = [a_{2i}, b_{2i+1}]$  be an effective even candidate. Then  $K_i$  isolates an even zero provided one of the following conditions hold:

- (E)<sup>d</sup>:  $t_{2i} \in \text{Zero}(f^d)$  and  $\frac{\partial f^u}{\partial x}$  is negative at  $a_{2i}$  or  $b_{2i}$ ,
- (E)<sup>u</sup>:  $t_{2i} \in \text{Zero}(f^u)$  and  $\frac{\partial f^d}{\partial x}$  is positive at  $a_{2i}$  or  $b_{2i}$ .

**Proof.** Say  $t_{2i}$  is a zero of  $f^d$  (the case where  $t_{2i} \in \text{Zero}(f^u)$  is similar). We have  $t_{2i+1} \in B_\xi$  for some  $\xi \in \text{Zero}(f)$ , and we also know that  $f' = \frac{\partial f}{\partial x}$  is positive at  $t_{2i+1}$ . There are just two cases: either (a)  $t_{2i}$  is in  $A_\xi$ , or (b)  $t_{2i}$  is in  $B_\xi$ . If (a) holds, then  $\xi$  is an even zero in  $[c, t_{2i+1}]$  (where  $c = a_{2i}$  or  $b_{2i}$ ), and our lemma is true.

So assume (b) and (E)<sup>d</sup>. From (E)<sup>d</sup> and the monotonicity (10), we know that  $f'$  is negative at  $c$ . If  $c = b_{2i}$  then we get a contradiction since (b) implies  $f'$  is positive over  $B_\xi \supseteq [t_{2i}, t_{2i+1}] \supseteq [c, t_{2i+1}]$ . If  $c = a_{2i}$ , the argument is more subtle. We know that  $\xi \in [t_{2j}, t_{2j+1}]$  for some  $j < i$  and  $t_{2j+1} < t_{2i}$  (for  $t_{2i}$  is an odd zero). Moreover,  $f'$  has constant sign in  $B_\xi \supseteq [t_{2j+1}, t_{2i+1}] \supseteq [c, t_{2i+1}]$ . Again this yields a contradiction.  $\square$

For the even effective candidates, we shall need a **constant sign property**:

$$\left. \begin{array}{l} \text{Let } t_{2j}, t_{2j+1} (j \geq 1) \text{ all be real zeros of } f^u \text{ or } f^d. \\ \text{If } t_{2j}, t_{2j+1} \in \text{Zero}(f^d) \text{ then } \frac{\partial f^u}{\partial x} \text{ is positive in } [a_{2j-1}, b_{2j-1}] \text{ and } [a_{2j+1}, b_{2j+1}]. \\ \text{If } t_{2j}, t_{2j+1} \in \text{Zero}(f^u) \text{ then } \frac{\partial f^d}{\partial x} \text{ is negative in } [a_{2j-1}, b_{2j-1}] \text{ and } [a_{2j+1}, b_{2j+1}]. \end{array} \right\} \quad (14)$$



Note that  $t_{2j-1} \in B_\zeta, t_{2j+1} \in B_\xi$  for some  $\zeta, \xi \in \text{Zero}_I(f)$ . And we know  $\frac{\partial f^u}{\partial x}(x) > 0$  ( $\frac{\partial f^d}{\partial x}(x) < 0$ ) for all  $x \in B_\eta(\eta = \xi, \zeta)$  when  $t \in \text{Zero}_I(f^d)$  ( $t = t_{2j-1}, t_{2j+1}$ ) ( $t \in \text{Zero}_I(f^u)$ ). So the constant sign can be reached. We strengthen this to a necessary and sufficient criterion:

**THEOREM 11 (Effective Isolation Criteria).** Let  $K_i = [a_{2i}, b_{2i+1}]$  be an effective candidate. If  $K_i$  is an even effective candidate, further assume that constant sign property holds. Then  $K_i$  is an isolating interval of  $f$  iff one of the following conditions hold:

(O)  $K_i$  is an effective odd candidate.

(E):  $K_i$  is an effective even candidate and,  $i = 0$  or  $i > 0$  and  $\frac{\partial f^u}{\partial x}$  (resp.,  $\frac{\partial f^d}{\partial x}$ ) has some zero in  $[b_{2i-1}, b_{2i+1}]$  if  $f^d$  (resp.,  $f^u$ ) has two distinct zeros in  $K_i$ .

**Proof.** As a preliminary remark, we note that  $K_i$  contains at most one zero of  $f$ .

( $\Leftarrow$ ) We first show that (O) or (E) implies that  $K_i$  is an isolating interval. Suppose (O) holds. We may assume that  $f^u$  has a zero in  $[a_{2i}, b_{2i}]$  and  $f^d$  has a zero in  $[a_{2i+1}, b_{2i+1}]$ . Thus  $[a_{2i}, b_{2i+1}]$  contains a candidate interval  $J_i = [t_{2i}, t_{2i+1}]$  satisfying the conditions of Lemma 5, and  $J_i$  has an odd zero of  $f$ . Suppose (E) holds. Without loss of generality, assume  $f^u$  has two distinct zeros in  $K_i$ . If  $i = 0$ , then clearly,  $K_i$  has a zero of  $f$ . Otherwise, these zeros must be  $t_{2i}$  and  $t_{2i+1}$ . By assumption,  $\frac{\partial f^d}{\partial x}$  has some zero in  $[b_{2i-1}, b_{2i+1}]$ ; but in fact this zero lies in  $[b_{2i-1}, t_{2i+1}] \subseteq J_i$  because  $[a_{2i+1}, b_{2i+1}]$  satisfies the constant sign property (14). Now Lemma 9 implies  $f$  has some zero in  $J_i \subseteq K_i$ .

( $\Rightarrow$ ) Suppose  $f$  has some zero in  $K_i$ . We must show that either (O) or (E) holds. From the definition of  $K_i$ , we know there are two distinct roots of  $f^u f^d$  in  $K_i$ . If  $f^u(t_{2i}) = 0$  iff  $f^d(t_{2i+1}) = 0$ , then clearly (O) holds. Otherwise,  $f^d(t_{2i}) = 0$  iff  $f^d(t_{2i+1}) = 0$ . If  $i = 0$ , it is clear. If  $i \geq 1$ , without loss of generality, assume that  $t_{2i}, t_{2i+1}$  are zeros of  $f^d$ . We must show that  $\frac{\partial f^u}{\partial x}$  has some zero  $z$  in  $[b_{2i-1}, b_{2i+1}]$ . By Lemma 9,  $\frac{\partial f^u}{\partial x}$  has some zero  $z$  in  $[t_{2i-1}, t_{2i+1}]$ . So it is enough to show that  $z$  cannot lie in  $[t_{2i-1}, b_{2i-1}]$ . But this is a consequence of the constant sign property.  $\square$

We can use Sturm theorem to check whether a polynomial ( $\frac{\partial f^u(x)}{\partial x}$  or  $\frac{\partial f^d(x)}{\partial x}$ ) has real root in a given interval or isolate the real roots of them directly. In most cases, we need not to use this since Lemma 10 holds for almost all the cases in practice.

### 3. Bounds of Triangular System

Consider a triangular polynomial system  $F_n$ :

$$F_n = \{f_1(x_1), f_2(x_1, x_2), \dots, f_n(x_1, \dots, x_n)\} \quad (15)$$

where  $f_i \in \mathbb{Z}[x_1, \dots, x_i]$ . Generalizing our univariate notation, if  $B \subseteq \mathbb{R}^n$ , let  $\text{Zero}_B(F_n)$  denote the set of real zeros of  $F_n$  restricted to  $B$ .

Let  $B = I_1 \times \dots \times I_n$  be an  $n$ -dimensional box,  $I_i = [a_i, b_i]$ , and  $\xi = (\xi_1, \dots, \xi_{n-1}) \in \square \xi = I_1 \times \dots \times I_{n-1}$  a real zero of  $F_{n-1} = \{f_1, \dots, f_{n-1}\} = 0$ . Consider

$$f(x) := f_n(\xi_1, \dots, \xi_{n-1}, x). \quad (16)$$

We have a three-fold goal in this section:

- Compute lower estimates on the evaluation bound  $EB_{I_n}(f)$ .
- Construct a sleeve  $(I_n, f^u, f^d)$  for  $f$  that satisfies the monotonicity property.
- Compute an upper estimate on the sleeve bound  $SB_{I_n}(f^u, f^d)$ .

### 3.1. Lower Estimate on Evaluation Bounds

We give two methods to compute lower estimates of  $EB_{I_n}(f)$ . The first method is based on a general result about multivariate zero bounds in (29); another is based on resultant computation.

Let  $\Sigma = \{p_1, \dots, p_n\} \subseteq \mathbb{Z}[x_1, \dots, x_n]$  be a zero dimensional equation system. Let  $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$  be one of these zeros. Suppose  $d_i = \deg(p_i)$  and

$$K := \max\{\sqrt{n+1}, \|p_1\|_2, \dots, \|p_n\|_2\},$$

where  $\|p\|_2$  is the 2-norm of  $p$ . Then we have the following result (29, p. 341):

**THEOREM 12.** Let  $(\xi_1, \dots, \xi_n)$  be a complex zero of  $\Sigma$ . For any  $i$ , if  $|\xi_i| \neq 0$  then

$$|\xi_i| > MRB(\Sigma) := (2^{3/2}NK)^{-D} 2^{-(n+1)d_1 \cdots d_n}. \quad (17)$$

where  $N := (1 + \sum_{i=1}^n d_i)$ ,  $D := (1 + \sum_{i=1}^n \frac{1}{d_i}) \prod_{i=1}^n d_i$ .

Note that this theorem defines a numerical value  $MRB(\Sigma)$  (the **multivariate root bound**) for  $\Sigma$ . Given  $F_n$  as in (15), consider the polynomial set

$$\widehat{F}_n := \{f_1, \dots, f_{n-1}, \frac{\partial f_n}{\partial X}, Y - f_n\} \quad (18)$$

in  $\mathbb{Z}[x_1, \dots, x_{n-1}, x, Y]$ , where  $f_n = f_n(x_1, \dots, x_{n-1}, x)$ .

**LEMMA 13.** Use the notations in (16). Let  $(\xi_1, \dots, \xi_{n-1})$  be a zero of  $F_{n-1}$ . Then the evaluation bound  $EB_{I_n}(f)$  of  $f(x) \in \mathbb{R}[x]$  satisfies  $EB_{I_n}(f) > MRB(\widehat{F}_n)$ .

**Proof.** As  $F_n$  is zero-dimensional, so is  $\widehat{F}_n$ , which is easily seen. If  $(\xi_1, \dots, \xi_n, y)$  is a zero of  $\widehat{F}_n$ , then  $f'(\xi_n) = 0$ . Moreover,  $y = f(\xi_n)$ . By definition of  $EB(f)$ , we have  $EB(f)$  is the minimum of all such non-zero  $|y|$ 's. By Theorem 12,  $EB_{I_n}(f) > MRB(\widehat{F}_n)$ .  $\square$

It is instructive to directly define the **evaluation bound** of a triangular system  $F_n$ : for  $B \subseteq \mathbb{R}^n$ , let  $B' = B \times \mathbb{R}$ . Then define  $EB_B(F_n)$  to be

$$\min\{|y| : (x_1, \dots, x_{n-1}, x, y) \in \text{Zero}_{B'}(\widehat{F}_n), y \neq 0\}, \quad (19)$$

assuming  $\min\{\} = \infty$ . Observe that (19) is a generalization of the corresponding univariate evaluation bound (2). For  $i = 2, \dots, n$ , we similarly have evaluation bounds  $EB_{B_i}(F_i)$  for  $F_i$ , where  $F_i = \{f_1, \dots, f_i\}$ .

This multivariate evaluation bound is a lower bound on the univariate one: with  $f$  given by (16). Since  $MRB(\widehat{F}_n)$  is easily computed, our algorithm can use  $MRB(\widehat{F}_n)$  as the lower bound on  $EB(F_n)$ .

In general,  $MRB(\widehat{F}_n)$  is not a good estimation (see Examples in Section 5). We propose a computational way to compute such a lower estimate via resultants. Consider  $\widehat{F}_n$  defined by (18). Let

$$e_i = \begin{cases} \text{res}_X(Y - f_n, \frac{\partial f_n}{\partial X}) & i = n, \\ \text{res}_{x_i}(e_{i+1}, f_i) & i = n-1, \dots, 1 \end{cases} \quad (20)$$

where  $\text{res}_x(p, q)$  is the resultant of  $p$  and  $q$  relative to  $x$ . Thus  $e_1 \in \mathbb{F}[Y]$ . If  $e_1 \neq 0$ , define

$$R(F_n) := \min\{|z| : e_1(z) = 0, z \neq 0\}.$$

If  $e_1$  has no real roots, let  $R(F_n) = \infty$ .

LEMMA 14. If  $e_1 \neq 0$ ,  $EB(F_n) \geq R(F_n)$ , and we can use  $R(F_n)$  as the evaluation bound.

**Proof.** From properties of the resultant,  $e_1(Y)$  is a linear combination of the polynomials in  $\widehat{F}_n$ . If  $(x_1, \dots, x_{n-1}, x, y) \in \mathbf{Zero}(\widehat{F}_n)$ ,  $y$  must be a zero of  $e_1(Y) = 0$ . From (20), we conclude  $EB(F_n) \geq R(F_n)$ .  $\square$

Therefore, we may isolate the real roots of  $e_1(Y) = 0$  and take  $\min\{l_1, -r_2\}$  as the evaluation bound for  $F_n$ , where  $(l_1, r_1)$  and  $(l_2, r_2)$  are the isolating intervals for the smallest positive root and the largest negative root of  $e_1(Y) = 0$  respectively.

We can use the multiresultant (see (3)) to optimize the evaluation bound computation.

### 3.2. Sleeve and Sleeve Bound

We assume a positive sign in  $I_i$ , that is,  $I_i > 0$  for  $i = 1, \dots, n$  and will show how to treat other cases in Section 4.

Given a polynomial  $g \in \mathbb{R}[x_1, \dots, x_n]$ , we may decompose it uniquely as  $g = g^+ - g^-$ , where  $g^+, g^- \in \mathbb{R}[x_1, \dots, x_n]$  each has only positive coefficients, and the support of  $g^+$  and  $g^-$  are both minimum. Here, the support of a polynomial  $g$  is the set of power products with non-zero coefficients in  $g$ .

Given  $f$  as in (16) and an isolating box  $\square\xi \in \square\mathbb{F}^{n-1}$  for  $\xi$ , following (18; 23), we define

$$\begin{aligned} f^u(x) &= f_n^u(\square\xi; x) = f_n^+(\mathbf{b}_{n-1}, x) - f_n^-(\mathbf{a}_{n-1}, x), \\ f^d(x) &= f_n^d(\square\xi; x) = f_n^+(\mathbf{a}_{n-1}, x) - f_n^-(\mathbf{b}_{n-1}, x), \end{aligned} \quad (21)$$

where  $f_n = f_n^+ - f_n^-$ ,  $\mathbf{a}_i = (a_1, \dots, a_i)$ ,  $\mathbf{b}_i = (b_1, \dots, b_i)$ , and  $\square\xi = [a_1, b_1] \times \dots \times [a_{n-1}, b_{n-1}]$ .

The bounding functions of the interval function of  $f(x)$  (see (11; 15)) are similar to our sleeve polynomials. The functions in (28) are not a sleeve. But in some special interval, they may have the some properties of our sleeve polynomials.

From the construction, it is clear that  $f^u \geq f \geq f^d$ . Moreover, both inequalities are strict if  $a_i = \xi_i = b_i$  does not hold for any  $i = 1, \dots, n - 1$ . Hence  $(I_n, f^u(x), f^d(x))$  is a sleeve for  $f(x)$  (18; 23). We further have:

LEMMA 15. Over any  $I_n = [l, r] > 0$ , we have:

- (i) (Monotonicity)  $\frac{\partial f^u}{\partial x} \geq \frac{\partial f}{\partial x} \geq \frac{\partial f^d}{\partial x}$ .
- (ii)  $f^u(x) - f^d(x)$  is monotonously increasing over  $I_n$ .

**Proof.** Let  $f(x) = f_n(\xi_1, \dots, \xi_{n-1}, x) = f_n^+(\xi_1, \dots, \xi_{n-1}, x) - f_n^-(\xi_1, \dots, \xi_{n-1}, x)$  and

$$\begin{aligned}
T_1(x) &= f^u(x) - f(x) \\
&= (f_n^+(b_1, \dots, b_{n-1}, x) - f_n^+(\xi_1, \dots, \xi_{n-1}, x)) + (f_n^-(\xi_1, \dots, \xi_{n-1}, x) - f_n^-(a_1, \dots, a_{n-1}, x)), \\
T_2(x) &= f(x) - f^d(x) \\
&= (f_n^+(\xi_1, \dots, \xi_{n-1}, x) - f_n^+(a_1, \dots, a_{n-1}, x)) + (f_n^-(b_1, \dots, b_{n-1}, x) - f_n^-(\xi_1, \dots, \xi_{n-1}, x)), \\
T_3(x) &= f^u(x) - f^d(x) \\
&= (f_n^+(b_1, \dots, b_{n-1}, x) - f_n^+(a_1, \dots, a_{n-1}, x)) + (f_n^-(b_1, \dots, b_{n-1}, x) - f_n^-(a_1, \dots, a_{n-1}, x)).
\end{aligned}$$

Since  $f_n^+, f_n^-$  are polynomials with positive coefficients and  $0 < a_i \leq \xi_i \leq b_i$  for all  $i$ ,  $f_n^+(b_1, \dots, b_{n-1}, x) - f_n^+(\xi_1, \dots, \xi_{n-1}, x)$ ,  $f_n^-(\xi_1, \dots, \xi_{n-1}, x) - f_n^-(a_1, \dots, a_{n-1}, x)$ , and hence  $T_1(x)$  are polynomials in  $X$  with positive coefficients. Similarly,  $T_2(x)$  and  $T_3(x)$  are polynomials with positive coefficients. For  $x > 0$ , we have  $\frac{\partial T_1(x)}{\partial x} = \frac{\partial f^u(x)}{\partial x} - \frac{\partial f(x)}{\partial x} \geq 0$ . Similarly, we can show that  $\frac{\partial T_2(x)}{\partial x} = \frac{\partial f(x)}{\partial x} - \frac{\partial f^d(x)}{\partial x} \geq 0$ , and  $\frac{\partial T_3(x)}{\partial x} = \frac{\partial f^u(x)}{\partial x} - \frac{\partial f^d(x)}{\partial x} \geq 0$ . Thus  $\frac{\partial f^u}{\partial x} \geq \frac{\partial f}{\partial x} \geq \frac{\partial f^d}{\partial x}$ . As consequence,  $f^u(x) - f^d(x)$  is monotone increasing in  $I_n$ .  $\square$

As an immediate corollary, we have

COROLLARY 16.  $SB_{I_n}(f^u, f^d) \leq f^u(r) - f^d(r)$ .

Our next goal is to give an upper bound on  $f^u(r) - f^d(r)$  as a function of

$$b := \max\{b_1, \dots, b_n\}, w := \max\{w_1, \dots, w_n\},$$

where  $w_i = b_i - a_i$ . Also let  $\mathbf{w} = (w_1, \dots, w_n)$ . For  $f \in \mathbb{R}[x_1, \dots, x_n]$ , write  $f = \sum_{\alpha} c_{\alpha} p_{\alpha}(x_1, \dots, x_n)$  where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , and  $p_{\alpha}(x_1, \dots, x_n)$  denotes the monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ . Let  $\|f\|_1 := \sum_{\alpha} |c_{\alpha}|$  denote its 1-norm. The inner product of two vectors, say  $\mathbf{w}$  and  $\alpha$ , is denoted  $\langle \mathbf{w}, \alpha \rangle$ . Let  $\mathbf{a}_i = (a_1, \dots, a_i)$ ,  $\mathbf{b}_i = (b_1, \dots, b_i)$ . We have the following result.

LEMMA 17. Let  $m = \sum_{i=1}^n \alpha_i \geq 1$ . Then

$$p_{\alpha}(\mathbf{b}_n) - p_{\alpha}(\mathbf{a}_n) \leq b^{m-1} \langle \alpha, \mathbf{w} \rangle \leq w m b^{m-1}.$$

**Proof.** We have

$$X^m - Y^m = (X - Y)(X^{m-1} + X^{m-2}Y + \dots + Y^{m-1}) \leq (X - Y)mX^{m-1}, \quad (22)$$

provided  $X \geq Y \geq 0$  and  $m \geq 1$ . Then, assuming each  $\alpha_i \geq 1$ ,

$$\begin{aligned}
& p_{\alpha}(b_1, \dots, b_n) - p_{\alpha}(a_1, \dots, a_n) \\
&= \sum_{i=1}^n \left\{ \left( \prod_{j=1}^{i-1} a_j^{\alpha_j} \right) (b_i^{\alpha_i} - a_i^{\alpha_i}) \left( \prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \\
&\leq \sum_{i=1}^n w_i \alpha_i \left\{ \left( \prod_{j=1}^{i-1} a_j^{\alpha_j} \right) (b_i^{\alpha_i-1}) \left( \prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \quad (\text{by (22)}) \\
&\leq \sum_{i=1}^n w_i \alpha_i \left\{ \left( \prod_{j=1}^{i-1} b_j^{\alpha_j} \right) (b_i^{\alpha_i-1}) \left( \prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} = b^{m-1} \sum_{i=1}^n w_i \alpha_i.
\end{aligned}$$

In general, if any  $\alpha_i = 0$ , the corresponding term in the summation could be omitted in the above derivation, and the proof remains valid.  $\square$

The above lemma extends linearly to a polynomial  $f$ :

COROLLARY 18. Let  $f = \sum_{\alpha} c_{\alpha} p_{\alpha}(x_1, \dots, x_n)$ ,  $c_{\alpha} > 0$ ,  $c_{\alpha} \in \mathbb{R}$ ,  $m = \deg(f) \geq 1$ . Then

$$f(\mathbf{b}_n) - f(\mathbf{a}_n) \leq b^{m-1} \sum_{\alpha} |c_{\alpha}| \langle \mathbf{w}, \alpha \rangle \leq wmb^{m-1} \|f\|_1.$$

THEOREM 19. Let  $(I_n, f^u, f^d)$  be a sleeve as in (21), and  $\square_{n-1}\xi = I_1 \times \dots \times I_{n-1}$  an isolating box for  $\xi \in \mathbb{R}^{n-1}$ , where  $I_i = [a_i, b_i] > 0$ ,  $I_n = [l, r] > 0$ , and  $w = \max_{i=1}^{n-1} \{b_i - a_i\}$ . Then

$$SB_I(f^u, f^d) \leq wm \|f_n\|_1 b^{m-1},$$

where  $m = \deg(f_n)$ ,  $b = \max\{b_1, \dots, b_{n-1}, r\}$ .

**Proof.** Let  $f(x) = \sum_{i=0}^m C_i(\xi_1, \dots, \xi_{n-1}) X^i$  where  $C_i \in \mathbb{Z}[x_1, \dots, x_{n-1}]$  has degree  $\leq m - i$ ,  $C_i = C_i^+ - C_i^-$ ,  $\mathbf{a} = (a_1, \dots, a_{n-1})$ , and  $\mathbf{b} = (b_1, \dots, b_{n-1})$ . We have

$$f^u(x) = \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^-(\mathbf{a})) X^i, f^d(x) = \sum_{i=0}^m (C_i^+(\mathbf{a}) - C_i^-(\mathbf{b})) X^i.$$

For  $x \in I_n$ , we have

$$\begin{aligned} f^u(x) - f^d(x) &= \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^+(\mathbf{a}) + C_i^-(\mathbf{b}) - C_i^-(\mathbf{a})) x^i \\ &\leq \sum_{i=0}^m w(m-i) b^{m-i-1} (\|C_i^+\|_1 + \|C_i^-\|_1) b^i \quad (\text{By Corollary 18}) \\ &< wmb^{m-1} \sum_{i=0}^m \|C_i\|_1 = wmb^{m-1} \|f_n\|_1. \end{aligned}$$

□

We give two corollaries to the above theorem.

COROLLARY 20. For a fixed  $F_n$  and  $I_n$ , when  $w \rightarrow 0$ ,  $SB_{I_n}(f^u, f^d) \rightarrow 0$ .

So when  $w \rightarrow 0$ ,  $f^u \rightarrow f$  and  $f^d \rightarrow f$ , which implies that, with sufficient refinement, the sleeve-evaluation inequality (3) will eventually hold. The next corollary gives an explicit condition to guarantee this:

COROLLARY 21. The sleeve-evaluation inequality (3) holds if  $w < \frac{EB_{I_n}(f)}{m \|f_n\|_1 b^{m-1}}$ .

#### 4. The Main Algorithm

In this section, we present our isolation algorithm: given  $F_n$  as in (15), to isolate the real zeros of  $F_n$  in a given  $n$ -dimensional box  $B = I_1 \times \dots \times I_n$ .

#### 4.1. Refinement of Isolating Box

Refining an isolation box is a basic subroutine in our algorithm. Let  $\square_n \xi = \square_{n-1} \xi \times [c, d] > 0$  be an isolating box for a zero  $\xi = (\xi_1, \dots, \xi_n)$  of  $F_n = 0$ ,  $([c, d], f^d, f^u)$  a sleeve associated with  $\square_n \xi$  satisfying (3) and (10),  $\square'_{n-1} \xi$  an isolating box of  $F_{n-1}$  satisfying  $\square'_{n-1} \xi \subsetneq \square_{n-1} \xi$ ,  $f(x) = f_n(\xi_1, \dots, \xi_{n-1}, x)$ , and  $\bar{f}^u(x) = f_n^u(\square'_{n-1} \xi, x)$ ,  $\bar{f}^d(x) = f_n^d(\square'_{n-1} \xi, x)$  (for definition, see (21)).

LEMMA 22. Let  $t_0, t_1$  be the real roots of  $f^u f^d = 0$  in  $[c, d]$  and  $t'_0 < t'_1$  the two smallest real roots of  $\bar{f}^u \bar{f}^d = 0$  in  $[c, d]$ . If  $\square'_{n-1} \neq [\xi_1, \xi_1] \times \dots \times [\xi_{n-1}, \xi_{n-1}]$ , then  $[t'_0, t'_1] \subset [t_0, t_1]$  and  $\xi \in \square'_{n-1} \xi \times [t'_0, t'_1]$ .

**Proof.** From  $\square'_{n-1} \xi \subsetneq \square_{n-1} \xi$ ,  $\square'_{n-1} \neq [\xi_1, \xi_1] \times \dots \times [\xi_{n-1}, \xi_{n-1}]$ , and (21), we have

$$f^d(x) < \bar{f}^d(x) < f(x) < \bar{f}^u(x) < f^u(x), \forall x \in [c, d].$$

It is not difficult to check that sleeve-evaluation inequality (3) and the monotonicity property (10) hold for the sleeve  $([c, d], \bar{f}^u, \bar{f}^d)$ . Wlog, we assume  $f^u(t_0) = 0, f^d(t_1) = 0$ . The proofs for other cases are similar. We have  $\bar{f}^u(t_0) < f^u(t_0) = 0$  and  $\bar{f}^u(\xi_n) > f(\xi_n) = 0$ . Then  $\bar{f}^u$  has at least one root in  $(t_0, \xi_n)$ . Since  $(t_0, \xi_n) \subset A_{\xi_n}$ , by Lemma 6,  $\bar{f}^u(x)$  has a unique real root in  $(t_0, \xi_n)$ . Let  $t'_0$  be this root. Then,  $t'_0 > t_0$ . Since  $\bar{f}^u(x) < f^u(x) < 0$ ,  $\bar{f}^u$  has no real roots in  $[c, t_0]$  and  $t'_0$  is the smallest root of  $\bar{f}^u \bar{f}^d = 0$  in  $[c, d]$ . Similarly, we could show that  $\bar{f}^d(x) = 0$  has at least one root in  $(\xi_n, t_1)$ . Let  $t'_1$  be the smallest of these roots. Then  $t'_0$  and  $t'_1$  are the two smallest roots of  $\bar{f}^u \bar{f}^d = 0$  in  $[c, d]$  and  $\xi_n \in (t'_0, t'_1) \subset [t_0, t_1]$ .  $\square$

The lemma tells us how to refine an isolating box  $K = I_1 \times \dots \times I_n$  of a triangular system  $F_n$  without using Theorem 11. The following algorithm is to refine  $K$  of  $F_n$  to  $\hat{K} = \hat{I}_1 \times \dots \times \hat{I}_n$  under the precision  $\epsilon$ .

```

Refine( $F_n, K, \epsilon$ )
Input:  $F_n, K, \epsilon$ .
Output:  $\hat{K} = \hat{I}_1 \times \dots \times \hat{I}_n$  with  $w = \max_{j=1}^n \{|\hat{I}_j|\} \leq \epsilon$ .
1. If  $n = 1$ , subdivide  $I_n$  until  $|I_n| < \epsilon$  and return  $I_n$ .
2. Let  $K_{n-1} = I_1 \times \dots \times I_{n-1}$ ,  $w = \max_{j=1}^n \{|I_j|\}$ .
   If  $w \leq \epsilon$ , return  $K$ . Else,  $\delta = \epsilon$ .
3. while  $w > \epsilon$ , do
   3.1.  $\delta = \delta/2$ .
   3.2. If  $K_{n-1}$  is a point,  $f(x) = f_n(\xi_1, \dots, \xi_{n-1}, x) \in \mathbb{F}[x]$ .
       Isolate its roots under  $\epsilon$ , return them.
   3.3.  $K_{n-1} := \text{Refine}(F_{n-1}, K_{n-1}, \delta)$ .
   3.4. Compute the sleeve:  $f^u(x) := f_n^u(K_{n-1}, x), f^d(x) := f_n^d(K_{n-1}, x)$ .
   3.5. Isolate the roots of  $f^u f^d$  in  $I_n$  with precision  $\delta$ .
   3.6. Denote the first two intervals as  $[c_1, d_1], [c_2, d_2]$ .
   3.7.  $w := d_2 - c_1$ .
4. Return  $\hat{K} := K_{n-1} \times [c_1, d_2]$ .

```

#### 4.2. Verifying Zeros

Let  $\alpha = (\alpha_1, \dots, \alpha_k)$  be a real root of the triangular system  $\Sigma_k = \{h_1, \dots, h_k\}$ ,  $B = I_1 \times \dots \times I_k$  an isolating box of  $\alpha$ , and  $g(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$ . We show how to check whether  $g(\alpha_1, \dots, \alpha_k) = 0$ .

We call  $\rho = \min\{|g(\alpha)| : g(\alpha) \neq 0, \forall \alpha \in \text{Zero}_B(\Sigma_k)\}$  the **zero bound** of  $g$  on  $\Sigma_k$ . Let

$$\Sigma_B = \{h_1, \dots, h_k, Y - g\}. \quad (23)$$

We have two methods to compute the zero bound. First, by Theorem 12,  $MRB(\Sigma_B)$  can be taken as the zero bound. Second, we may compute the zero bound by resultant computation. Let  $r_{k+1} = Y - g(x_1, \dots, x_k)$  and  $r_i = \text{res}(h_i, r_{i+1}, x_i)$  for  $i = k, \dots, 1$ . Then  $r_1(Y)$  is a univariate polynomial in  $Y$ . If  $r_1 \not\equiv 0$ , chose a lower bound  $\rho$  for all the absolute values of the nonzero real roots of  $r_1$ . It is clear that  $\rho$  is smaller than the absolute value of any nonzero root of  $r_1(Y) = 0$ .

We give the following algorithm.

```

ZeroTest( $\Sigma_n, B, g(x_1, \dots, x_n)$ )
Input:  $\Sigma_n, B = I_1 \times \dots \times I_n, g(x_1, \dots, x_n)$ .
Output: True if  $g(\alpha) = 0$  or FALSE otherwise.
1.  $\delta = \max_{j=1}^n \{|I_j|\}$ .
2.  $g^u = g^+(b_1, \dots, b_n) - g^-(a_1, \dots, a_n)$ ,  $g^d = g^+(a_1, \dots, a_n) - g^-(b_1, \dots, b_n)$ .
3. If  $g^d = g^u$ , then  $g = g^d = g^u$ .
   If  $g^d = 0$  return TRUE; else return FALSE. end
4. If  $g^u g^d \geq 0$ , then  $g \neq 0$  and return FALSE. end
5. Compute the zero bound  $\rho$ .
6. If  $|g^u| < \rho$ , and  $|g^d| < \rho$ , then  $g < \rho$  and
   hence  $g(\alpha) = 0$  and return TRUE. end
7.  $\delta = \delta/2$ ,  $B = \text{Refine}(\Sigma_n, B, \delta)$ , and goto step 2.

```

#### 4.3. Isolation Algorithm

We now give the real root isolation algorithm RootIsoTS for a triangular system.

**Remarks:** Algorithm RootIsoTS can be improved or made more complete in the following ways.

- The assumption  $B_n > 0$  is reasonable. If we want to obtain the real roots of  $f$  in the interval  $I = (a, b) < 0$ , we may consider  $g(x) = f(-x)$  in the interval  $(-b, -a)$ . If  $0 \in (a, b)$ , we can consider the two parts,  $(a, 0)$  and  $(0, b)$  respectively, since we can check whether 0 is a root of  $f(x) = 0$ .
- If  $f(a) = 0$  or  $f(b) = 0$ , we can ignore the first or last element in  $SL_{f,I}$  to form effective candidate intervals of  $f$ . When  $f(a) = 0$ , the first effective candidate interval may or may not be the isolating interval of  $f$ , we need to check it by Theorem 11. And we need to use the first isolating interval in  $SL_{f,I}$  to decide whether the first effective candidate interval is isolating if the first three elements in  $SL_{f,I}$  are all isolating intervals of  $f^u$  (or  $f^d$ ).

RootIsoTS

Input:  $F_n, B_n = \prod_{i=1}^n I_i (I_i = [l_i, r_i] > 0)$ ,  $\epsilon > 0$ .

Output: An isolating set  $\square \mathbf{Zero}_{B_n}(F_n)$ .

1. Compute  $\square \mathbf{Zero}_{B_1}(F_1)$  for  $F_1$  to precision  $\epsilon$ .  
 $Result := \square \mathbf{Zero}_{B_1}(F_1)$ .  $New := \emptyset$ .  
 If  $Result = \emptyset$ , return  $Result$ , end
2. For  $i$  from 2 to  $n$ , do
  - 2.1. Compute  $EB_i := EB(F_i)$  for  $F_i$ .
  - 2.2.  $\delta := \epsilon$ .
  - 2.3. while  $Result \neq \emptyset$ , do
    - 2.3.1. Choose an element  $\square_{i-1}\xi$  from  $Result$ .  
 $Result := Result \setminus \{\square_{i-1}\xi\}$ .
    - 2.3.2. Compute the sleeve:  $f^u(x) = f_i^u(\square_{i-1}\xi, x)$ ,  $f^d(x) = f_i^d(\square_{i-1}\xi, x)$ .
    - 2.3.3. While  $f^u(r_i) - f^d(r_i) \geq EB_i$ ,  
 $\delta := \delta/2$ .  
 $\square_{i-1}\xi := \text{Refine}(F_{i-1}, \square_{i-1}\xi, \delta)$ .  
 Recompute  $f^u(x)$  and  $f^d(x)$ .
    - 2.3.4. Isolate the real roots of  $f^u f^d$  in  $I_i$ .
    - 2.3.5 If the set derived from 2.3.4 is empty and  $f^u(a)f^d(a) < 0$ , then  $f(x) \equiv 0$ .  
 The input system is not zero dimension. end
    - 2.3.6. Compute the parity of these roots.
    - 2.3.7. Construct the effective candidate intervals.
    - 2.3.8. for each effective candidate interval  $K$ ,
      - 2.3.8.1. Check whether  $K$  is isolating.  
 If  $K$  is odd,  $K$  is isolating;  
 If  $K$  is even:  
     If Lemma 10 holds,  $K$  is isolating;  
     Else, ensure (14).  $K$  is isolating iff Theorem 11 (E) holds.
      - 2.3.8.2. If  $K$  is isolating, then  
 $K := \text{Refine}(F_i, K, \epsilon)$ .  
 $New := New \cup \{\square_{i-1}\xi \times K\}$ .
  - 2.4. If  $New = \emptyset$ , return  $New$ , end
  - 2.5.  $Result := New$ .  $New := \emptyset$ .
3. return  $Result$ .

- If we want to find all real roots of  $f$ , we first isolate the real roots of  $f$  in  $(0, 1)$ , then isolate the real roots of  $g(x) = X^n * f(1/x)$  in  $(0, 1)$ , and check whether 1 is a root of  $f$ . As a result, we can find all the roots of  $f(x) = 0$  in  $(0, +\infty)$ . We can find the roots of  $f(x) = 0$  in  $(-\infty, 0)$  by isolating the roots of  $f(-x) = 0$  in  $(0, +\infty)$ . Finally, check whether 0 is a root of  $f(x) = 0$ .
- In step 2.3.5, it is clear that  $f(x) \equiv 0$  when  $\mathbf{Zero}_I(f^u f^d) = \emptyset$  and  $f^u(x)f^d(x) < 0$ ,  $\forall x \in I$ . This can be used to check whether the given system is zero-dimensional or not. When a system is not zero-dimensional, we can also isolate its zero-dimensional subsystem.

#### 4.4. Examples and Experimental Results

We first gave two working examples. The timings are collected on a PC with a 3.2G CPU, 512M memory, and Windows OS.



**Example 1:** Consider the system  $F_2 = \{f_1, f_2\}$  where

$$\begin{aligned} f_1 &= x^4 - 3x^2 - x^3 + 2x + 2, \\ f_2 &= y^4 + xy^3 + 3y^2 - 6x^2y^2 + 4xy + 2xy^2 - 4x^2y + 4x + 2. \end{aligned}$$

Set the precision to be  $2^{-4}$ . Isolating the real roots of  $f_1$  to precision  $2^{-4}$ , we obtain the following isolating intervals:  $[[\frac{-23}{16}, \frac{-11}{8}], [\frac{-5}{8}, \frac{-9}{16}], [\frac{11}{8}, \frac{23}{16}], [\frac{25}{16}, \frac{13}{8}]]$ . Next consider  $\square_1\xi = [\frac{11}{8}, \frac{23}{16}]$ , where  $\xi \in \text{Zero}(f_1)$ . We will isolate the real roots of  $f_2(\xi, y) = 0$  in  $[0, 2]$ .

We derive  $EB_2 = \frac{1}{2}$  by resultant computation. The sleeve computed using the interval  $\square_1\xi$  is

$$\begin{aligned} f^u(y) &= -\frac{175}{32}y^2 - \frac{29}{16}y + y^4 + \frac{23}{16}y^3 + \frac{31}{4}, \\ f^d(y) &= -\frac{851}{128}y^2 - \frac{177}{64}y + y^4 + \frac{11}{8}y^3 + \frac{15}{2}. \end{aligned}$$

The sleeve bound of  $([0, 2], f^u, f^d)$  is  $SB = f^u(2) - f^d(2) = \frac{59}{8}$ . Since the sleeve-evaluation inequality (3) does not hold, we refine  $\square_1\xi$ . Let  $\square_1\xi = \text{Refine}(f_1, \square_1\xi, \frac{1}{2}) = [\frac{181}{128}, \frac{363}{256}]$ . We have the new sleeve

$$\begin{aligned} f^u(y) &= -\frac{50475}{8192}y^2 - \frac{9529}{4096}y + y^4 + \frac{363}{256}y^3 + \frac{491}{64}, \\ f^d(y) &= -\frac{204331}{32768}y^2 - \frac{39097}{16384}y + y^4 + \frac{181}{128}y^3 + \frac{245}{32} \end{aligned}$$

with sleeve bound  $SB = f^u(2) - f^d(2) = \frac{949}{2048} < \frac{1}{2} = EB_2$ . The sleeve  $([0, 2], f^u, f^d)$  is faithful (4) since  $f^u(0) = \frac{491}{64} > \frac{1}{2}$ ,  $f^d(0) = \frac{245}{32} > \frac{1}{2}$ ,  $f^u(2) = \frac{2927}{512} > \frac{1}{2}$ ,  $f^d(2) = \frac{10759}{2048} > \frac{1}{2}$ . Isolating  $f^u f^d$  in  $[0, 2]$  to precision  $2^{-8}$ , we obtain  $SL_{f_2, [0, 2]}: [[\frac{165}{128}, \frac{331}{256}], [\frac{395}{256}, \frac{99}{64}]]$  both with parities 1. These intervals are both isolating intervals of  $f^d$ . It forms an isolating interval of  $f_2(\xi, y)$  by Lemma 8. So there is an even root of  $f_2(\xi, y)$  in  $[0, 2]$  by Theorem 11. It is in  $[\frac{165}{128}, \frac{99}{64}]$ . So  $[\frac{11}{8}, \frac{23}{16}] \times [\frac{165}{128}, \frac{99}{64}]$  is an isolating box of triangular system  $F_2$ .

The isolating box does not satisfy our output precision requirement. Refine the isolating box with Refine, we obtain  $[\frac{181}{128}, \frac{5793}{4096}] \times [\frac{1423}{1024}, \frac{2947}{2048}]$ .

Eventually, we obtain all the isolating boxes for  $F_2 = 0$  in 0.141s with RootIsoTS.

Using Theorem 12 to compute  $MRB(F_2)$ , we have  $MRB(F_2) > \frac{1}{2^{289}}$  and the computing time is 9.282s. By Corollary 21, this precision is enough for us to isolate the roots of  $F_2$ .

**Example 2:** Consider the following system from (11).

$$\begin{aligned} f_1 &= -12z^2 - 3yz + xz - 27z - 4y^2 - 11xy - 5y + 29x^2 + 11x - 27; \\ f_2 &= -25z^2 - 23yz + 23xz + 4z + 2y^2 + 7xy + 21y + 4x^2 - 15x - 30; \\ f_3 &= -14z^2 + 27yz - 29xz + 11z + 4y^2 - 31xy + 22y - 12x^2 - 28x - 9. \end{aligned}$$

We first transform the system to a triangular system with WSolve package (25) in 0.141s. The isolating time for the roots of the triangular system under the precision  $2^{-20}$  is 0.406s. The C program in (11) uses 0.62s on a SUN4 with a 400 MHz CPU and 2GB of memory.

We implemented RootIsoTS in Maple 10 and tested our program with three sets of examples on a PC with a 3.2G CPU, 512M memory, and Windows OS.

The coefficients of the polynomials are within  $-100$  to  $100$ . The precision is  $\frac{1}{2^{10}}$ . We use the method mentioned in the **Remarks** for RootIsoTS to compute *all the real solutions*. We estimate the evaluation bounds by resultant computation. The most time-consuming parts are the computation of the evaluation bounds and the refinement for the isolating boxes.

The first set of examples are sparse polynomials and the results are given in Table 1. The *type* of a triangular system  $F_n = \{f_1, \dots, f_n\}$  is a list  $(d_1, \dots, d_n)$  where  $d_i = \deg_{x_i}(f_i)$ . The column started with TYPE is the type of the tested triangular systems. TIME is the average running time for each triangular system in seconds. NS is the average number of real solutions for each triangular system. NT is the number of tested triangular systems. NE is the number of terms in each polynomial.

TYPE	TIME	NS	NT	NE
(3, 3)	0.04862	2.04	100	(4, 10)
(9, 7)	0.52717	3.99	100	(10, 10)
(21, 21)	108.9115	5.45	20	(10, 10)
(3, 3, 3)	0.15783	3.48	100	(4, 10, 10)
(9, 7, 5)	16.20573	8.36	100	(10, 10, 10)
(3, 3, 3, 3)	1.69115	5.64	100	(4, 10, 10, 10)
(3, 3, 3, 3, 3)	159.1199	8.0	10	(4, 10, 10, 10, 10)

**Table 1.** Timings for sparse triangular systems

The second set of examples are dense polynomials and the results are in Table 2. A triangular system  $F_n = \{f_1, \dots, f_n\}$  of type  $(d_1, \dots, d_n)$  is called *dense* if  $f_i = \sum_{k=0}^{d_i} c_k x_i^k$  and  $\deg_{x_j}(c_k) = d_j - k$  for all  $k$  and  $i > j$ .

TYPE	TIME	NS	NT	NE
(3, 3)	0.05355	1.91	100	(3.99, 8.02)
(9, 8)	1.87486	4.26	100	(9.94, 43.98)
(11, 11)	8.782	4.5	80	(11.975, 72.5)
(16, 14)	50.22	6.0	100	(16.9, 127.13)
(21, 15)	164.23	6.22	100	(21.91, 176.8)
(3, 3, 3)	0.387	2.91	100	(3.99, 7.77, 13.01)
(5, 4, 4)	2.97	4.88	100	(5.99, 14.72, 24.24)
(5, 5, 5)	33.22	5.61	80	(5.9, 17.7, 42.1)
(8, 7, 6)	592.18	7.6	10	(8.9, 36.0, 79.8)
(3, 3, 3, 3)	119.94	6.96	50	(4.0, 8.1, 12.8, 20.9)
(5, 5, 5, 3)	551.44	3.4	10	(6.0, 32.1, 42.3, 21.5)

**Table 2.** Timings for dense triangular systems

The third set of examples are triangular systems with multiple roots and the results are given in Table 3. A triangular system of type  $(d_1, \dots, d_n)$  is generated as follows:  $f_1$  is a random polynomial in  $x_1$  and with degree  $d_1$  in  $x_1$  and  $f_i = a_i^2(b_i x_i + c_i)^{\lfloor \frac{d_i+1}{2} \rfloor - \lfloor \frac{d_i}{2} \rfloor}$  for  $i = 2, \dots, n$ , where  $a_i$  is a random polynomial in  $x_1, \dots, x_i$  and with degree  $\lfloor d_i/2 \rfloor$  in  $x_i$ ,  $b_i, c_i$  are random polynomials in  $x_1, \dots, x_{i-1}$ , and  $\lfloor d \rfloor$  is the maximal integer which is less than  $d$ . In Table 3, NM is the average number of multiple roots for the tested systems.

TYPE	TIME	NS	NM	NT	NE
(5, 5)	0.712	3.71	1.57	100	(5.9, 34.4)
(9, 8)	0.604	3.1	3.1	100	(9.9, 18.9)
(13, 11)	32.44	6.55	3.92	100	(13.9, 107.6)
(23, 21)	466.0	6.15	3.75	20	(24.0, 183.4)
(3, 3, 3)	3.213	5.59	3.24	100	(3.9, 13.0, 31.7)
(9, 7, 5)	425.9	12.95	8.15	20	(9.9, 60.8, 100.3)
(3, 3, 3, 3)	130.6	11.15	6.1	20	(4.0, 12.2, 33.7, 62.9)

**Table 3.** Timings for dense triangular systems with multiple roots

From the above experimental results, we could conclude that our algorithm is capable of handling quite large triangular systems.

## 5. Triangular Systems Without Multiple Roots

As mentioned in the preceding section, one of the most time-consuming part of the algorithm is the computation of the evaluation bound. In this section, we will propose a complete root isolation algorithm for zero-dimensional triangular systems without multiple roots, which does not need to compute the evaluation bound. The idea of the algorithm is: for this kind of systems, we could give a criterion for the roots of the system to be isolated by the roots of the sleeve functions.

### 5.1. Root Isolation of Univariate Equation without Multiple Roots

Consider a univariate polynomial  $f(x) \in \mathbb{R}[x]$  without multiple roots, that is,  $f$  is square free. We will isolate its real roots in a given interval  $I = [a, b] \in \square\mathbb{F}$ .

We call  $(I, f^u, f^d)$  of  $f$  a *normal sleeve* if it satisfies condition (4) and the monotonicity property (10). As mentioned in the Remarks after Algorithm RootIsoTS, to assume that a sleeve is normal is reasonable. The following results show that Corollary 4 is valid in this case without the evaluation-sleeve inequality.

**LEMMA 23.** Let  $(I, f^d, f^u)$  be a normal sleeve for  $f$ . Then we have  $\#(\text{ZERO}_I(f^u f^d)) = \text{even}$ . If  $\text{ZERO}(f^u f^d) = (t_0, \dots, t_{2m-1})$  and  $t_i \leq t_{i+1}$ , then each real root of  $f$  in  $I$  must be contained in an *candidate interval* of  $f$ , that is, an interval like  $(t_{2i}, t_{2i+1})$ ,  $(0 \leq i \leq m-1)$ .

**Proof.** Since  $(I, f^d, f^u)$  is a normal sleeve, we have (4). Wlog, we assume that  $f(a) > 0, f^d(a) > 0, f^u(a) > 0$  and  $f(b) < 0, f^d(b) < 0, f^u(b) < 0$ . Let  $\xi_0, \dots, \xi_s$  be the roots of  $f(x) = 0$  in  $I$ . Then, we have  $f^u(x) > f(x) \geq 0$  for  $x \in [a, \xi_0]$  and  $f^d(\xi_0) < f(\xi_0) = 0$ . Then,  $f^u(x)$  has no roots in  $[a, \xi_0]$  and  $f^d(x) = 0$  must have roots in  $J = [a, \xi_0]$ . We will show that  $\#\text{ZERO}_J(f^d f^u) = \#\text{ZERO}_J(f^d)$  is odd. Note that a univariate polynomial changes its sign after passing through an odd root and keeps its sign after passing through an even root. Since we considered multiplicities in  $\text{ZERO}_J$  and  $f^d(a) > 0, f^d(\xi_0) < 0$ ,  $\#\text{ZERO}_J(f^d)$  must be odd. As a consequence,  $\xi_0$  is in a candidate interval. Since  $f(x) = 0$  has no multiple roots in  $I$ , there exists a number  $c > \xi_0$  such that  $f(x) < 0$  and  $f^u(x) > 0$  on  $(\xi_0, c]$ . We can similarly show that  $\#\text{ZERO}_K(f^d f^u) = \#\text{ZERO}_K(f^u)$  is even for  $K = [c, \xi_1]$ . Then,  $\xi_1$  is also in a candidate interval. Similarly,  $\xi_s$  is in a candidate interval  $(t_{2u}, t_{2u+1})$ . We thus have  $f^d(x) < f(x) \leq 0$  on  $[\xi_s, b]$ . Also,  $f^u(\xi_s) > f(\xi_s) = 0$  and

$f^u(b) < 0$ . Then for  $L = [\xi_s, b]$ ,  $\#\text{ZERO}_L(f^d f^u) = \#\text{ZERO}_L(f^u)$  is odd and with  $t_{2u+1}$  as the first root.

We proved that in  $[a, \xi_0]$  and  $[\xi_s, b]$ , the numbers of roots are odd; in  $[\xi_i, \xi_{i+1}]$ ,  $i = 0, \dots, s-1$ , the numbers of roots are even. As a consequence,  $\#\text{ZERO}_I(f^u f^d)$  is even and each root of  $f(x) = 0$  is in a candidate interval.  $\square$

The above lemma shows that each root of  $f = 0$  is in a candidate interval of  $f^d f^u = 0$ . But, one candidate interval may contain more than one roots of  $f = 0$ . The following lemma gives a sufficient criterion for a candidate interval to be an isolating one.

LEMMA 24. Let  $(I, f^d, f^u)$  be a normal sleeve for  $f$  and  $J = [t_{2i}, t_{2i+1}]$  a candidate interval. If  $J$  is an odd interval (definition see (9)) and

$$\left. \begin{array}{l} 1. \frac{\partial f^u}{\partial x} \text{ has no roots in } [t_{2i}, t_{2i+1}] \text{ if } t_{2i} \in \text{Zero}(f^d), \\ 2. \frac{\partial f^d}{\partial x} \text{ has no roots in } [t_{2i}, t_{2i+1}] \text{ if } t_{2i} \in \text{Zero}(f^u) \end{array} \right\} \quad (24)$$

then  $J$  is an isolating interval of a root of  $f(x) = 0$ .

**Proof.** Since  $J$  is an odd interval, it contains at least one root of  $f(x) = 0$ . If  $J$  contains more than one roots of  $f(x) = 0$ , the function  $y = f(x)$  must have an extremal point  $x_0$  and  $x_0$  must be a root of  $\frac{\partial f}{\partial x}(x) = 0$ . On the other hand, we will show that for  $f$  satisfying the conditions in the lemma,  $\frac{\partial f}{\partial x}$  cannot have a root in  $J$ , which means  $f$  has at most one root in  $J$ , and hence prove the lemma. Wlog, we assume that  $f^d(t_{2i}) = f^u(t_{2i+1}) = 0$ . Let  $\xi$  be a root of  $f(x) = 0$  in  $J$ . Since  $f^u(\xi) > 0$  and  $f^u(t_{2i+1}) = 0$ , there exists an  $\eta \in (\xi, t_{2i+1})$  such that  $\frac{\partial f^u}{\partial x}(\eta) < 0$ . Since  $\frac{\partial f^u}{\partial x}$  has no roots in  $J$ , we have  $\frac{\partial f^u}{\partial x} < 0$  on  $J$ . From the monotonicity property (10), we have  $\frac{\partial f^d}{\partial x} \leq \frac{\partial f}{\partial x} \leq \frac{\partial f^u}{\partial x}$ , and hence  $\frac{\partial f}{\partial x} < 0$  on  $J$ . Therefore,  $\frac{\partial f}{\partial x} = 0$  have no roots in  $J$ .  $\square$

The following lemma shows that the conditions in Lemma 24 are also necessary in certain sense.

LEMMA 25. Let  $f \in R[x]$  be square free and  $(I, f^d, f^u)$  a normal sleeve for  $f$  constructed with formula (21). Then, when  $f^u$  and  $f^d$  sufficiently approximate  $f$ , each candidate interval is odd and satisfies condition (24).

**Proof.** Wlog, we assume that  $\xi$  is the first root of  $f = 0$  in  $I$ ,  $C = (t_{2k}, t_{2k+1})$  is the candidate interval containing  $\xi$ ,  $f^d(a) > 0$ ,  $f^u(a) > 0$ , and  $f^d(t_{2k}) = 0$ . Since  $f$  has no multiple roots, we have  $\frac{\partial f}{\partial x}(\xi) < 0$ . Use the notations  $A_\xi$  and  $B_\xi$  introduced in (7). When  $f^u$  and  $f^d$  sufficiently approximate  $f$ , for example, when the sleeve-evaluation inequality (3) holds,  $f = 0$  has no extremal points in  $\bar{A}_\xi = [a_\xi, \xi]$  and  $f^d = 0$  has no roots in  $[a, a_\xi]$ . Since  $\frac{\partial f}{\partial x}(\xi) < 0$ , for  $x \in \bar{A}_\xi$ , we have  $\frac{\partial f}{\partial x}(x) < 0$  and  $|\frac{\partial f}{\partial x}(x)| > \rho$  for a positive number  $\rho$ .

We first show that  $C$  must be an odd interval when  $f^d, f^u$  sufficiently approximate  $f$ . From the way to construct  $f^u$  and  $f^d$ , the coefficients of  $\frac{\partial f^u}{\partial x}, \frac{\partial f^d}{\partial x}$  will sufficiently approximate that of  $\frac{\partial f}{\partial x}$  when  $f^u, f^d$  sufficiently approximate  $f$ . Then, when  $f^d, f^u$  sufficiently approximate  $f$ ,  $\frac{\partial f^u}{\partial x}$  and  $\frac{\partial f^d}{\partial x}$  will sufficiently approximate  $\frac{\partial f}{\partial x}$ . Since  $\frac{\partial f}{\partial x}(x) < 0$  and  $|\frac{\partial f}{\partial x}(x)| > \rho$  for  $x \in \bar{A}_\xi$ , when  $f^u, f^d$  sufficiently approximate  $f$ , we have  $\frac{\partial f^u}{\partial x}(x) < 0$ ,

$\frac{\partial f^d}{\partial x}(x) < 0$  for  $x \in \bar{A}_\xi$ . As a consequence,  $f^d$  has only one root in  $A_\xi$  and  $C$  is the first candidate interval. Since we assume that the sleeve-evaluation inequality (3) holds,  $C$  must be an odd interval. So we proved that  $C$  is the first candidate interval and is odd when  $f^d, f^u$  sufficiently approximate  $f$ . Other cases can be proved similarly and we proved that all candidate intervals must be odd when  $f^d, f^u$  sufficiently approximate  $f$ .

Since  $\frac{\partial f^u}{\partial x}(x) < 0$  for  $x \in \bar{A}_\xi$ , condition (24) will be satisfied when  $C$  is contained in  $\bar{A}_\xi$ , which is possible when  $f^d, f^u$  sufficiently approximate  $f$ .  $\square$

Similar to the results in Section 2.3, we can obtain an effective version of the criteria given in Lemma 24. Let  $\text{ZERO}_I(f^u f^d)$  be the sorted list given in (8), and  $[a_i, b_i]$  an isolating interval of  $t_i$ , where any two distinct intervals  $[a_i, b_i]$  and  $[a_j, b_j]$  are disjoint. We still call  $[a_{2i}, b_{2i+1}]$  an *effective candidate* of  $f$  in  $I$ , which is called odd if  $[t_{2i}, t_{2i+1}]$  is an odd interval. We have

**THEOREM 26.** Let  $f \in \mathbb{R}[x]$  be square free,  $(I, f^u, f^d)$  a normal sleeve for  $f$ , and  $K$  the set of effective candidates of  $f$  in  $I$ . When  $f^d$  and  $f^u$  sufficiently approximate  $f$  and  $[a_i, b_i]$  sufficiently approximates  $t_i$ , each  $J \in K$  is an odd interval satisfying

$$\left. \begin{array}{l} 1. \frac{\partial f^u}{\partial x} \text{ has no roots in } [a_{2i}, b_{2i+1}] \text{ if } t_{2i} \in \text{Zero}(f^d); \\ 2. \frac{\partial f^d}{\partial x} \text{ has no roots in } [a_{2i}, b_{2i+1}] \text{ if } t_{2i} \in \text{Zero}(f^u). \end{array} \right\} \quad (25)$$

As a consequence,  $K$  is a set of isolating intervals for the roots of  $f$  in  $I$ .

**Proof.** By Lemma 25, when  $f^d$  and  $f^u$  sufficiently approximate  $f$ , each  $J \in K$  is an odd interval satisfying (24). Wlog, consider the first case in (24). We have  $|\frac{\partial f^u}{\partial x}(x)| > 0$  for  $x \in [t_{2i}, t_{2i+1}]$ . Since  $[a_{2i}, b_{2i}]$  and  $[a_{2i+1}, b_{2i+1}]$  are isolating intervals of  $t_{2i}$  and  $t_{2i+1}$  respectively, we can refine them so that  $|\frac{\partial f^u}{\partial x}(x)| > 0$  for  $x \in [a_{2i}, b_{2i+1}]$  and condition (25) is satisfied. By Lemma 24, these intervals are isolating intervals for some roots of  $f$ . By Lemma 23, these intervals are isolating intervals for all roots of  $f$ .  $\square$

## 5.2. Root Isolation Algorithm and Experiment Results

The basic idea to isolate the real roots of a triangular system is to construct and refine the sleeves until all the effective candidates are odd intervals and condition (25) is satisfied. After this, the effective candidate intervals are isolating intervals. Algorithm RootIsoSQFree is based on this idea.

We implemented Algorithm RootIsoSQFree in Maple on a PC with a 3.2G CPU, 512M memory, and Windows OS. Table 4 contains the data for fourteen types of triangular sets. The meaning of the parameters can be found in Section 4.4.

For triangular systems of types (9, 8), (21, 15), and (8, 7, 6), Algorithm RootIsoSQFree are 57, 1748, 4907 times faster than Algorithm RootIsoTS (Table 2) on a sample set of 100 problems for each type. Therefore, in terms of efficiency, the improvements of Algorithm RootIsoSQFree comparing to that of Algorithm RootIsoTS is huge. Also, we can see that Algorithm RootIsoSQFree is good enough to isolate the roots for large scale systems efficiently.

RootIsoSQFree

**Input:**  $F_n$ : a zero dimensional triangular system without multiple roots;

$$B_n = \prod_{i=1}^n I_i(I_i = [l_i, r_i] > 0), \epsilon > 0.$$

**Output:** An isolating set  $\square \mathbf{Zero}_{B_n}(F_n)$ .

1. Compute  $\square \mathbf{Zero}_{B_1}(F_1)$  for  $F_1$  to precision  $\epsilon$ .  
*Result* :=  $\square \mathbf{Zero}_{B_1}(F_1)$ . *New* :=  $\emptyset$ .  
 If *Result* =  $\emptyset$ , return *Result*, end
2. For  $i$  from 2 to  $n$ , do
  - 2.1.  $\delta := \epsilon$ .
  - 2.2. while *Result*  $\neq \emptyset$ , do
    - 2.2.1. Choose an element  $\square_{i-1}\xi$  from *Result*.  
*Result* := *Result*  $\setminus \{\square_{i-1}\xi\}$ .
    - 2.2.2. Compute the sleeve:  
 $f^u(x) = f_i^u(\square_{i-1}\xi, x)$ ,  $f^d(x) = f_i^d(\square_{i-1}\xi, x)$ .
    - 2.2.3. Isolate the real roots of  $f^u f^d$  in  $I_i$  with precision  $\epsilon > 0$ .
    - 2.2.4. Construct the set  $K$  of effective candidate intervals.
    - 2.2.5. While there is a  $J \in K$  s.t.  $J$  is not odd or Condition (25) doesn't hold,  
 $\delta := \delta/2$ .  
 $\square_{i-1}\xi := \text{Refine}(F_{i-1}, \square_{i-1}\xi, \delta)$ .  
 Reconstruct  $f^u(x)$ ,  $f^d(x)$  and recompute an effective candidate set  $K$ .
    - 2.2.6.  $K := \text{Refine}(F_i, K, \epsilon)$ . *New* := *New*  $\cup \{\square_{i-1}\xi \times K\}$ .
  - 2.3. If *New* =  $\emptyset$ , return *New*, end
  - 2.4. *Result* := *New*. *New* :=  $\emptyset$ .
3. return *Result*.

TYPE	TIME	NS	NT	NE
(9, 8)	0.03282	4.39	100	(9.9, 44.67)
(21, 15)	0.09391	5.75	100	(21.85, 135.37)
(25, 21)	0.19454	6.66	100	(25.88, 251.81)
(51, 41)	0.93559	9.08	100	(51.78, 898.49)
(119, 70)	4.33518	11.77	100	(119.38, 2543.39)
(219, 180)	54.33796	15.33	100	(218.7, 16387.83)
(8, 7, 6)	0.12077	8.08	100	(8.96, 35.86, 83.64)
(19, 17, 14)	1.22715	14.58	100	(19.94, 170.11, 676.38)
(39, 37, 31)	19.44103	24.63	100	(39.82, 737.1, 5954.7)
(139, 77, 41)	70.086375	36.25	40	(139.25, 3066.475, 13177.05)
(9, 7, 5, 3)	0.14828	8.05	100	(9.95, 35.85, 55.73, 34.8)
(19, 17, 15, 13)	9.49561	29.81	100	(19.96, 170.14, 811.96, 2368.19)
(59, 37, 25, 23)	71.62045	28.1	20	(59.45, 737.0, 3259.15, 17458.95)
(11, 9, 8, 7, 5, 3, 3)	3.41567	41.31	100	(11.9, 54.7, 163.31, 328.42, 250.94, 83.57, 119.38)

**Table 4.** Timings for dense triangular systems without multiple roots

## 6. Conclusion

This paper provides a complete numerical algorithm of isolating the real roots for arbitrary zero-dimensional triangular systems. The key idea is to use a sleeve satisfying

the sleeve-evaluation inequality to isolate the roots for a univariate polynomial with algebraic number coefficients. Even with our current simple implementation, the algorithm is shown to be quite effective. One of the most time-consuming part of the algorithm is the computation of the evaluation bound. We further propose a complete root isolation algorithm for zero-dimensional triangular systems without multiple roots, which does not need to compute the evaluation bound and is shown to be much faster than the general algorithm.

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