Ambient Isotopic Meshing for Implicit Algebraic Surfaces with Singularities
(Plenary Talk)

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Abstract. A complete method is proposed to compute an ambient isotopic meshing for an implicit algebraic surface with singularities. By ambient isotopic, we mean a meshing with correct topology and any given precision. We use symbolic computation to guarantee the correctness and use numerical computation whenever possible to enhance the efficiency. Nontrivial examples are given to show the effectiveness of the algorithm.

1 The Main Results

To determine the topology of an algebraic surface and to use triangular meshes to approximately represent the surface are basic operations in computer graphics and geometric modeling. A recent survey on this topic can be found in [2].

A meshing is called **isotopic** if it has the same topology and the same geometry as the surface. A meshing is called **ambient isotopic** or **certified** if it is isotopic and approximates the surface to any given precision. It is known that isotopy is stronger than homeomorphism [2].

Precisely, an **isotopic meshing** for a surface $\mathcal{S} \subset \mathbb{R}^3$ consists of a triangular polyhedron $\mathscr{G}$ and a continuous mapping $\gamma : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ which, for any fixed $t \in [0,1]$, is a homeomorphism $\gamma(\cdot,t)$ from R^3 to itself, and which continuously deforms $\mathscr{G}$ into S: $\gamma(\cdot,0) = id$, $\gamma(\mathscr{G},1) = \mathcal{S}$.

An isotopic meshing $\mathscr{G}$ for a surface $\mathcal{S} \subset \mathbb{R}^3$ is called an **ambient isotopic meshing** if, for a given number $\epsilon > 0$, $\mathscr{G}$ gives an ϵ-approximation for $\mathcal{S}$ in the following sense $\| P - \gamma(P,1) \| \le \epsilon$ for all $P \in \mathscr{G}$. Such a meshing is also called an **ϵ-meshing**.

We use intervals to isolate real numbers: let $\square\mathbb{Q}$ denote the set of intervals of the form $[a,b]$ where $a < b \in \mathbb{Q}$. The **length** of an interval box $\mathbf{B}_n = [a_1,b_1] \times \cdots \times [a_n,b_n] \in \square\mathbb{Q}^n$ is defined to be $|\mathbf{B}_n| = \max_i(b_i - a_i)$.

Our main result can be summarized as follows.

Theorem 1. *Let $f(x,y,z)$ be a square free polynomial with rational numbers as coefficients. For any algebraic surface $\mathcal{S} : f(x,y,z) = 0$, a given number $\epsilon > 0$, and a bounding box $\mathbf{B} \in \square\mathbb{Q}^3$, we have an algorithm to compute an* **ϵ-meshing** *for $\mathcal{S}_\mathbf{B} = \mathcal{S} \cap \mathbf{B}$.*

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We first give a sketch of our algorithm and will explain some of the key steps later in this extended abstract.

Algorithm 2. **AMeshSur**($f(x, y, z), \mathbf{B}_3, \epsilon$). *The input is the same as described in Theorem 1. The output is an* ϵ**-meshing** *for* $\mathcal{S}_{\mathbf{B}}$.

S1 Compute the strong projection curve $\mathcal{C} : g(x, y) = 0$ and an ϵ-meshing for $\mathcal{C}$.
S2 Meshing the singular part of $\mathcal{S}$.
S3 Meshing the non-singular part of $\mathcal{S}$.
S4 Merge the meshes obtained in Steps **S2** and **S3**.

We will explain how to compute the strong projection curve in Section 2. The methods to compute an ϵ-meshing for a plane curve can be found in [4]. We will briefly show how to mesh the singular parts of $\mathcal{S}$ in Section 3. We use modified methods from [3,11] to mesh the non-singular part of $\mathcal{S}$. Details of our results can be found in [5,7,8].

There exist four main approaches to compute isotopic meshings for surfaces: the marching cube method, the Morse theory method, the Delaunay refinement method, and the CAD (Cylindrical Algebraic Decomposition) based method [2]. Of these methods, only the CAD based methods are capable of treating surfaces with singularities.

We give a method to compute a certified meshing for implicit algebraic surfaces with singularities. The method is a hybrid one based on the CAD approach. We use symbolic computation to guarantee the correctness and use numerical computation whenever possible to enhance the efficiency. To our knowledge, this is the first method to compute an ambient isotopic meshing for surfaces with singularities.

To use a triangular polyhedron to approximate a surface, we usually need polyhedrons with thousands of faces. A strategy to reduce the number of meshes is to use quadratic surfaces to construct certified approximation. This has been done for algebraic curves [9,10] and is an interest problem for algebraic surfaces.

2 Strong Projection Curve

The basic idea for the CAD based methods to compute the topology of a surface $\mathcal{S}$ is to project $\mathcal{S}$ to the xy-plane, compute the topology of the projection curve, and obtain the topology of $\mathcal{S}$ by lifting the topology of the projection curve to the space. To compute the ambient meshing for a surface, we need a strong projection curve, which will be discussed in this section.

Let $\mathcal{S} : f(x, y, z) = 0$ be an algebraic surface, where $f(x, y, z) \in \mathbb{Q}[x, y, z]$ is square free. A point P_0 is a **critical point** of $\mathcal{S}$ if $f(P_0) = f_z(P_0) = 0$.

Let

$$G_1(x, y) = \text{sqrfree}(\text{Res}(f, \frac{\partial f}{\partial z}, z)). \tag{1}$$

The plane curve $G_1(x, y) = 0$ is called the **projection curve** of $\mathcal{S}$.

A point is called z**-extremal** of a surface or a space curve if the surface or curve achieves a local extremum value at this point in the z-direction. In order to compute the ambient meshing for a surface, we need to project the z-extremal points of $\mathcal{S}$ and the space curve $f(x,y,z) = G_1(x,y) = 0$ to the plane. We have

Lemma 1. *Let $f(x,y,z) = \prod_i f_i(x,y,z)$ be a square free polynomial and f_i irreducible polynomials. A necessary condition for the surface $f(x,y,z) = 0$ to have a z-extremal point is*

$$G_2(x,y) = \prod_i \mathrm{Res}(f_i, \frac{\partial f_i}{\partial x}, z) \prod_i \mathrm{Res}(f_i, \frac{\partial f_i}{\partial y}, z) = 0 \tag{2}$$

where only the nonzero resultants are included.

The following example shows that we need to consider the irreducible factors. Let $f = (z-y)(z-x)(x^2+y^2+z^2-1)$. Then $\mathrm{Res}(f, f_x, z) = \mathrm{Res}(f, f_y, z) \equiv 0$. But the surface indeed has an z-extremal point at $(0,0,1)$.

We also need to consider the z-extremal points of spatial curves defined by $g(x,y) = f(x,y,z) = 0$, where g and f are polynomials. For this purpose, we need to decompose the curve into irreducible ones. The leading coefficient of g (f) as a univariate polynomial in y (z) is called the **initial** of g (f). Any spatial curve $f(x,y,z) = g(x,y) = 0$ can be decomposed into the union of irreducible curves represented by irreducible chains algorithmically [12,5]. The initials of these irreducible chains are univariate polynomial in x. We have:

Lemma 2. *Let $g(x,y), f(x,y,z)$ be an irreducible chain and*

$$\begin{aligned} I(x) &= \text{ product of the initials of } f, g. \\ T(x) &= \mathrm{Res}(\mathrm{Res}(h, f, z), g, y) \text{ where } h(x,y,z) = f_x g_y - f_y g_x. \end{aligned} \tag{3}$$

Let E be the set of z-extremal points of the curve $\mathcal{C} : f = g = 0$. Then $Proj_x(E) \subset V(T(x)) \cup V(I(x))$. Furthermore, if $T(x) \equiv 0$, then the curve defined by f, g is contained in several planes perpendicular to the z-axis.

The following example shows that we need to decompose the curve into irreducible ones. Let $f = z(x^2+z^2-1), g = y$. Then $\mathrm{Res}(f_x g_y - f_y g_x, f, z) \equiv 0$. But the curve indeed has a z-extremal point at $(0,0,1)$.

The plane curve

$$g(x,y) = G_1(x,y)G_2(x,y)T(x)I(x) \tag{4}$$

is called the **strong projection curve** of surface $\mathcal{S}$, where G_1, G_2, T and I are defined in (1), (2), and (3) respectively. In the case of (3), we will include the nonzero projections for all irreducible components of $G_1(x,y)G_2(x,y) = f(x,y,z) = 0$.

The purpose to introduce the concept of strong projection curve is that on a region containing no points of $g(x,y) = 0$, the surface $\mathcal{S}$ and the space curve $g(x,y) = f(x,y,z) = 0$ have no z-extremal points. This property allows us to estimate the z-values of a surface patch or a curve segment over a region R with their values on the boundary of R.

3 Segregating Box for Points and Curve Segments

A key idea to mesh the singular part of $\mathcal{S} : f(x, y, z) = 0$ is to compute the segregating boxes for the critical points and critical curve segments of $\mathcal{S}$. The idea of segregating boxes for points of plane curves was originally introduced in [1]. Here, we give a new interval based method to compute it and extend the concept to critical curve segments.

The critical parts of $\mathcal{S}$ consists of the **critical curve** defined by $g(x, y) = f(x, y, z) = 0$ and the **critical points** defined by $h(x) = g(x, y) = f(x, y, z) = 0$, where g is defined in (4) and $h(x) = \mathrm{Res}(g, g_y, y)$.

We may compute isolating boxes for the singular points of $\mathcal{S}$ with the method given in [8]. Let $P = (\alpha, \beta, \gamma)$ be a critical point on $\mathcal{S}$ such that $f(\alpha, \beta, z) = 0$ has a finite number of solutions. Then $\mathbf{B}$ is called a **segregating box** of P if $\mathcal{S}$ does not intersect with the top and bottom faces of $\mathbf{B}$. A segregating box can be computed as follows. We may compute an isolating interval $[u, v]$ of γ as the solution of $f(\alpha, \beta, z) = 0$. Let $\mathbf{B}_2 = [a, b] \times [c, d]$ be a box containing (α, β). We may compute the inclusion function $\square f(\mathbf{B}_2, u), \square f(\mathbf{B}_2, v)$ and subdivide $\mathbf{B}_2$ until $0 \notin \square f(\mathbf{B}_2, u)$ and $0 \notin \square f(\mathbf{B}_2, v)$. This process will terminate since $f(\alpha, \beta, u) f(\alpha, \beta, v) \neq 0$.

In a similar way, we may assume that the strong projection curve $\mathcal{C} : g(x, y) = 0$ of $\mathcal{S}$ does not meet the top and bottom of $\mathbf{B}_2$. As a consequence, $\mathcal{S}$ intersects $\mathbf{B}$ on the four side faces and the critical curves of $\mathcal{S}$ intersects $\mathbf{B}$ on the left and right faces: $[a, a] \times [c, d] \times [u, v]$ and $[b, b] \times [c, d] \times [u, v]$. With these conditions, it is not difficult to compute the topology and meshing for $\mathcal{S}$ inside $\mathbf{B}$. We can subdivide the boxes until $|\mathbf{B}| < \epsilon$ and the meshings thus obtained are ϵ-meshings for $\mathcal{S}$.

Now, we introduce briefly how to mesh $\mathcal{S}$ near a critical curve segment. A box $\mathbf{B} = [a, b] \times [c, d] \times [u, v]$ is called a **segregating box** for a critical curve segment S inside $\mathbf{B}$ if $\mathcal{S}$ is the only smooth curve branch of the critical curve inside $\mathbf{B}$ and $\mathcal{S}$ does not intersect with the top and bottom faces of $\mathbf{B}$.

We may compute a segregating box for a critical curve segment S as follows. Let P_a and P_b be the intersection points of S with planes $x = a$ and $x = b$ respectively. Then we may compute an isolating box $\mathbf{B}_a = [y_{a,1}, y_{a,2}] \times [z_{a,1}, z_{a,2}]$ for P_a by solving the equations $g(a, y) = f(a, y, z) = 0$ with [6]. Compute $\mathbf{B}_b = [y_{b,1}, y_{b,2}] \times [z_{b,1}, z_{b,2}]$ similarly. We may set the segregating box of S to be $\mathbf{B}_P = [a, b] \times [\min\{y_{a,1}, y_{b,1}\}, \max\{y_{a,2}, y_{b,2}\}] \times [\min\{z_{a,1}, z_{b,1}\}, \max\{z_{a,2}, z_{b,2}\}]$. This is

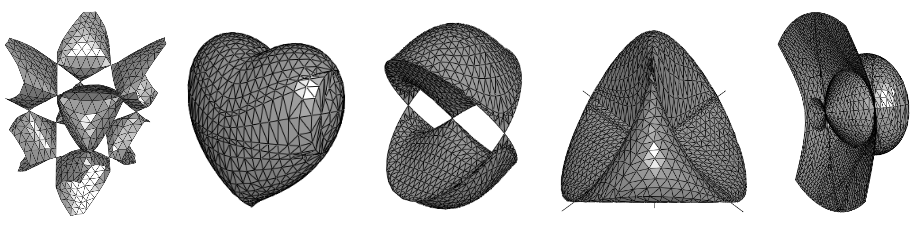

Fig. 1. Meshing for surfaces with singular points and singular curves

true because, due to the conditions in the strong projection curve, S is monotone both in y and z directions. If $|\mathbf{B}_P| > \epsilon$, we can further subdivide the boxes. After the segregating boxes for the critical curve segments are constructed, we can compute a meshing for the surface inside $\mathbf{B}$.

Now, we have computed the meshing and topological structure for $\mathcal{S}$ inside the segregating boxes for the critical point and critical curve of $\mathcal{S}$. Outside these boxes, $\mathcal{S}$ has no singular points and can be meshed by modifying the methods in [3,11].

Figure 1 are the meshings for five surfaces computed with the implementation of our algorithm in Maple. Equations defining these surfaces can be found in [5].

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