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Young type inequalities for matrices

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Abstract: First, some Young type inequalities for scalars were given. Then on the base of them, corresponding Young type inequalities for matrices were established.

Key words: unitarily invariant norms; Young type inequality; positive semidefinite matrices; singular values

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矩阵的 Young 型不等式

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摘要: 首先给出了若干标量 Young 型不等式. 然后在此基础上, 建立了相应的矩阵 Young 型不等式.

关键词: 酉不变范数; Young 型不等式; 半正定矩阵; 奇异值

0 Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm, the trace norm, and the spectral norm of A are defined by $\|A\|_2 = \sqrt{\sum\limits_{j=1}^{n} s_j^2(A)}$, $\|A\|_1 = \sum\limits_{j=1}^{n} s_j(A)$, and $\|A\|_{sp} = s_1(A)$, respectively, where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_{n-1}(A) \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

The classical Young inequality for nonnegative real numbers says that if $a, b \geqslant 0$ and $0 \leqslant v \leqslant 1$, then

$$a^v b^{1-v} \leqslant va + (1-v)b \tag{0.1}$$

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with equality if and only if $a=b$. If $p,q\geqslant 0$ such that $\frac{1}{p}+\frac{1}{q}=1$, then the inequality (0.1) can be written as $ab\leqslant \frac{a^p}{p}+\frac{b^q}{q}$.

Several matrix versions of the Young inequality(0.1) have been recently established[1-9]. It seems that matrix versions of the Young inequality have aroused considerable interest. The main purpose of this paper is to give some Young type inequalities for matrices.

1 Young type inequalities for scalars

We begin this section with the Young type inequalities for scalars.

Lemma 1.1 Suppose that $a,b\geqslant 0$. If $0\leqslant v\leqslant \frac{1}{2}$, then

$$\left[(va)^v b^{1-v}\right]^2+v^2(a-b)^2\leqslant v^2a^2+(1-v)^2b^2. \tag{1.1}$$

If $\frac{1}{2}\leqslant v\leqslant 1$, then

$$\{a^v[(1-v)b]^{1-v}\}^2+(1-v)^2(a-b)^2\leqslant v^2a^2+(1-v)^2b^2. \tag{1.2}$$

Proof If $0\leqslant v\leqslant \frac{1}{2}$. Then, by inequality (0.1), we have

$$\begin{aligned} v^2a^2+(1-v)^2b^2-v^2(a-b)^2 &= b\left[2v(va)+(1-2v)b\right] \\ &\geqslant b(va)^{2v}b^{1-2v}=\left[(va)^v b^{1-v}\right]^2, \end{aligned}$$

and so

$$v^2a^2+(1-v)^2b^2\geqslant \left[(va)^v b^{1-v}\right]^2+v^2(a-b)^2.$$

If $\frac{1}{2}\leqslant v\leqslant 1$, then

$$\begin{aligned} v^2a^2+(1-v)^2b^2-(1-v)^2(a-b)^2 &= a[(2v-1)a+2(1-v)^2b] \\ &\geqslant aa^{2v-1}[(1-v)b]^{2-2v}=\{a^v[(1-v)b]^{1-v}\}^2, \end{aligned}$$

and so

$$v^2a^2+(1-v)^2b^2\geqslant \{a^v[(1-v)b]^{1-v}\}^2+(1-v)^2(a-b)^2.$$

This completes the proof.

Hirzallah and Kittaneh[7] obtained a refinement of the scalar Young's inequality as follows:

$$\left(a^v b^{1-v}\right)^2+r_0^2(a-b)^2\leqslant (va+(1-v)b)^2, \tag{1.3}$$

where $r_0=\min\{v,1-v\}$. In Kittaneh and Manasrah's paper[9], the following related refinement of the scalar Young's inequality was obtained

$$\left(a^v b^{1-v}\right)^2+r_0(a-b)^2\leqslant va^2+(1-v)b^2. \tag{1.4}$$

When comparing Lemma 1.1 with the inequalities (1.3) and (1.4), it is easy to observe that both the left-hand and the right-hand sides of Lemma 1.1 are greater than or equal to the corresponding sides in (1.3) and (1.4), respectively. It should be noticed that neither Lemma 1.1 nor (1.3) and (1.4) is uniformly better than the other.

2 Young type inequalities for matrices

A matrix Young inequality in [3] says that if $A, B \in M_n$ are positive semidefinite, then

$$s_j\left(A^v B^{1-v}\right) \leqslant s_j\left(vA + (1-v)B\right)$$

for $j = 1, \cdots, n$. The above singular value inequality of Ando entailed the norm inequality

$$\left\|A^v B^{1-v}\right\| \leqslant \left\|vA + (1-v)B\right\|.$$

Kosaki[4], Bhatia and Parthasarathy[6] proved that if $A, B, X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$\left\|A^v X B^{1-v}\right\|_2^2 \leqslant \left\|vAX + (1-v)XB\right\|_2^2. \tag{2.1}$$

Based on the refined Young inequality (1.3), it has been shown in [7] that if $A, B, X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$\left\|A^v X B^{1-v}\right\|_2^2 + r_0^2\left\|AX - XB\right\|_2^2 \leqslant \left\|vAX + (1-v)XB\right\|_2^2. \tag{2.2}$$

Obviously, (2.2) is an improvement of (2.1) for the Hilbert-Schmidt norm. In Kittaneh and Manasrah's paper[9], the following Young inequalities for matrices were obtained

$$\left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|_2 + 2r_0\left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}\right) \leqslant \|AX + XB\|_2,$$

$$\left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|_2^2 + 2r_0\left\|AX - XB\right\|_2^2 \leqslant \left\|AX + XB\right\|_2^2.$$

In this section, we give the trace norm, the Hilbert-Schmidt norm, and determinant versions of Young type inequalities based on the Young type inequalities (1.1) and (1.2). To do this, we need the following lemma.

Lemma 2.1[10] Let $A, B \in M_n$. Then

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A)\, s_j(B).$$

Theorem 2.2 Let $A, B \in M_n$ be positive semidefinite. If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$v^v\left\|A^v B^{1-v}\right\|_1 \leqslant \sqrt{v^2\|A\|_2^2 + (1-v)^2\|B\|_2^2 - v^2\left(\|A\|_2 - \|B\|_2\right)^2}. \tag{2.3}$$

If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$(1-v)^{1-v}\left\|A^v B^{1-v}\right\|_1 \leqslant \sqrt{v^2\|A\|_2^2 + (1-v)^2\|B\|_2^2 - (1-v)^2\left(\|A\|_2 - \|B\|_2\right)^2}. \tag{2.4}$$

Proof If $0 \leqslant v \leqslant \frac{1}{2}$, by the inequality (1.1), we have

$$[(v s_j(A))^v s_j(B)^{1-v}]^2 + v^2\left(s_j(A) - s_j(B)\right)^2 \leqslant v^2 s_j^2(A) + (1-v)^2 s_j^2(B)$$

for $j=1,\cdots,n$. Thus, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\operatorname{tr}(v^2A^2+(1-v)^2B^2) &= v^2\operatorname{tr}A^2+(1-v)^2\operatorname{tr}B^2\\ &=\sum_{j=1}^n\left(v^2s_j^2(A)+(1-v)^2s_j^2(B)\right)\\ &\geqslant\sum_{j=1}^n[(vs_j(A))^v s_j(B)^{1-v}]^2\\ &\quad+v^2\left(\sum_{j=1}^n s_j^2(A)+\sum_{j=1}^n s_j^2(B)-2\sum_{j=1}^n s_j(A)s_j(B)\right)\\ &\geqslant\sum_{j=1}^n\left[v^v s_j(A^v)s_j\left(B^{1-v}\right)\right]^2\\ &\quad+v^2\left(\|A\|_2^2+\|B\|_2^2-2\left(\sum_{j=1}^n s_j^2(A)\right)^{\frac12}\left(\sum_{j=1}^n s_j^2(B)\right)^{\frac12}\right)\\ &=v^{2v}\sum_{j=1}^n\left[s_j(A^v)s_j\left(B^{1-v}\right)\right]^2+v^2\left(\|A\|_2-\|B\|_2\right)^2. \end{aligned}\tag{2.5}$$

On the other hand, we also have

$$v^2\operatorname{tr}A^2+(1-v)^2\operatorname{tr}B^2=v^2\|A\|_2^2+(1-v)^2\|B\|_2^2.\tag{2.6}$$

Therefore, it follows from (2.5) and (2.6) that

$$v^2\|A\|_2^2+(1-v)^2\|B\|_2^2-v^2\left(\|A\|_2-\|B\|_2\right)^2\geqslant v^{2v}\sum_{j=1}^n\left[s_j(A^v)s_j\left(B^{1-v}\right)\right]^2.\tag{2.7}$$

By Lemma 2.1 and (2.7), we have

$$v^v\left\|A^vB^{1-v}\right\|_1\leqslant\sqrt{v^2\|A\|_2^2+(1-v)^2\|B\|_2^2-v^2\left(\|A\|_2-\|B\|_2\right)^2}.$$

If $\frac12\leqslant v\leqslant1$, then by the inequality (1.2) and the same method above, we have the inequality (2.4). This completes the proof.

Theorem 2.3 Let $A,B,X\in M_n$ such that A and B are positive semidefinite. If $0\leqslant v\leqslant\frac12$, then

$$\begin{aligned}\|vAX+(1-v)XB\|_2^2\geqslant{}& v^2\|AX-XB\|_2^2+v^{2v}\|A^vXB^{1-v}\|_2^2\\ &+2v(1-v)\|A^{1/2}XB^{1/2}\|_2^2.\end{aligned}\tag{2.8}$$

If $\frac12\leqslant v\leqslant1$, then

$$\begin{aligned}\|vAX+(1-v)XB\|_2^2\geqslant{}&(1-v)^2\|AX-XB\|_2^2+(1-v)^{2-2v}\|A^vXB^{1-v}\|_2^2\\ &+2v(1-v)\|A^{1/2}XB^{1/2}\|_2^2.\end{aligned}\tag{2.9}$$

Proof Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U,V\in M_n$ such that $A=UDU^*$ and $B=VDV^*$, where

$D = \mathrm{diag}\left(\lambda_1, \cdots, \lambda_n\right), \quad E = \mathrm{diag}\left(\mu_1, \cdots, \mu_n\right)$, and $\lambda_i, \mu_i \geqslant 0, i = 1, \cdots, n$.

Let $Y = U^* X V = (y_{ij})$. Then

$$vAX + (1-v)\, XB = U\left(vDY + (1-v)\, YE\right) V^* = U\left(\left(v\lambda_i + (1-v)\,\mu_j\right) y_{ij}\right) V^*,$$
$$AX - XB = U\left(\left(\lambda_i - \mu_j\right) y_{ij}\right) V^*, A^{1/2} X B^{1/2} = U(\lambda_i^{1/2}\mu_j^{1/2} y_{ij}) V^*$$

and $A^v X B^{1-v} = U\left(\lambda_i^v \mu_j^{1-v} y_{ij}\right) V^*$. If $0 \leqslant v \leqslant \frac{1}{2}$, by inequality (1.1), we have

$$\begin{aligned}\|vAX + (1-v)\, XB\|_2^2 &= \sum_{i,j=1}^{n} \left(v\lambda_i + (1-v)\,\mu_j\right)^2 |y_{ij}|^2 \\ &= \sum_{i,j=1}^{n} \left(v^2\lambda_i^2 + (1-v)^2\mu_j^2 + 2v(1-v)\,\lambda_i\mu_j\right)|y_{ij}|^2 \\ &\geqslant v^2 \sum_{i,j=1}^{n} \left(\lambda_i - \mu_j\right)^2 |y_{ij}|^2 + v^{2v} \sum_{i,j=1}^{n} \left(\lambda_i^v \mu_j^{1-v}\right)^2 |y_{ij}|^2 \\ &\quad + \sum_{i,j=1}^{n} 2v(1-v)\,\lambda_i\mu_j\, |y_{ij}|^2 \\ &\geqslant v^2 \|AX - XB\|_2^2 + v^{2v}\|A^v X B^{1-v}\|_2^2 + 2v(1-v)\,\|A^{1/2} X B^{1/2}\|_2^2.\end{aligned}$$

If $\frac{1}{2} \leqslant v \leqslant 1$, then by the inequality (1.2) and the same method above, we have the inequality (2.9). This completes the proof.

Remark 2.4 The inequality (2.8) is related to the inequality (2.1). It should be noticed that neither (2.1) nor (2.8) is uniformly better than the other.

Theorem 2.5 Let $A, B \in M_n$ be positive definite. If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$\begin{aligned}\det\left(vA + (1-v)\, B\right)^2 \geqslant v^{2nv} \det\left(A^v B^{1-v}\right)^2 + v^{2n} \det\left(A - B\right)^2 \\ + \left(2v(1-v)\right)^n \det B^{1/2} A B^{1/2}.\end{aligned} \tag{2.10}$$

If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$\begin{aligned}\det\left(vA + (1-v)\, B\right)^2 \geqslant (1-v)^{2n(1-v)} \det\left(A^v B^{1-v}\right)^2 + (1-v)^{2n} \det\left(A - B\right)^2 \\ + \left(2v(1-v)\right)^n \det B^{1/2} A B^{1/2}.\end{aligned} \tag{2.11}$$

Proof To prove the determinant inequality, note that by the inequality (1.1), we have

$$v^{2v}[s_j^v(B^{-1/2}AB^{-1/2})]^2 + v^2(s_j(B^{-1/2}AB^{-1/2}) - 1)^2 \leqslant v^2 s_j^2(B^{-1/2}AB^{-1/2}) + (1-v)^2$$

for $j = 1, \cdots, n$. Therefore,

$$\det(vB^{-1/2}AB^{-1/2} + (1-v)I)^2 = \prod_{j=1}^{n} (v s_j(B^{-1/2}AB^{-1/2}) + 1 - v)^2$$
$$= \prod_{j=1}^{n} (v^2 s_j^2(B^{-1/2}AB^{-1/2}) + (1-v)^2 + 2v(1-v) s_j(B^{-1/2}AB^{-1/2}))$$

$$\geqslant \prod_{j=1}^{n}(v^{2v}s_j^{2v}(B^{-1/2}AB^{-1/2})+v^2(s_j(B^{-1/2}AB^{-1/2})-1)^2+2v(1-v)s_j(B^{-1/2}AB^{-1/2}))$$

$$\geqslant v^{2nv}\prod_{j=1}^{n}s_j^{2v}(B^{-1/2}AB^{-1/2})+v^{2n}\prod_{j=1}^{n}(s_j(B^{-1/2}AB^{-1/2})-1)^2$$

$$+(2v(1-v))^n\prod_{j=1}^{n}s_j(B^{-1/2}AB^{-1/2})$$

$$=v^{2nv}\det(B^{-1/2}AB^{-1/2})^{2v}+v^{2n}\det(B^{-1/2}AB^{-1/2}-I)^2+(2v(1-v))^n\det B^{-1/2}AB^{-1/2}.$$

Thus, we have

$$\det(vA+(1-v)B)^2\geqslant v^{2nv}\det\left(A^vB^{1-v}\right)^2+v^{2n}\det(A-B)^2+(2v(1-v))^n\det B^{1/2}AB^{1/2}.$$

If $\frac{1}{2}\leqslant v\leqslant 1$, then by the inequality (1.2) and the same method above, we have the inequality (2.11). This completes the proof.

Remark 2.6 If $A,\ B\in M_n$ are positive definite, a determinant version of the arithmetic-geometric mean inequality is known[11]:

$$\det\left(\frac{A+B}{2}\right)^2\geqslant\det(AB). \tag{2.12}$$

Obviously, the inequality (2.10) or (2.11) is a generalization of the inequality (2.12).

[References]

[1] BHATIA R, KITTANEH F. On singular values of a product of operators [J]. SIAM J Matrix Anal Appl, 1990, 11: 271-277.

[2] KITTANEH F. On some operator inequalities [J]. Linear Algebra Appl, 1994, 208/209: 19-28.

[3] ANDO T. Matrix Young inequality [J]. Oper Theory Adv Appl, 1995, 75: 33-38.

[4] KOSAKI H. Arithmetic-geometric mean and related inequalities for operators [J]. J Funct Anal, 1998, 156: 429-451.

[5] ZHAN X. Inequalities for unitarily invariant norms [J]. SIAM J Matrix Anal Appl, 1998, 20: 466-470.

[6] BHATIA R, PARTHASARATHY K R. Positive definite functions and operator inequalities [J]. Bull London Math Soc, 2000, 32: 214-228.

[7] HIRZALLAH O, KITTANEH F. Matrix Young inequalities for the Hilbert-Schmidt norm [J]. Linear Algebra Appl, 2000, 308: 77–84.

[8] BHATIA R, KITTANEH F. Notes on matrix arithmetic–geometric mean inequalities [J]. Linear Algebra Appl, 2000, 308: 203-211.

[9] KITTANEH F, MANASRAH Y. Improved Young and Heinz inequalities for matrices [J]. J Math Anal Appl, 2010, 361: 262-269.

[10] ZHAN X. Matrix theory [M]. Beijing: Higher Education Press, 2008. (in Chinese)

[11] HORN R A, JOHNSON C R. Matrix analysis [M]. Cambridge: Cambridge University Press, 1985: 467.