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Pricing option with transaction costs under the subdiffusive Black-Scholes model

GU Hui[1], ZHANG Yun-xiu[1,2]

(1. *Department of Mathematics, East China Normal University, Shanghai* 200241, *China*;
2. *Department of Mathematics, Nanjing Forest University, Nanjing* 210037, *China*)

Abstract: This paper dealt with the problem of discrete time option pricing by the subdiffusive Black-Scholes model with transaction costs. A subdiffusive geometric Brownian motion was introduced as the model of underlying asset prices exhibiting subdiffusive dynamics. In the presence of transaction costs, by a mean self-financing delta-hedging argument in a discrete time setting, a pricing formula for the European call option in discrete time setting was obtained.

Key words: option pricing; transaction costs; subdiffusive dynamics
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次扩散BS模型下带交易费的期权定价

顾 惠[1], 张云秀[1,2]

(1. 华东师范大学 数学系, 上海 200241;
2. 南京林业大学 应用数学系, 南京 210037)

摘要: 研究次扩散 BS 模型下的离散带交易费的期权定价问题. 引入作为标的股票价格的次扩散几何布朗运动. 在存在交易费的情况下, 利用离散时间平均自融资 delta 对冲策略得到欧式看涨期权的定价公式.

关键词: 期权定价; 交易费; 次扩散动力学

0 Introduction

A classic and still most popular model of market is the Black-Scholes(BS) model, presented in 1973 by F. Black and M. Scholes[1]. The model is based on a diffusion process called geometric Brownian motion(GBM), which assumes the price of the underlying asset $\bar{S}(t)$ satisfies

$$\bar{S}(t) = S_0 \exp\{\mu t + \sigma B(t)\},\ S_0 > 0, \tag{0.1}$$

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第一作者: 顾惠, 男, 博士研究生, 研究方向为分形几何、金融数学. E-mail: ghui314@163.com.

or the equivalent form of stochastic differential equation

$$\mathrm{d}\bar{S}(t) = \left(\mu + \frac{\sigma^2}{2}\right)\bar{S}(t)\,\mathrm{d}t + \sigma\bar{S}(t)\,\mathrm{d}B(t), \quad \bar{S}(0) = S_0 > 0,$$

with constant drift μ and volatility σ, here $B(t)$ being the standard Brownian motion.

Empirical research shows that GBM cannot describe many characteristic features of markets, such as: long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, periods of constant values, etc. Therefore, there have been a number of generalizations of the BS model. Marcin Magdziarz[2] introduced a so-called subdiffusive BS model to describe properly financial data exhibiting periods of constant values. He proved that the subdiffusive BS model is arbitrage free but incomplete, and obtained the corresponding price formula of European options.

In this paper we deals with the problem of discrete time option pricing by the subdiffusive BS model with transaction costs. The paper is organized as follows: in Section 1, we give some properties of subdiffusive BS model and in Section 2, we compute the formula for the European call option.

1 Subdiffusive Black-Scholes model

1.1 α-stable subordinator and its inverse

Definition 1 For $\alpha \in (0,1)$, the α-stable subordinator $\{U_\alpha(\tau)\}_{\tau\geqslant 0}$ is a Lévy process with nonnegative increments and the Laplace transform: $\mathbb{E}\mathrm{e}^{-uU_\alpha(\tau)} = \mathrm{e}^{-\tau u^\alpha}$; The inverse α-stable subordinator $\{T_\alpha(t)\}_{t\geqslant 0}$ is the first-passage time process: $T_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}$.

$T_\alpha(t)$ is of course non-decreasing and since $U_\alpha(\tau)$ is a pure-jump process with càdlàg trajectories, the sample paths of $T_\alpha(t)$ are continuous. Additionally, every jump of $U_\alpha(\tau)$ corresponds to a flat period of $T_\alpha(t)$. To see more about the the α-stable subordinator and its inverse, please refer [3–5].

Proposition 2 (1) The α-stable subordinator $U_\alpha(\tau)$ is $1/\alpha$-self-similar. That is, for every $c > 0$, $U_\alpha(c\tau) \overset{d}{=} c^{\frac{1}{\alpha}}U_\alpha(\tau)$, where "$\overset{d}{=}$" denotes "equality of all finite dimensional distributions". Correspondingly, the inverse process $T_\alpha(t)$ is α-self-similar.

(2) For any $n \in \mathbf{N}, \lambda \in \mathbf{R}$, $\mathbb{E}(T_\alpha^n(t)) = \frac{n!t^{n\alpha}}{\Gamma(n\alpha+1)}$ and $\mathbb{E}\mathrm{e}^{\lambda T_\alpha(t)} = E_\alpha(\lambda t^\alpha) < +\infty$, where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

(3) For any $n \in \mathbf{N}$, $0 \leqslant s < t < +\infty$, $\mathbb{E}\left(|T_\alpha(t) - T_\alpha(s)|^n\right) = n!\int_A \prod_{j=1}^n U(\,\mathrm{d}x_j - x_{j-1})$, where $A = \{(x_0, x_1, \cdots x_n) : x_0 = 0, s < x_1 < x_2 < \cdots < x_n \leqslant t\}$ and $U(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}$.

Proof (1), (2) are the results in [6, 7], and (3) can be immediately obtained from Proposition 1 in [8].

1.2 Subdiffusion process

$T_\alpha(t)$ is the inverse α-stable subordinator defined in Definition 1., $\{B(t)\}_{t\geqslant 0}$ is the standard Brownian motion assumed to be independent of both $\{T_\alpha(t)\}$ and $\{U_\alpha(\tau)\}$. As a result, the compound process $Z_\alpha(t) = B(T_\alpha(t))$ is $\alpha/2$-self-similar and $\mathbb{E}(Z_\alpha(t)) = 0$, $\mathbb{E}(Z_\alpha^2(t)) = \frac{t^\alpha}{\Gamma(\alpha+1)}$.

For $\beta > 0$, a random function $X(x)$ is said to be $o(x^\beta)$ if $\lim\limits_{x\searrow 0} \frac{\mathbb{E}(|X(x)|^n)}{x^{n\beta}} = 0$ for every $n \in \mathbf{N}$.

Proposition 3　If $0 < \beta, \beta_1, \beta_2 < +\infty$, $n \in \mathbf{N}$, then

$$o(x^{\beta_1}) \cdot o(x^{\beta_2}) = o(x^{\beta_1+\beta_2}), \text{ in particular } \left(o(x^\beta)\right)^n = o(x^{n\beta}),$$
$$o(x^{\beta_1}) + o(x^{\beta_2}) = o(x^{\min\{\beta_1,\beta_2\}}).$$

Lemma 4　For $n \in \mathbf{N}, 0 \leqslant s < t < +\infty$, there exists positive numbers a_n and b_n such that

$$\mathbb{E}\left(|T_\alpha(t) - T_\alpha(s)|^n\right) \leqslant a_n (t-s)^{n\alpha} \textit{ and } \mathbb{E}\left(|Z_\alpha(t) - Z_\alpha(s)|^n\right) \leqslant b_n (t-s)^{\frac{n\alpha}{2}}.$$

Proof　It follows from Proposition 2 (3) that

$$\begin{aligned}
&\mathbb{E}\left(|T_\alpha(t) - T_\alpha(s)|^n\right)\\
=\,& n! \int_s^t U'(x_1) \int_{x_1}^t U'(x_2 - x_1) \cdots \int_{x_{n-1}}^t U'(x_n - x_{n-1})\, \mathrm{d}x_n \cdots \mathrm{d}x_2\, \mathrm{d}x_1\\
\leqslant\,& n! \int_s^t U'(x_1) \int_{x_1}^{t-s+x_1} U'(x_2 - x_1) \cdots \int_{x_{n-1}}^{t-s+x_{n-1}} U'(x_n - x_{n-1})\, \mathrm{d}x_n \cdots \mathrm{d}x_2\, \mathrm{d}x_1\\
=\,& n! \frac{(t-s)^{(n-1)\alpha}(t^\alpha - s^\alpha)}{\Gamma^n(\alpha+1)}\\
\leqslant\,& \frac{n!}{\Gamma^n(\alpha+1)}(t-s)^{n\alpha}.
\end{aligned}$$

Since $T_\alpha(t)$ is non-decreasing, $B(\tau)$ is 1/2-self-similar with stationary increments and the processes are independent, we get that

$$\mathbb{E}\left(|Z_\alpha(t) - Z_\alpha(s)|^{2n}\right) = \mathbb{E}\left(B^{2n}(1)\right) \mathbb{E}\left(|T_\alpha(t) - T_\alpha(s)|^n\right) \leqslant c_n (t-s)^{n\alpha}$$

and so

$$\mathbb{E}\left(|Z_\alpha(t) - Z_\alpha(s)|^n\right) \leqslant \left[\mathbb{E}\left(|Z_\alpha(t) - Z_\alpha(s)|^{2n}\right)\right]^{\frac{1}{2}} \leqslant c_n^{\frac{1}{2}} (t-s)^{\frac{n\alpha}{2}} = b_n (t-s)^{\frac{n\alpha}{2}},$$

where $b_n = c_n^{\frac{1}{2}} = \left(\frac{(2n)!}{2^n \Gamma^n(\alpha+1)}\right)^{\frac{1}{2}}$.

Corollary 5　$\Delta T_\alpha(t) = o(\Delta t^{\alpha-\varepsilon})$, $\Delta Z_\alpha(t) = o(\Delta t^{\frac{\alpha}{2}-\varepsilon})$ for arbitrary $\varepsilon \in (0, \frac{\alpha}{2})$.

Lemma 6　For $\lambda \in \mathbf{R}, t \geqslant 0, n \in \mathbf{N}$, $\mathbb{E}\left(\mathrm{e}^{n\lambda|Z_\alpha(t)|}\right) < +\infty$.

Proof

$$\begin{aligned}
\mathbb{E}\left(|Z_\alpha(t)|^k\right) &\leqslant \left[\mathbb{E}\left(|Z_\alpha(t)|^{2k}\right)\right]^{\frac{1}{2}} = \left[\mathbb{E}\left(|B(1)|^{2k}\right) \cdot \mathbb{E}\left(|T_\alpha(t)|^k\right)\right]^{\frac{1}{2}} = \left[\frac{t^{k\alpha}(2k)!}{2^k \Gamma(k\alpha+1)}\right]^{\frac{1}{2}}\\
&= \left(\frac{t^\alpha}{2}\right)^{\frac{k}{2}} \left[\frac{(2k)!}{\Gamma(k\alpha+1)}\right]^{\frac{1}{2}}, \quad k \in \mathbf{N} \cup \{0\}.
\end{aligned}$$

Denote $Y(t) = \mathrm{e}^{\lambda|Z_\alpha(t)|}$, then

$$\begin{aligned}
\mathbb{E}(Y^n(t)) = \mathbb{E}\mathrm{e}^{n\lambda|Z_\alpha(t)|} &= \sum_{k=0}^\infty \frac{(n\lambda)^k}{k!} \mathbb{E}\left(|Z_\alpha(t)|^k\right)\\
&\leqslant \sum_{k=0}^\infty \left(n\lambda\sqrt{\frac{t^\alpha}{2}}\right)^k \frac{1}{k!}\sqrt{\frac{(2k)!}{\Gamma(k\alpha+1)}} =: \sum_{k=0}^\infty a_k,
\end{aligned}$$

since

$$\frac{a_{k+1}}{a_k} = n\lambda\sqrt{\frac{t^\alpha}{2}}\frac{\sqrt{(2k+2)(2k+1)}}{k+1}\sqrt{\frac{\mathrm{B}(\alpha, k\alpha+1)}{\Gamma(\alpha)}} \to 0 \quad (k \to \infty),$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function. This means $\mathbb{E}(Y^n(t)) < +\infty$. Moreover, from the proof we can see that $\mathbb{E}\,(Y(t)^n)$ are uniformly bounded on any compact subset of $[0, +\infty)$.

Replacing the time t of (0.1) by the time-change process $T_\alpha(t)$, we get a subdiffusion process named time-changed GBM, and that is

$$S_t = \bar{S}(T_\alpha(t)) = S_0 \exp(\mu T_\alpha(t) + \sigma Z_\alpha(t)), \quad S_0 > 0. \tag{1.1}$$

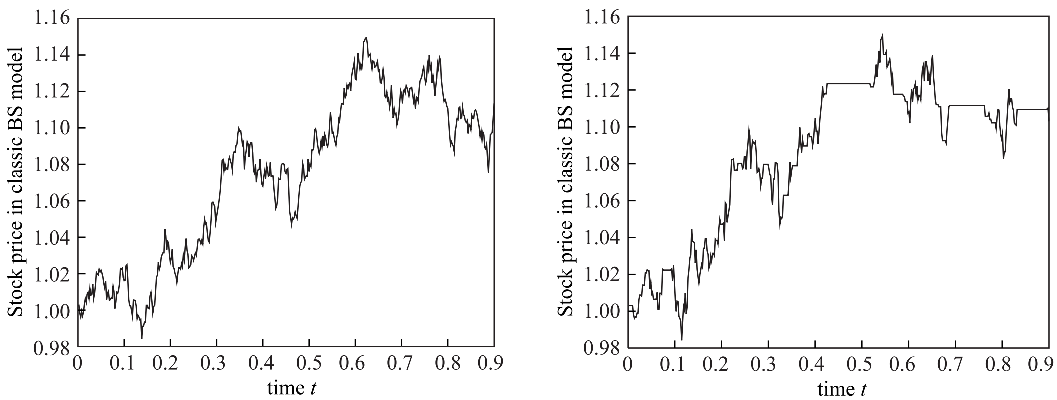

Fig. 1 The left figure is a typical trajectory of the stock price $\bar{S}(t)$ in the classic BS model and the right a trajectory of the stock price S_t defined by (1.1) for $\mu = 0.05, \sigma = 0.1, S_0 = 1$, and $\alpha = 0.8$.

1.3 Model and market assumptions

We follow the other usual assumptions used in the classic BS model but with the following exceptions:

(1) The price S_t of the underlying stock at time t is given by (1.1), i.e. $S_t = S_0 \exp\left(\mu T_\alpha(t) + \sigma Z_\alpha(t)\right)$, with $\mu, \sigma, S_0 > 0$ and $\alpha \in (\frac{2}{3}, 1)$.

(2) Trading takes place only at discrete time points $\Delta t,\ 2\Delta t,\ 3\Delta t \cdots, T$, where $\Delta t > 0$ is a fixed small time step.

(3) The transaction costs are proportional to the value of the transaction in the underlying stock. If D shares of the underlying stock are bought ($D > 0$) or sold ($D < 0$) at the price S_t, the transaction cost is given by $\frac{k}{2}|D|S_t$, where k is a positive constant.

(4) The portfolio P_t consists of $D(t)$ units of the underlying stock and riskless bonds with value $Q(t)$, i.e. $P_t = D(t)S_t + Q(t)$. Since there is no portfolio that replicates the European call option in a market with propositional transaction costs[9], we only require the hedging portfolio to replicate the value of option at each trading time point.

(5) The traders' behavior is assumed to be bounded rational, their decisions can be explained both by their reaction to the past stock price, according to a standard speculative behavior, and by imitation of other traders' past decisions. It is well known that the delta-hedging strategy plays a central role in the theory of option pricing and that it is popularly used

on the trading floor. The traders are assumed to follow, anchor, and imitate the Black-Scholes delta-hedging strategy to price an option.

2　Pricing formula under transaction costs

In this section we derive a discrete-time pricing formula for the European call option under the above assumptions.

Let $C = C(t, S_t)$ be the value of a European call option on the underlying stock S_t at time t. K is the strike price, T is the maturity time and r is the continuous rate of return of a riskless bond. We obtain the main result as follows.

Theorem 7　At each trading time point t, the value of the European call option $C = C(t, S_t)$ satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\tilde{\sigma}(t)^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0$$

with the boundary condition $C(T, S_T) = \max(S_T - K, 0)$. And the value of the option is

$$C(t, S_t) = S_t N(d_1) - K\mathrm{e}^{-r(T-t)} N(d_2), \tag{2.1}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\tilde{\sigma}(t)^2}{2}\right)(T-t)}{\tilde{\sigma}(t)\sqrt{T-t}}, \quad d_2 = d_1 - \tilde{\sigma}(t)\sqrt{T-t}, \tag{2.2}$$

$$\tilde{\sigma}^2(t) = \sigma^2\left[M_\alpha^2(t, \Delta t) + kM_\alpha^1(t, \Delta t)\sigma^{-1}\right]\Delta t^{-1}, \tag{2.3}$$

$M_\alpha^1(t, \Delta t)$ and $M_\alpha^2(t, \Delta t)$ denote respectively the first and second moments of the random variable $|\Delta Z_\alpha(t)|$, and $N(\cdot)$ is the cumulative normal density function.

Proof　Let $\varepsilon > 0$ be an arbitrary and sufficient small number. The difference of S_t over time interval $[t, t + \Delta t)$ of length Δt is

$$\begin{aligned}\Delta S_t &= S_{t+\Delta t} - S_t = S_t\left(\mathrm{e}^{\mu\Delta T_\alpha(t)+\sigma\Delta Z_\alpha(t)} - 1\right)\\ &= S_t\left(\mu\Delta T_\alpha(t) + \sigma\Delta Z_\alpha(t) + \frac{1}{2}\left[\mu\Delta T_\alpha(t) + \sigma\Delta Z_\alpha(t)\right]^2\right)\\ &\quad + \frac{1}{6}S_t\mathrm{e}^{\theta[\mu\Delta T_\alpha(t)+\sigma\Delta Z_\alpha(t)]}\left[\mu\Delta T_\alpha(t) + \sigma\Delta Z_\alpha(t)\right]^3,\end{aligned}$$

where $\theta = \theta(t, \Delta t) \in (0, 1)$ is a random variable corresponding to the process S.

$\frac{1}{6}\mathrm{e}^{\theta[\mu\Delta T_\alpha(t)+\sigma\Delta Z_\alpha(t)]} \leqslant \frac{1}{6}\mathrm{e}^{\mu T_\alpha(T)}\mathrm{e}^{\sigma|Z_\alpha(t)|}\mathrm{e}^{\sigma|Z_\alpha(t+\Delta t)|}$. It follows from Proposition 2(2), 3, Corollary 5 and Lemma 6 that $\Delta t^{2\varepsilon}\frac{1}{6}\mathrm{e}^{\theta[\mu\Delta T_\alpha(t)+\sigma\Delta Z_\alpha(t)]} = o(\Delta t^{\varepsilon})$ and

$$\frac{1}{6}\mathrm{e}^{\theta[\mu\Delta T_\alpha(t)+\sigma\Delta Z_\alpha(t)]}\left[\mu\Delta T_\alpha(t) + \sigma\Delta Z_\alpha(t)\right]^3 = o(\Delta t^{\frac{3}{2}\alpha-\varepsilon}).$$

We have

$$\frac{\Delta S_t}{S_t} = \mu\Delta T_\alpha(t) + \sigma\Delta Z_\alpha(t) + \frac{1}{2}\sigma^2\left(\Delta Z_\alpha(t)\right)^2 + o(\Delta t^{\frac{3}{2}\alpha-\varepsilon}). \tag{2.4}$$

Applying the Taylor series expansion to $C(t, S_t)$ and using (2.4) we obtain that

$$
\begin{aligned}
\Delta C(t,S_t) &= \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S_t}\Delta S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\Delta S_t^2 + o(\Delta t^{\frac{3}{2}\alpha-\varepsilon}) \\
&= \frac{\partial C}{\partial t}\Delta t + \mu S_t \frac{\partial C}{\partial S_t}\Delta T_\alpha(t) + \sigma S_t \frac{\partial C}{\partial S_t}\Delta Z_\alpha(t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}\left(\Delta Z_\alpha(t)\right)^2 \\
&\quad + \frac{1}{2}\sigma^2 S_t \frac{\partial C}{\partial S_t}\left(\Delta Z_\alpha(t)\right)^2 + o(\Delta t^{\frac{3}{2}\alpha-\varepsilon}), \\
\Delta\left(\frac{\partial C}{\partial S_t}\right) &= \frac{\partial^2 C}{\partial S_t \partial t}\Delta t + \frac{\partial^2 C}{\partial S_t^2}\Delta S_t + \frac{1}{2}\frac{\partial^3 C}{\partial S_t^3}\Delta S_t^2 + o(\Delta t^{\frac{3}{2}\alpha-\varepsilon}),
\end{aligned}
$$

From (2.1) to (2.3) we can check that $\frac{\partial^2 C}{\partial S_t^2}$, $\frac{\partial^3 C}{\partial S_t^3}$, $\frac{\partial^2 C}{\partial S_t \partial t}$ is $o(\Delta t^{\frac{1}{2}-\frac{\alpha}{4}-\varepsilon})$, thus

$$
\left|\Delta\left(\frac{\partial C}{\partial S_t}\right)\right| S_{t+\Delta t} = \sigma S_t^2 \left|\frac{\partial^2 C}{\partial S_t^2}\right| |\Delta Z_\alpha(t)| + o(\Delta t). \tag{2.5}
$$

Moreover, by Assumptions (2) and (3) in Subsection 1.3, the change of the value to the portfolio $P_t = D_t S_t + Q_t$ over the time interval $(t, t+\Delta t]$ is

$$
\Delta P_t = D_t \Delta S_t + \Delta Q_t - \frac{k}{2}|\Delta D_t| S_{t+\Delta t} = D_t \Delta S_t + r Q_t \Delta t - \frac{k}{2}|\Delta D_t| S_{t+\Delta t} + o(\Delta t), \tag{2.6}
$$

as the number of shares D_t is held fixed during $[t, t+\Delta t)$.

By Assumption (4), $C(t, S_t)$ is replicated by the portfolio $P(t)$ and values of the option equal values of the replicating portfolio at time point Δt, $2\Delta t$, $3\Delta t, \cdots$ That is, $C(t, S_t) = D_t S_t + Q_t$. Taking $D_t = \frac{\partial C}{\partial S_t}$ and using (2.4)–(2.6) we obtain

$$
\begin{aligned}
\Delta P_t &= \frac{\partial C}{\partial S_t}\left(\mu S_t \Delta T_\alpha(t) + \sigma S_t \Delta Z_\alpha(t) + \frac{1}{2}\sigma^2 S_t \left(\Delta Z_\alpha(t)\right)^2\right) + r Q_t \Delta t \\
&\quad - \frac{k}{2}\left|\Delta\left(\frac{\partial C}{\partial S_t}\right)\right| S_{t+\Delta t} + o(\Delta t) \\
&= \frac{\partial C}{\partial S_t}\left(\mu S_t \Delta T_\alpha(t) + \sigma S_t \Delta Z_\alpha(t) + \frac{1}{2}\sigma^2 S_t \left(\Delta Z_\alpha(t)\right)^2\right) + r\left(C(t,S_t) - \frac{\partial C}{\partial S_t}S_t\right)\Delta t \\
&\quad - \frac{k}{2}\sigma S_t^2 \left|\frac{\partial^2 C}{\partial S_t^2}\right| |\Delta Z_\alpha(t)| + o(\Delta t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta P - \Delta C = \left(rC - rS_t\frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t}\right)\Delta t - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}\left(\Delta Z_\alpha(t)\right)^2 \\
- \frac{k}{2}\sigma S_t^2 \left|\frac{\partial^2 C}{\partial S_t^2}\right| |\Delta Z_\alpha(t)| + o(\Delta t).
\end{aligned} \tag{2.7}
$$

It follows (2.7) and Assumptions (4), (5) in Subsection 1.3 that

$$\begin{aligned}\mathbb{E}(\Delta P-\Delta C)=&\left(rC-rS_t\frac{\partial C}{\partial S_t}-\frac{\partial C}{\partial t}\right)\Delta t-\frac{M_\alpha^2(t,\Delta t)}{2}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}\\&-\frac{kM_\alpha^1(t,\Delta t)}{2}\sigma S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right|+o(\Delta t)\\=&\left(rC-rS_t\frac{\partial C}{\partial S_t}-\frac{\partial C}{\partial t}-\frac{M_\alpha^2(t,\Delta t)}{2\Delta t}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}-\frac{kM_\alpha^1(t,\Delta t)}{2\Delta t}\sigma S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right|\right)\Delta t\\&+o(\Delta t)=0,\end{aligned}\tag{2.8}$$

where $M_\alpha^1(t,\Delta t)$ and $M_\alpha^2(t,\Delta t)$ denote the first and second moments of the random variable $|\Delta Z_\alpha(t)|$, respectively, i.e.

$$\begin{aligned}M_\alpha^1(t,\Delta t)&=\mathbb{E}\left(|B(1)|\right)\mathbb{E}\left(\Delta T_\alpha^{\frac{1}{2}}(t)\right)=\sqrt{\frac{2}{\pi}}\mathbb{E}\left(\Delta T_\alpha^{\frac{1}{2}}(t)\right),\\M_\alpha^2(t,\Delta t)&=\mathbb{E}\left(B^2(1)\right)\mathbb{E}\left(\Delta T_\alpha(t)\right)=\frac{(t+\Delta t)^\alpha-t^\alpha}{\Gamma(\alpha+1)}\end{aligned}$$

by using independence of $B(\tau)$ and $T_\alpha(t)$ and Proposition 2(2). To get numerical approximation of $\mathbb{E}\left(\Delta T_\alpha^{\frac{1}{2}}(t)\right)$, one can see [7] for details. Thus, it follows from (2.8) that

$$rC=rS_t\frac{\partial C}{\partial S_t}+\frac{\partial C}{\partial t}+\frac{M_\alpha^2(t,\Delta t)}{2\Delta t}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}+\frac{kM_\alpha^1(t,\Delta t)}{2\Delta t}\sigma S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right|.\tag{2.9}$$

Denote $\left[M_\alpha^2(t,\Delta t)+kM_\alpha^1(t,\Delta t)\sigma^{-1}\mathrm{sign}\left(\frac{\partial^2 C}{\partial S_t^2}\right)\right]^{\frac{1}{2}}\Delta t^{-\frac{1}{2}}\sigma$ by $\tilde\sigma(t)$. It is known that $\frac{\partial^2 C}{\partial S_t^2}$ is always positive for the simple European call option in the absence of transaction costs, if we postulate the same behavior of $\frac{\partial^2 C}{\partial S_t^2}$ here, then

$$\tilde\sigma(t)^2=\sigma^2\left[M_\alpha^2(t,\Delta t)+kM_\alpha^1(t,\Delta t)\sigma^{-1}\right]\Delta t^{-1}\tag{2.10}$$

Therefore, it follows from (2.9) and (2.10) that

$$\frac{\partial C}{\partial t}+rS_t\frac{\partial C}{\partial S_t}+\frac{1}{2}\tilde\sigma(t)^2S_t^2\frac{\partial^2 C}{\partial S_t^2}-rC=0$$

and so

$$C(t,S_t)=S_tN(d_1)-K\mathrm{e}^{-r(T-t)}N(d_2),$$

where $N(\cdot)$ is the cumulative normal density function,

$$d_1=\frac{\ln(\frac{S_t}{K})+(r+\frac{\tilde\sigma(t)^2}{2})(T-t)}{\tilde\sigma(t)\sqrt{T-t}}\quad\text{and}\quad d_2=d_1-\tilde\sigma(t)\sqrt{T-t}.$$

Remark 8 Make $\alpha\nearrow 1$, then $T_\alpha(t)$ and $Z_\alpha(t)$ degenerate to t and the standard Brownian motion $B(t)$ with stationary increments. So $M_\alpha^1(t,\Delta t)\to\sqrt{\frac{2\Delta t}{\pi}}, M_\alpha^2(t,\Delta t)\to\Delta t$ and $\tilde\sigma(t)^2\to\sigma^2\left(1+\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{\Delta t}}\right)$, this is the result in Leland[9].

Remark 9 In order to apply the subdiffusive Black-Scholes model to real market data, it is crucial to give parameters estimation procedures. One can refer [10] to see details for the estimation of the parameter α.

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