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Multiple solutions for $p(x)$-Laplacian problems in R^N

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Abstract: By using the fountain theorem and the dual fountain theorem, respectively, the existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbf{R}^N$ were studied, assumed that one of the perturbation terms $f_1(x,u)$, $f_2(x,u)$ is superlinear and satisfies the Ambrosetti-Rabinowitz type condition and the other one is sublinear. The discussion was based on variable exponent Lebesgue and Sobolev spaces.

Key words: variable exponent Sobolev spaces; $p(x)$-Laplacian; $(\mathrm{PS})_c^*$ condition; fountain theorem; dual fountain theorem

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R^N 上的 $p(x)$-Laplace 问题的多解性

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摘要: 在扰动项 $f_1(x,u)$, $f_2(x,u)$ 中, 其中一项是超线性并且满足 Ambrosetti-Rabinowitz 条件, 另一项为次线性的情形下, 分别利用“喷泉定理”和“对偶喷泉定理” 研究了无界区域 $\mathbf{R}^N$ 上的 $p(x)$-Laplace 方程解的存在性和多解性问题. 此问题是基于变指数 Lebesgue 和 Sobolev 空间进行讨论的.

关键词: 变指数 Sobolev 空间; $p(x)$-Laplacian; $(\mathrm{PS})_c^*$ 条件; 喷泉定理; 对偶喷泉定理

0 Introduction

In the recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics and calculus of variations. For more information on modelling physical phenomena by equations

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involving $p(x)$-growth condition we refer to [1–3]. The appearance of such physical models was facilitated by the development of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in literature as early as 1931 in an article by Orlicz[4]. The spaces $L^{p(x)}$ are special cases of Orlicz spaces L^{φ} originated by Nakano[5] and developed by Musielak and Orlicz[6,7], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)\mathrm{d}x < \infty$ for a suitable φ. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the studies of Tsenov[8], Sharapudinov[9] and Zhikov[10,11].

Recently, Fan and Han[12], by using the fountain theorem and the dual fountain theorem, respectively, studied the existence of solutions for $p(x)$-Laplacian equations in $\mathbf{R}^N$, assumed that perturbation terms $f_1(x,u), f_2(x,u)$ satisfy concave-convex nonlinearity and obtained the existence and multiplicity of solutions. More recently, applying the mountain pass theorem, Fu[13] constrained the nonlinear term $f(x,u)$ in superlinear case and obtained the existence of solutions for $p(x)$-Laplacian problem in an exterior domain Ω in $\mathbf{R}^N$ with the Dirichlet boundary condition. Fu and Zhang[14] got at least two non-trivial weak solutions for the $p(x)$-Laplacian problem in $\mathbf{R}^N$ via global minimizer method and the mountain pass theorem, assumed that the nonlinear term $f(x,u)$ in sublinear case. Fu and Zhang[15], with the aid of the symmetric mountain pass theorem, got the multiplicity of solutions when one of the perturbation terms $f_1(x,u), f_2(x,u)$ is superlinear and satisfies the Ambrosetti-Rabinowitz type condition and the other one is sublinear. Motivated by their works, the aim of this paper is to discuss the existence and multiplicity of the following $p(x)$-Laplacian equation in $\mathbf{R}^N$

$$-\Delta_{p(x)}u + |u|^{p(x)-2}u = f_1(x,u) + f_2(x,u), \quad x \in \mathbf{R}^N, \quad u \in W^{1,p(x)}(\mathbf{R}^N), \tag{0.1}$$

where $N \geqslant 2$, $p : \mathbf{R}^N \to \mathbf{R}$ is Lipschitz continuous, $1 < p^- \leqslant p^+ < N$, $f_i : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ satisfies Carathedory conditions $(i = 1, 2)$. Our goal will be to obtain the existence and multiplicity of solutions for (0.1) in the generalized Sobolev spaces $W^{1,p(x)}(\mathbf{R}^N)$ by using the fountain theorem and the dual fountain theorem, respectively, under $f_1(x,u)$ and $f_2(x,u)$ satisfy appropriate conditions. These results extend some of the results in [12–15]. We point out the presence in (0.1) of the $p(x)$-Lapalce operator. This is a natural extension of the p-Laplace operator. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than p-Laplace operator, for example it is inhomogeneous. For related results involving the Laplace operator, see [16, 17].

1 Preliminary

Let Ω be an open domain of $\mathbf{R}^N$, denote:

$$L_+^{\infty}(\Omega) = \{p \in L^{\infty}(\Omega) : \operatorname*{ess\,inf}_{x\in\Omega} p(x) \geqslant 1\}.$$

For $p \in L_+^{\infty}(\Omega)$, let

$$p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x), \quad p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

On the basic properties of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, we refer to [18–22]. In the following, we display some facts which we will use later.

Denoted by $P(\Omega)$ the set of all measurable real functions defined on Ω, elements in $P(\Omega)$ which are equal to each other almost everywhere are considered as one element.

For $p \in L^\infty_+(\Omega)$, define

$$L^{p(x)}(\Omega) = \{u \in P(\Omega) : \int_\Omega |u|^{p(x)}\,\mathrm{d}x < \infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,\mathrm{d}x \leqslant 1\Big\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Denoted by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

In this paper we will use the following equivalent norm on $W^{1,p(x)}(\Omega)$:

$$\|u\| = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} + \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}.$$

Proposition 1.1[23,24] The space $L^{p(x)}(\Omega)$ is a separable, uniformly convex Banach space, and has conjugate space $L^{q(x)}(\Omega)$, where $1/q(x) + 1/p(x) = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \leqslant \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)} \leqslant 2|u|_{p(x)}|v|_{q(x)}.$$

Proposition 1.2[12] The spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ all are separable and reflexive Banach spaces.

Proposition 1.3[12,23,25] Denote

$$\rho(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x, \quad \forall\, u \in L^{p(x)}(\Omega).$$

Then

(1) For $u \neq 0, |u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(3) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \geqslant \rho(u) \geqslant |u|_{p(x)}^{p^+}$;

(4) $|u_k|_{p(x)} \to 0 \Leftrightarrow \rho(u_k) \to 0$; $|u_k|_{p(x)} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

Proposition 1.4[12,23,25] Denote

$$I(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x, \quad \forall\, u \in L^{p(x)}(\Omega).$$

Then

(1) For $u \neq 0, \|u\| = \lambda \Leftrightarrow I(\frac{u}{\lambda}) = 1$;

(2) $\|u\| < 1 (= 1; > 1) \Leftrightarrow I(u) < 1 (= 1; > 1)$;

(3) $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leqslant I(u) \leqslant \|u\|^{p^+}$; $\|u\| < 1 \Rightarrow \|u\|^{p^-} \geqslant I(u) \geqslant \|u\|^{p^+}$;

(4) $\|u_k\| \to 0 \Leftrightarrow I(u_k) \to 0$; $\|u_k\| \to \infty \Leftrightarrow I(u_k) \to \infty$.

Proposition 1.5[12,23,26] If $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function and satisfies

$$|f(x,s)| \leqslant a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \text{for any } x \in \Omega, s \in \mathbf{R},$$

where $a(x) \in L^{p_2(x)}(\Omega)$, and $b \geqslant 0$ is a constant, $p_1(x)$, $p_2(x) \in L_+^\infty(\Omega)$, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 1.6[12] If $p : \Omega \to \mathbf{R}$ is Lipschitz continuous and $p^+ < N$, then for $q \in L_+^\infty(\Omega)$ with $p(x) \leqslant q(x) \leqslant p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$, here

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geqslant N. \end{cases}$$

For $\alpha, \beta \in P(\Omega)$, use the symbol $\alpha \ll \beta$ to denote $\operatorname{ess} \inf\limits_{x \in \overline{\Omega}} (\beta(x) - \alpha(x)) > 0$.

Proposition 1.7[12] Let Ω be a bounded domain in $\mathbf{R}^N$, $p \in C(\overline{\Omega}), p^+ < N$. Then for any $q \in L_+^\infty(\Omega)$ with $q \ll p^*$, there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.

2 Assumptions and statements of the main result

For simplicity we write $X = W^{1,p(x)}(\mathbf{R}^N)$, denote by $u_n \rightharpoonup u$ the weak convergence of sequence u_n to u in X, denote by c and c_i the generic positive constants.

$u \in X$ is called a weak solution of (0.1) if

$$\int_{\mathbf{R}^N} (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv) \,\mathrm{d}x = \int_{\mathbf{R}^N} (f_1(x,u) + f_2(x,u)) v \,\mathrm{d}x, \quad \forall\, v \in X.$$

The functional associated to problem (0.1) is

$$\varphi(u) = \int_{\mathbf{R}^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \,\mathrm{d}x - \int_{\mathbf{R}^N} F_1(x,u) \,\mathrm{d}x - \int_{\mathbf{R}^N} F_2(x,u) \,\mathrm{d}x, \quad \forall\, v \in X,$$

where $F_i(x,u)$ is denoted by

$$F_i(x,u) = \int_0^u f_i(x,s) \,\mathrm{d}s, \quad i = 1, 2.$$

By Lemma 2.2, Lemma 2.3 below, it is easy to see that the functional φ defined above is of class $C^1(X, \mathbf{R})$, then

$$\langle \varphi'(u), v \rangle = \int_{\mathbf{R}^N} (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv) \,\mathrm{d}x - \int_{\mathbf{R}^N} (f_1(x,u) + f_2(x,u)) v \,\mathrm{d}x,$$

so the critical points of φ are just the weak solutions of problem (0.1). Write

$$J(u)=\int_{\mathbf{R}^N}\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}\,\mathrm{d}x,\quad \Psi_i(u)=\int_{\mathbf{R}^N}F_i(x,u)\,\mathrm{d}x,\quad i=1,2,$$

then $\varphi=J-\Psi_1-\Psi_2$. Now we state the assumptions on perturbation terms $f_1(x,u)$, $f_2(x,u)$ for problem (0.1) are as follows:

(F$_1$) $f_i:\mathbf{R}^N\times\mathbf{R}\to\mathbf{R}$ satisfies Caratheodory conditions and

$$|f_i(x,u)|\leqslant a_i(x)|u|^{\alpha_i(x)-1},$$

where $0\leqslant a_i(x)\in L^{r_i(x)}(\mathbf{R}^N)\bigcap L^\infty(\mathbf{R}^N)$, $r_i,\alpha_i\in L_+^\infty(\mathbf{R}^N)$, $1<\alpha_2^-\leqslant\alpha_2^+<p^-\leqslant p^+<\alpha_1^-$, $\alpha_1\ll p^*$, $\frac{1}{r_i(x)}+\frac{\alpha_i(x)}{p^*(x)}=1$. denote $a=\inf_{x\in\mathbf{R}^N}(p(x)-\alpha_2(x))$.

(F$_2$) There is a positive constant $\mu>p^+$ satisfying $a_0=\inf_{x\in\mathbf{R}^N}(\mu-p(x))>0$ such that

$$0<\mu F_1(x,u)\leqslant f_1(x,u)u,\quad \forall x\in\mathbf{R}^N, u\neq 0.$$

(F$_3$) $\exists\,\delta>0$, $0<b_0\in C(\mathbf{R}^N,\mathbf{R})$, $q_0\in L_+^\infty(\mathbf{R}^N)$, $q_0^+<p^-$ such that

$$f_2(x,t)\geqslant b_0(x)t^{q_0(x)-1}\text{ for }x\in\mathbf{R}^N,\ 0<t\leqslant\delta,$$

and $F_2(x,t)\geqslant 0$ for $x\in\mathbf{R}^N$.

(F$_4$) $f_i(x,-u)=-f_i(x,u)$, $\quad\forall(x,u)\in\mathbf{R}^N\times\mathbf{R}$, i=1,2.

Our main results as follows:

Theorem 2.1 Under assumptions (F$_1$)–(F$_4$), then

(1) Problems (0.1) have solutions $\{\pm u_k\}_{k=1}^\infty$ such that $\varphi(\pm u_k)\to+\infty$ as $k\to\infty$.

(2) Problems (0.1) have solutions $\{\pm v_k\}_{k=1}^\infty$ such that $\varphi(\pm v_k)<0$, $\varphi(\pm v_k)\to 0$ as $k\to\infty$.

Before giving our proofs we first give several lemmas that will be used later.

Lemma 2.2[12] $J\in C^1(X,\mathbf{R})$ and the derivative operator, denote by J',

$$J'(u)v=\int_{\mathbf{R}^N}(|\nabla u|^{p(x)-2}\nabla u\nabla v+|u|^{p(x)-2}uv)\,\mathrm{d}x,\quad\forall\,u,v\in X.$$

J is a convex functional, $J':X\to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S+) type, namely, $u_n\rightharpoonup u$ and $\limsup_{n\to\infty}J'(u_n)(u_n-u)\leqslant 0$ implies $u_n\to u$.

Lemma 2.3[12] Suppose

$$|f(x,u)|\leqslant\sum_{i=1}^m b_i(x)|u|^{q_i(x)-1},\quad\forall(x,u)\in\mathbf{R}^N\times\mathbf{R},$$

where $b_i(x)\geqslant 0$, $b_i\in L^{r_i(x)}(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$, $r_i,q_i\in L_+^\infty(\mathbf{R}^N), q_i\ll p^*$, and there are $s_i\in L_+^\infty(\mathbf{R}^N)$ such that

$$p(x)\leqslant s_i(x)\leqslant p^*(x),\quad\frac{1}{r_i(x)}+\frac{q_i(x)}{s_i(x)}=1.$$

Then $\Psi\in C^1(X,\mathbf{R})$ and Ψ,Ψ' are weakly-strongly continuous, i.e., $u_n\rightharpoonup u$ implies $\Psi(u_n)\to\Psi(u)$ and $\Psi'(u_n)\to\Psi'(u)$.

Remark 2.4[12] Under the conditions in Lemmas 2.2 and 2.3, $\varphi' = J' - \Psi_1' - \Psi_2'$ is the sum of a (S+) type map and a weakly-strongly continuous map, so φ' is of (S+) type. To verify that φ satisfies (PS) condition on X, it is enough to verify that any (PS) sequence is bounded.

As X is a separable and reflexive Banach space, there exist (see [27]) $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

(i) $\langle f_n, e_m\rangle = \delta_n^m$, as $n = m$, $\delta_n^m = 1$; as $n \neq m$, $\delta_n^m = 0$;

(ii) $X = \overline{\text{span}}\{e_n : n \in N\},\ X^* = \overline{\text{span}}^{W^*}\{f_n : n \in N\}$.

For $k = 1, 2, \cdots$, denote

$$X_k = \text{span}\{e_k\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j\geqslant k} X_j}. \tag{2.1}$$

Lemma 2.5[12] Assume that $\Psi : X \to \mathbf{R}$ is weakly-strongly continuous and $\Psi(0) = 0, \gamma > 0$ is a given positive number. Set

$$\beta_k = \sup_{u\in Z_k, \|u\|\leqslant\gamma} |\Psi(u)|,$$

then $\beta_k \to 0$ as $k \to \infty$.

We will use the following the fountain theorem and the dual fountain theorem to prove Theorem 2.1.

Proposition 2.6[12] (Fountain theorem, see [12, 28]) Assume

(A1) X is a Banach space, $\varphi \in C^1(X, \mathbf{R})$ is an even functional, the subspaces X_k, Y_k and Z_k are defined by (2.1).

If for each $k \in N$, there exist $\rho_k > r_k > 0$ such that

(A2) $\inf\limits_{u\in Z_k, \|u\|=r_k} \varphi(u) \to +\infty$ as $k \to \infty$;

(A3) $\max\limits_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \leqslant 0$;

(A4) φ satisfies $(\text{PS})_c$ condition for every $c > 0$. Then φ has a sequence of critical values tending to $+\infty$.

Proposition 2.7[12] (Dual Fountain theorem, see [12, 28, 29]) Assume (A1) is satisfied and there is a $k_0 > 0$ so as to for each $k \geqslant k_0$, there exist $\rho_k > r_k > 0$ such that

(B1) $\inf\limits_{u\in Z_k, \|u\|=\rho_k} \varphi(u) \geqslant 0$;

(B2) $b_k := \max\limits_{u\in Y_k, \|u\|=r_k} \varphi(u) < 0$;

(B3) $d_k := \inf\limits_{u\in Z_k, \|u\|\leqslant\rho_k} \varphi(u) \to 0$ as $k \to \infty$;

(B4) φ satisfies $(\text{PS})_c^*$ condition for every $c \in [d_{k_0}, 0)$. Then φ has a sequence of negative critical values converging to 0.

Remark 2.8[12] φ satisfying the $(\text{PS})_c^*$ condition means that any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to \infty, u_{n_j} \in Y_{n_j}, \varphi(u_{n_j}) \to c$ and $(\varphi|Y_{n_j}|)(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ contains a subsequence converging to critical point of φ.

Lemma 2.9 If the assumptions in Theorem 2.1 hold, then φ satisfies $(\text{PS})_c^*$ condition for every $c \in \mathbf{R}$.

Proof Suppose $\{u_{n_j}\}_{n_j\geqslant 1}\subset X$ such that

$$n_j\to\infty,\quad u_{n_j}\in Y_{n_j},\quad \varphi(u_{n_j})\to c,\quad (\varphi|Y_{n_j})'(u_{n_j})\to 0.$$

Then for n_j large enough, we can find $M_1>0$ such that

$$|\varphi(u_{n_j})|\leqslant M_1. \tag{2.2}$$

Since $(\varphi|Y_{n_j})'(u_{n_j})\to 0$, we have $\langle\varphi'(u_{n_j}),u_{n_j}\rangle\to 0$. In particular, the sequence $\{\langle\varphi'(u_{n_j}),u_{n_j}\rangle\}_{n_j\geqslant 1}$ is bounded. Thus there exists $M_2>0$ such that

$$|\langle\varphi'(u_{n_j}),u_{n_j}\rangle|\leqslant M_2. \tag{2.3}$$

We claim that the sequence $\{u_{n_j}\}_{n_j\geqslant 1}$ is bounded.

$$\begin{aligned}\varphi(u_{n_j})&=\int_{\mathbf{R}^N}\Big(\frac{|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)}}{p(x)}-F_1(x,u_{n_j})-F_2(x,u_{n_j})\Big)\,\mathrm{d}x\\&=\int_{\mathbf{R}^N}\Big(\Big(\frac{1}{p(x)}-\frac{1}{\mu}\Big)(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})-F_1(x,u_{n_j})+\frac{1}{\mu}f_1(x,u_{n_j})u_{n_j}\\&\quad-F_2(x,u_{n_j})+\frac{1}{\mu}f_2(x,u_{n_j})u_{n_j}+\frac{1}{\mu}(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})\\&\quad-\frac{1}{\mu}f_1(x,u_{n_j})u_{n_j}-\frac{1}{\mu}f_2(x,u_{n_j})u_{n_j}\Big)\,\mathrm{d}x.\end{aligned}$$

From the assumptions (F_1) and (F_2), implies

$$\begin{aligned}\varphi(u_{n_j})&\geqslant\int_{\mathbf{R}^N}\Big(\frac{a_0}{\mu p^+}(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})-\Big(\frac{a_2(x)}{\mu}+\frac{a_2(x)}{\alpha_2(x)}\Big)|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x\\&\quad+\Big\langle\varphi'(u_{n_j}),\frac{u_{n_j}}{\mu}\Big\rangle\\&\geqslant\int_{\mathbf{R}^N}\Big(\frac{a_0}{2\mu p^+}(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})\Big)\,\mathrm{d}x+\Big\langle\varphi'(u_{n_j}),\frac{u_{n_j}}{\mu}\Big\rangle\\&\quad+\int_{\mathbf{R}^N}\Big(\frac{a_0}{2\mu p^+}(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)a_2(x)|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x.\end{aligned}$$

We claim that there exists $c_0>0$ such that

$$\int_{\mathbf{R}^N}\Big(\frac{a_0}{2\mu p^+}(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)a_2(x)|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x\geqslant -c_0.$$

In fact, denote $L=\max\Big\{1,\Big(\frac{2Ap^+(\alpha_2^-+\mu)}{a_0\alpha_2^-}\Big)^{\frac{1}{a}}\Big\},\Omega_1=\{x\in\mathbf{R}^N:|u(x)|\geqslant L\},\Omega_2=\mathbf{R}^N\backslash\Omega_1$, where $A=\sup\limits_{x\in\mathbf{R}^N}a_2(x)$. For $\forall u\in W^{1,p(x)}(\mathbf{R}^N)$

$$\begin{aligned}&\int_{\Omega_1}\Big(\frac{a_0}{2\mu p^+}|u_{n_j}|^{p(x)}-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)a_2(x)|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x\\&\quad\geqslant\int_{\Omega_1}\Big(\frac{a_0}{2\mu p^+}|u_{n_j}|^{p(x)}-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)A|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x\geqslant 0.\end{aligned}$$

By Young's inequality, for $\forall\,\varepsilon\in(0,1)$, we have

$$a_2(x)|u_{n_j}|^{\alpha_2(x)}\leqslant\Big(\frac{a_2(x)}{\varepsilon}\Big)^{r_2(x)}\frac{1}{r_2(x)}+(\varepsilon|u_{n_j}|^{\alpha_2(x)})^{\frac{p^*(x)}{\alpha_2(x)}}\frac{\alpha_2(x)}{p^*(x)}.$$

Then we obtain that

$$\begin{aligned}&\int_{\Omega_2}\Big(\frac{a_0}{2\mu p^+}|u_{n_j}|^{p(x)}-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)a_2(x)|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x\\ &\quad\geqslant\int_{\Omega_2}\Big(\frac{a_0}{2\mu p^+}|u_{n_j}|^{p(x)}-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)\frac{|a_2(x)|^{r_2(x)}}{\varepsilon^{r_2(x)}r_2(x)}-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)\frac{|u_{n_j}|^{p^*(x)}\varepsilon^{\frac{p^*(x)}{\alpha_2(x)}}\alpha_2(x)}{p^*(x)}\Big)\,\mathrm{d}x.\end{aligned}$$

Let $\varepsilon<\min\Big\{1,\Big(\frac{a_0\alpha_2^-p^{*-}L^{p^--p^{*+}}}{2\alpha_2^+p^+(\mu+\alpha_2^-)}\Big)^{\frac{\alpha_2^+}{p^{*-}}}\Big\}$, then

$$\int_{\Omega_2}\Big(\frac{a_0}{2\mu p^+}|u_{n_j}|^{p(x)}-\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)a_2(x)|u_{n_j}|^{\alpha_2(x)}\Big)\,\mathrm{d}x\geqslant-\int_{\Omega_2}\Big(\frac{1}{\mu}+\frac{1}{\alpha_2^-}\Big)\frac{|a_2(x)|^{r_2(x)}}{\varepsilon^{r_2(x)}r_2(x)}\,\mathrm{d}x=-c_0.$$

From (2.2), (2.3) and Proposition 1.4, we conclude that

$$\begin{aligned}M_1+\frac{1}{\mu}M_2&\geqslant\varphi(u_{n_j})-\Big\langle\varphi'(u_{n_j}),\frac{u_{n_j}}{\mu}\Big\rangle\\ &\geqslant\int_{\mathbf{R}^N}\Big(\frac{a_0}{2\mu p^+}(|\nabla u_{n_j}|^{p(x)}+|u_{n_j}|^{p(x)})\Big)\,\mathrm{d}x-c_0\geqslant\frac{a_0}{2\mu p^+}\|u_{n_j}\|^{p^-}-c_0,\end{aligned}$$

which implies that the sequence $\{u_{n_j}\}_{n_j\geqslant1}\subset X$ is bounded. By passing a subsequence if necessary, we can assume $u_{n_j}\rightharpoonup u_0$ in X as $n_j\to+\infty$. As $X=\overline{\bigcup_{n_j}Y_{n_j}}$, we can choose $v_{n_j}\in Y_{n_j}$ such that $v_{n_j}\to u_0$. Hence

$$\begin{aligned}\lim_{n_j\to\infty}\varphi'(u_{n_j})(u_{n_j}-u_0)&=\lim_{n_j\to\infty}\varphi'(u_{n_j})(u_{n_j}-v_{n_j})+\lim_{n_j\to\infty}\varphi'(u_{n_j})(v_{n_j}-u_0)\\ &=\lim_{n_j\to\infty}(\varphi|Y_{n_j})'(u_{n_j})(v_{n_j}-u_0)=0.\end{aligned}$$

Noting that $\varphi'=J'-\Psi_1'-\Psi_2'$ is of (S+) type, we have

$$u_{n_j}\to u_0\quad\text{and}\quad\varphi'(u_{n_j})\to\varphi'(u_0).$$

Next, we prove $\varphi'(u_0)=0$. Taking arbitrary $w_k\in Y_k$, when $n_j\geqslant k$, we have

$$\begin{aligned}\langle\varphi'(u_0),w_k\rangle&=\langle\varphi'(u_0)-\varphi'(u_{n_j}),w_k\rangle+\langle\varphi'(u_{n_j}),w_k\rangle\\ &=\langle\varphi'(u_0)-\varphi'(u_{n_j}),w_k\rangle+\langle(\varphi|Y_{n_j})'(u_{n_j}),w_k\rangle.\end{aligned}$$

Taking limit on the right-hand side of the equation above, we obtain

$$\langle\varphi'(u_0),w_k\rangle=0,\quad\forall\,w_k\in Y_k.$$

So we have $\varphi'(u_0)=0$. Therefore, φ satisfies $(\mathrm{PS})_c^*$ condition for every $c\in\mathbf{R}$. Thus we complete the proof.

Proof of conclusion (1) Let us verify for φ the conditions in the fountain theorem 2.6 item by item. Obviously, because of the assumptions of (F_4), φ is an even functional. Similar to the process of verifying the $(PS)_c^*$ condition in the proof of Lemma 2.9, we can get the boundedness of $\{\|u_n\|\}$, then φ satisfies $(PS)_c$ condition by Remark 2.4. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A2) and (A3) hold.

(A2) For $k = 1, 2, \cdots$, write

$$\theta_i^k = \sup_{v \in Z_k, \|v\| \leqslant 1} \int_{\mathbf{R}^N} \frac{a_i(x)}{\alpha_i(x)} |v|^{\alpha_i(x)} \,\mathrm{d}x, \quad i = 1, 2. \tag{2.4}$$

Then $\theta_i^k > 0$ and $\theta_i^k \to 0$ as $k \to \infty$. Choosing $u \in Z_k$ and $\|u\| \geqslant 1$, we have

$$\begin{aligned}
\varphi(u) &= \int_{\mathbf{R}^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \,\mathrm{d}x - \int_{\mathbf{R}^N} F_1(x, u) \,\mathrm{d}x - \int_{\mathbf{R}^N} F_2(x, u) \,\mathrm{d}x \\
&\geqslant \int_{\mathbf{R}^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} \,\mathrm{d}x - \int_{\mathbf{R}^N} \frac{a_1(x)}{\alpha_1(x)} |u|^{\alpha_1(x)} \,\mathrm{d}x - \int_{\mathbf{R}^N} \frac{a_2(x)}{\alpha_2(x)} |u|^{\alpha_2(x)} \,\mathrm{d}x \\
&\geqslant \frac{1}{p^+} \|u\|^{p^-} - \|u\|^{\alpha_1^+} \theta_1^k - \|u\|^{\alpha_2^+} \theta_2^k.
\end{aligned}$$

For sufficiently large k, we have $\theta_2^k < \frac{1}{2p^+}$. As $\alpha_2^+ < p^-$, we get

$$\varphi(u) \geqslant \frac{1}{2p^+} \|u\|^{p^-} - \|u\|^{\alpha_1^+} \theta_1^k.$$

Taking $r_k = \big(\frac{p^-}{2p^+ \alpha_1^+ \theta_1^k}\big)^{\frac{1}{\alpha_1^+ - p^-}}$, let $u \in Z_k$ and $\|u\| = r_k$, for sufficiently large k,

$$\varphi(u) \geqslant \Big(\frac{p^-}{2p^+ \alpha_1^+}\Big)^{\frac{p^-}{\alpha_1^+ - p^-}} \frac{\alpha_1^+ - p^-}{2p^+ \alpha_1^+} \Big(\frac{1}{\theta_1^k}\Big)^{\frac{p^-}{\alpha_1^+ - p^-}}.$$

Now $\theta_1^k \to 0$ and $\alpha_1^+ > p^-$ implies

$$\inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to +\infty \text{ as } k \to \infty.$$

(A3) Condition (F_2) implies $F_1(x, tu) \geqslant t^\mu F_1(x, u)$, $\forall\, t \geqslant 1$. Choosing $v \in Y_k$ such that $\|v\| = 1$, then for any $t > 1$, we have

$$\begin{aligned}
\varphi(tv) &= \int_{\mathbf{R}^N} \frac{|\nabla tv|^{p(x)} + |tv|^{p(x)}}{p(x)} \,\mathrm{d}x - \int_{\mathbf{R}^N} F_1(x, tv) \,\mathrm{d}x - \int_{\mathbf{R}^N} F_2(x, tv) \,\mathrm{d}x \\
&\leqslant \frac{1}{p^-} t^{p^+} - \int_{\mathbf{R}^N} F_1(x, tv) \,\mathrm{d}x \leqslant \frac{1}{p^-} t^{p^+} - t^\mu \int_{\mathbf{R}^N} F_1(x, v) \,\mathrm{d}x.
\end{aligned}$$

By virtue of $\int_{\mathbf{R}^N} F_1(x, v) \,\mathrm{d}x > 0$ and $\mu > p^+$, there exist $\rho_k > r_k$ such that $t = \rho_k$ concludes $\varphi(tv) \leqslant 0$, and then

$$\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leqslant 0.$$

Thus we complete the proof of Theorem 2.1(1).

Proof of conclusion (2) We use the dual fountain Theorem 2.7 to prove it. Obviously, because of the assumptions of (F_4), φ is an even functional and satisfies $(PS)_c^*$ condition (see

Lemma 2.9). We need only to prove there exist $\rho_k > r_k > 0$ such that for sufficiently large k we have (B1)–(B3).

(B1) Let θ_i^k be defined by (2.4), When $v \in Z_k$ and $\|v\| = 1$ and $0 < t < 1$, we have

$$\varphi(tv) \geqslant \frac{1}{p^+} t^{p^+} - \theta_1^k t^{\alpha_1^-} - \theta_2^k t^{\alpha_2^-}.$$

For sufficiently large k we have $\theta_1^k < \frac{1}{2p^+}$, as $\alpha_1^- > p^+$, thus

$$\varphi(tv) \geqslant \frac{1}{2p^+} t^{p^+} - \theta_2^k t^{\alpha_2^-}. \tag{2.5}$$

Taking $\rho_k = (4p^+\theta_2^k)^{\overline{\frac{1}{p^+-\alpha_2^-}}}$, then for sufficiently large k, let $t = \rho_k < 1, v \in Z_k$ with $\|v\| = 1$, we have $\varphi(tv) \geqslant 0$. So for sufficiently large k,

$$\inf_{u \in Z_k, \|u\| = \rho_k} \varphi(u) \geqslant 0.$$

(B2) Since $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, we may choose $\{Y_k : k = 1, 2, \cdots\}$, a sequence of finite dimensional vector subspaces of $W_0^{1,p(x)}(\Omega)$ defined by (2.1), such that $Y_k \subset C_0^\infty(\Omega)$ for all k. As the norms on Y_k are equivalent each other, there exists $\epsilon_k \in (0, 1)$ such that $v \in Y_k$ with $\|v\| \leqslant \epsilon_k$ implies $|v|_{L^\infty(\Omega)} \leqslant \delta$. Set

$$S_{\epsilon_k}^k = \{v \in Y_k : \|v\| = \epsilon_k\},$$

the compactness of $S_{\epsilon_k}^k$ with condition (F_3) concludes the existence of a constant d_k such that

$$\int_\Omega \frac{b_0(x)|v|^{q_0(x)}}{q_0(x)}\,\mathrm{d}x \geqslant d_k, \quad \forall\, v \in S_{\epsilon_k}^k.$$

For $v \in S_{\epsilon_k}^k$ and for any $0 < t < 1$, we have

$$\begin{aligned}
\varphi(tv) &\leqslant \int_\Omega \frac{|\nabla tv|^{p(x)} + |tv|^{p(x)}}{p(x)}\,\mathrm{d}x - \int_\Omega F_2(x, tv)\,\mathrm{d}x \\
&\leqslant \frac{1}{p^-} t^{p^-} \epsilon_k^{p^-} - \int_{\mathbf{R}^N} \frac{b_0(x) t^{q_0(x)} |v|^{q_0(x)}}{q_0(x)}\,\mathrm{d}x \\
&\leqslant \frac{1}{p^-} t^{p^-} \epsilon_k^{p^-} - t^{q_0^+} d_k.
\end{aligned}$$

From $q_0^+ < p^-$, there exists $t_k \in (0, \rho_k)$ such that $\varphi(tv) < 0$. Let $r_k = t_k \epsilon_k < \rho_k$, hence we get

$$b_k := \max_{u \in Y_k, \|u\| = r_k} \varphi(u) < 0.$$

(B3) Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k := \inf_{u \in Z_k, \|u\| \leqslant \rho_k} \varphi(u) \leqslant b_k := \max_{u \in Y_k, \|u\| = r_k} \varphi(u) < 0.$$

From (2.5), for $v \in Z_k$, $\|v\| = 1$, $0 \leqslant t \leqslant \rho_k$ and $u = tv$

$$\varphi(u) = \varphi(tv) \geqslant \frac{1}{2p^+} t^{p^+} - \theta_2^k t^{\alpha_2^-} \geqslant -\theta_2^k t^{\alpha_2^-} \geqslant -\theta_2^k \rho_k^{\alpha_2^-} \geqslant -\theta_2^k,$$

hence

$$d_k := \inf_{u\in Z_k, \|u\|\leqslant \rho_k} \varphi(u) \to 0 \quad \text{as} \quad k\to\infty.$$

Thus we complete the proof of Theorem 2.1(2).

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