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Oscillations of a class of third order nonlinear neutral functional differential equations

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Abstract: The oscillations of a class of third order nonlinear neutral functional differential equations was discussed, by using the generalized Riccati transformation and the integral averaging technique; and some new sufficient conditions for oscillations or tends to zero of all solutions of the equations were obtained. The example to illustrate the main results were given. The results extend and improve some known results.

Key words: third order neutral differential equation; oscillation criteria; Kamenev type; Philos type

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三阶非线性中立型泛函微分方程的振动性

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摘要: 利用广义 Riccati 变换和积分平均技巧, 研究一类三阶中立型泛函微分方程的振动性质, 建立了保证此方程一切解振动或者收敛于零的若干新的充分条件, 推广和改进了一些已有结果, 并给出了应用实例.

关键词: 三阶中立型微分方程; 振动准则; Kamenev 型; Philos型

0 Introduction

Recently, the oscillatory behavior of solutions of third order functional differential equations has attracted many researchers. For recent results we refer the reader in particular to [1-5] and the references cited therein. We know that neutral differential equations have applications in many problems such as the vibrating masses attached to an elastic bar and some variational problems (see [6]). Theoretically, the oscillation analysis of neutral equations are more complicated than that of delay equations (of [7-9]). In this paper, we study the oscillatory behavior of solutions of the third order neutral differential equation

$$(r(t)[x(t)+p(t)x(\tau(t))]'')'+q(t)f(x(\sigma(t)))g(x'(t))=0,\quad t>t_0. \tag{E}$$

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Throughout this paper, we always assume that

(H$_1$) $r(t),\ q(t)\in \mathbf{C}([t_0,\infty),(0,\infty)),\ \int_{t_0}^{\infty}\frac{1}{r(t)}\mathrm{d}t=\infty$;

(H$_2$) $p(t)\in \mathbf{C}([t_0,\infty)),\ 0\leqslant p(t)\leqslant p<1$;

(H$_3$) $\tau(t),\ \sigma(t)\in \mathbf{C}([t_0,\infty),(0,\infty)),\ \tau(t)\leqslant t,\ \sigma(t)\leqslant t,\ \lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$;

(H$_4$) $f\in \mathbf{C}(\mathbf{R},\mathbf{R}),\ \frac{f(u)}{u}\geqslant K>0,\ u\neq 0$;

(H$_5$) $g\in \mathbf{C}(\mathbf{R},[L,\infty)), L>0$.

For the equation (E), corresponding second order neutral differential equation, namely

$$(r(t)[x(t)+p(t)x(\tau(t))]')'+q(t)f(x(\sigma(t)))g(x'(t))=0,\quad t>t_0, \tag{E$_1$}$$

has been considered by Yang and Zhu in [10]. Corresponding second order delay differential equation, namely

$$x''(t)+q(t)f(x(\sigma(t)))g(x'(t))=0,\quad t>t_0, \tag{E$_2$}$$

has been also considered by Rogovchenko in [11]. In this paper, our aim is to study the third order neutral equation (E), which have not been considered yet in the literature, and establish the oscillation criteria which extend and improve the results in [11,12].

We put $y(t)=x(t)+p(t)x(\tau(t))$. By a solution of the Eq. (E) we mean a function $x(t)\in \mathbf{C}^1[T_x,\infty)$, $T_x\geqslant t_0$, which has the property $r(t)y''(t)\in \mathbf{C}^1[T_x,\infty)$ and satisfies Eq. (E) on $[T_x,\infty)$. We consider only those solution $x(t)$ of (E) which satisfy $\sup\{|x(t)|: t\geqslant T\}>0$ for all $T\geqslant T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise it is called nonoscillatory.

1 Main results

Before starting our main results, we begin with the following lemmas which will play an important role in the proof of main results.

Lemma 1[11] Assume that $u(t)>0,\ u'(t)>0,\ u''(t)\leqslant 0,\ t\geqslant t_0$, then for every $\alpha\in(0,1)$, there exists $T_\alpha\geqslant t_0$, such that $u(\sigma(t))\geqslant \alpha\frac{\sigma(t)}{t}u(t),\ t>T_\alpha$.

Lemma 2 Assume that $u(t)>0,\ u'(t)>0,\ u''(t)>0,\ u'''(t)\leqslant 0,\ t>T_\alpha$, then there exist $\beta\in(0,1)$, and $T_\beta>T_\alpha$, such that

$$u(t)\geqslant \beta t u'(t),\quad t>T_\beta. \tag{1}$$

Proof Set $y(t)=(t-T_\alpha)u(t)-\frac{1}{2}(t-T_\alpha)^2u'(t),\ t\geqslant T_\alpha$. Then $y(T_\alpha)=0$, and

$$y'(t)=u(t)-\frac{1}{2}(t-T_\alpha)^2u''(t). \tag{2}$$

We claim that $y'(t)>0$. In fact, since u'' is nonincreasing, by Taylor's theorem, we get

$$u(t)\geqslant u(T_\alpha)+(t-T_\alpha)u'(T_\alpha)+\frac{1}{2}(t-T_\alpha)^2u''(t).$$

This implies that $y'(t)\geqslant u(T_\alpha)+(t-T_\alpha)u'(T_\alpha)>0$. Since $y(T_\alpha)=0$, we have $y(t)\geqslant 0$ for $t\geqslant T_\alpha$, i.e. $u(t)\geqslant \frac{t-T_\alpha}{2}u'(t),\ t\geqslant T_\alpha$. So there exist $\beta\in(0,1)$ and $T_\beta>T_\alpha$, such that (1) holds. The proof is complete.

Lemma 3 Assume that $x(t)$ is an eventually positive solution of equation (E). Define the function

$$y(t) = x(t) + p(t)x(\tau(t)). \tag{3}$$

Then for $t \geqslant T \geqslant t_0$ sufficiently large, (A) $y(t) > 0,\ y'(t) > 0,\ y''(t) > 0$, or (B) $y(t) > 0,\ y'(t) < 0,\ y''(t) > 0$.

Proof Let $x(t)$ is an eventually positive solution of equation (E). Then there exists $t_1 \geqslant t_0$, such that $x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0,\ t \geqslant t_1$. It is easy to see that $y(t) > x(t) > 0$, and $(r(t)y''(t))' = -q(t)f(x[\sigma(t)])g(x'(t)) < 0,\ t \geqslant t_1$. Then $r(t)y''(t)$ is a nonincreasing function for $t \geqslant t_1$ and of a constant sign. Thus there exists $t_2 \geqslant t_1$, such that $y''(t) < 0$ or $y''(t) > 0$, $t \geqslant t_2$.

Now, we assume that $y''(t) < 0$, then there exists constant $M > 0$, such that

$$r(t)y''(t) \leqslant -M < 0, \quad t \geqslant t_2. \tag{4}$$

Integrating (4) over $[t_2, t]$, we get

$$y'(t) \leqslant y'(t_2) - M\int_{t_2}^{t} \frac{1}{r(s)}\mathrm{d}s.$$

Letting $t \to \infty$ in the above. From the condition (H_1) we get $y'(t) \to \infty$, i.e. $y'(t)$ is eventually negative. But $y''(t) < 0$ and $y'(t) < 0$ holds eventually, thus, there exists a $T \geqslant t_2$, such that $y(t) < 0$ for $t \geqslant T$. This contradicts with the fact that $y(t) > 0$. Then $y''(t) > 0$. The proof is complete.

Lemma 4 Let $x(t)$ be a positive solution of equation (E) and the corresponding $y(t)$ satisfies (B). If

$$\int_{t_0}^{\infty}\int_{v}^{\infty}(\frac{1}{r(u)}\int_{u}^{\infty} q(s)\mathrm{d}s)\mathrm{d}u\mathrm{d}v = \infty. \tag{5}$$

Then $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} y(t) = 0$.

Proof Suppose that $x(t)$ is a positive solution 泛函 of equation (E), and that $y(t)$ satisfy (B), i.e., $y(t) > 0,\ y'(t) < 0,\ t \geqslant T \geqslant t_0$. So there exists constant $l \geqslant 0$, such that $\lim\limits_{t\to\infty} y(t) = l$. Now we claim that $l = 0$. In fact, if $l > 0$, then for every $\varepsilon > 0$, there exists $t_1 \geqslant T$, such that $l < y(t) < l + \varepsilon,\ t \geqslant t_1$. We can choose $0 < \varepsilon < \frac{l(1-p)}{p}$, we obtain

$$x(t) = y(t) - p(t)x(\tau(t)) > l - py(\tau(t)) > l - p(l + \varepsilon). \tag{6}$$

Let $m = \frac{l-p(l+\varepsilon)}{l+\varepsilon}$, then $m > 0$. From (6) we have

$$x(t) > m(l + \varepsilon) > my(t). \tag{7}$$

Combining (H_4), (H_5), (7) and (E), yields

$$(r(t)y''(t))' + KLmq(t)y(\sigma(t)) \leqslant 0, \quad t \geqslant t_1. \tag{8}$$

Integrating (8) from t to ∞, we obtain

$$-r(t)y''(t) + KLm\int_{t}^{\infty} q(s)y(\sigma(s))\mathrm{d}s \leqslant 0.$$

Noting that $y(\sigma(t)) > l,\ t \geqslant t_2 \geqslant t_1$, we get

$$-y''(t) + \frac{KLlm}{r(t)} \int_t^{\infty} q(s)\mathrm{d}s \leqslant 0, \quad t \geqslant t_2. \tag{9}$$

Integrating (9) from t to ∞ again, we have

$$y'(t) + KLlm \int_t^{\infty} \Big(\frac{1}{r(u)} \int_u^{\infty} q(s)\mathrm{d}s\Big)\mathrm{d}u \leqslant 0. \tag{10}$$

Integrating (10) from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} \int_v^{\infty} \Big(\frac{1}{r(u)} \int_u^{\infty} q(s)\mathrm{d}s\Big)\mathrm{d}u\mathrm{d}v \leqslant \frac{y(t_2)}{KLml}. \tag{11}$$

This contradicts (5). Then $l = 0$. Since $0 < x(t) \leqslant y(t)$ implies $\lim\limits_{t\to\infty} x(t) = 0$. The proof is complete.

We now present some new oscillation results for equation (E) by using integral averages condition of Kamenev-type.

Theorem 1 Suppose $n > 1$ and that (5) holds. If there exists a function $\rho \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \Big[\rho(s)Q(s) - \frac{(\rho'(s))^2 r(s)}{4\rho(s)}\Big]\mathrm{d}s = \infty, \tag{12}$$

where

$$Q(s) = \alpha\beta KL(1-p)\frac{\sigma^2(s)}{s}q(s), \tag{13}$$

α and β are defined by Lemmas 1 and 2. Then every solution $x(t)$ of equation (E) is oscillatory or $\lim\limits_{t\to\infty} x(t) = 0$.

Proof Assume that (E) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0,\ x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geqslant t_1 \geqslant \max\{T, T_\beta\}$. Since the proof for the case $x(t) < 0$ for $t \geqslant t_1 \geqslant \max\{T, T_\beta\}$ is similar. Define the function $y(t)$ as in (3). From Lemma 3, there are two possible cases.

First consider that $y(t)$ is (A) type. Note that $x(t) = y(t) - p(t)x(\tau(t)) \geqslant y(t) - py(\tau(t)) \geqslant (1-p)y(t)$. From (H_4), (H_5) and (E), we get

$$(r(t)y''(t))' + KL(1-p)q(t)y(\sigma(t)) \leqslant 0, \quad t \geqslant t_1. \tag{14}$$

Let

$$w(t) = \rho(t)\frac{r(t)y''(t)}{y'(t)}, \quad t \geqslant t_1. \tag{15}$$

Then

$$w'(t) \leqslant -\frac{KL(1-p)\rho(t)q(t)y(\sigma(t))}{y'(t)} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{w^2(t)}{\rho(t)r(t)}. \tag{16}$$

From Lemma 1 and Lemma 2, for $t \geqslant t_1$, we have

$$w'(t) \leqslant -\rho(t)Q(t) - \Big[\frac{w(t)}{\sqrt{\rho(t)r(t)}} - \frac{1}{2}\sqrt{\frac{r(t)}{\rho(t)}}\rho'(t)\Big]^2 + \frac{(\rho'(t))^2 r(t)}{4\rho(t)} \leqslant -\Big[\rho(t)Q(t) - \frac{(\rho'(t))^2 r(t)}{4\rho(t)}\Big].$$

Where $Q(t)$ is as in (13). Thus

$$\int_{t_1}^{t}(t-s)^n\Big[\rho(s)Q(s)-\frac{(\rho'(s))^2r(s)}{4\rho(s)}\Big]\mathrm{d}s\leqslant-\int_{t_1}^{t}(t-s)^n w'(s)\mathrm{d}s,\quad t\geqslant t_1.$$

Noting that

$$\int_{t_1}^{t}(t-s)^n w'(s)\mathrm{d}s=n\int_{t_1}^{t}(t-s)^{n-1}w(s)\mathrm{d}s-w(t_1)(t-t_1)^n,$$

we have

$$\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n G(s)\mathrm{d}s\leqslant w(t_1)\Big(1-\frac{t_1}{t}\Big)^n-\frac{n}{t^n}\int_{t_1}^{t}(t-s)^{n-1}w(s)\mathrm{d}s,$$

where $G(s)=\rho(s)Q(s)-\frac{(\rho'(s))^2r(s)}{4\rho(s)}$. Thus

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n G(s)\mathrm{d}s<\infty,$$

which contradicts (12). If that $y(t)$ is (B) type. From Lemma 4 we have $\lim\limits_{t\to\infty}x(t)=\lim\limits_{t\to\infty}y(t)=0$. The proof is complete.

Next we present some new oscillatory criteria for equation (E) by using the integral averaging condition of Philos-type. So we introduce a class of functions P. Let

$$D=\{(t,s):t\geqslant s\geqslant t_0\},\quad D_0=\{(t,s):t>s\geqslant t_0\}.$$

We say that a function $H\in\mathbf{C}(D,\mathbf{R})$ belongs to a function class P, denoted by $H\in$ P, if it satisfies

(i) $H(t,t)=0,\ t\geqslant t_0;\ H(t,s)>0,\ (t,s)\in D_0$;

(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable, and such that $-\frac{\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)},\ (t,s)\in D_0$.

Theorem 2 Assume that (5) holds, and there exist functions $H\in$ P and $\rho\in\mathbf{C}^1([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[H(t,s)\rho(s)Q(s)-\frac{1}{4}\rho(s)r(s)h_1^2(t,s)\Big]\mathrm{d}s=\infty,\tag{17}$$

where $Q(s)$ is as in (13), and

$$h_1(t,s)=h(t,s)-\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}.\tag{18}$$

Then every solution $x(t)$ of equation (E) is oscillatory or $\lim\limits_{t\to\infty}x(t)=0$.

Proof Assume that (E) has a nonoscillatory solution $x(t)$. As the proof of Theorem 1, we may assume that $x(t)>0,\ x(\tau(t))>0$ and $x(\sigma(t))>0$ for $t\geqslant t_1\geqslant t_\beta$. Define the function $y(t)$ as in (3). There are two types: (A) type or (B) type.

First consider that $y(t)$ is (A) type. Define the function $w(t)$ as in (15). Then $w(t)>0$ and (16) holds. From Lemma 1 and 2, we obtain

$$w'(t)\leqslant-\rho(t)Q(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{1}{\rho(t)r(t)}w^2(t),\tag{19}$$

where $Q(s)$ is as in (13). Let $A(t)=\frac{\rho'(t)}{\rho(t)},\ B(t)=\frac{1}{\rho(t)r(t)}$, we have

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)\rho(s)Q(s)\mathrm{d}s \leqslant \int_{t_1}^{t} H(t,s)[-w'(s)+A(s)w(s)-B(s)w^2(s)]\mathrm{d}s\\&=-H(t,s)w(s)|_{t_1}^{t}+\int_{t_1}^{t}\Big\{\frac{\partial H(t,s)}{\partial s}w(s)+H(t,s)[A(s)w(s)-B(s)w^2(s)]\Big\}\mathrm{d}s\\&=H(t,t_1)w(t_1)-\int_{t_1}^{t}[\sqrt{H(t,s)}h_1(t,s)w(s)+H(t,s)B(s)w^2(s)]\mathrm{d}s\\&=H(t,t_1)w(t_1)-\int_{t_1}^{t}\Big[\sqrt{H(t,s)B(s)}w(s)+\frac{1}{2}\frac{h_1(t,s)}{\sqrt{B(s)}}\Big]^2\mathrm{d}s+\int_{t_1}^{t}\frac{h_1^2(t,s)}{4B(s)}\mathrm{d}s,\end{aligned}\tag{20}$$

where $h_1(t,s)$ is as in (18). Thus

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\Big[H(t,s)\rho(s)Q(s)-\frac{h_1^2(t,s)}{4B(S)}\Big]\mathrm{d}s\leqslant w(t_1),$$

which contradicts condition (17).

Next, if that $y(t)$ is (B) type. From Lemma 4 we have $\lim\limits_{t\to\infty}x(t)=\lim\limits_{t\to\infty}y(t)=0$. The proof is complete.

Remark 1 Theorem 3 in [11] and Theorem 1 in [12] are extended to third order neutral equation (E) by above Theorem 2.

In Theorem 2, when (17) is difficult to verify, we have the following Theorem.

Theorem 3 Let all the assumption, except (17), of Theorem 2 hold. Further, let

$$0<\inf_{s\geqslant T}\Big[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T)}\Big]\leqslant\infty,\tag{21}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\rho(s)r(s)h_1^2(t,s)\mathrm{d}s<\infty.\tag{22}$$

Let $\varphi\in\mathbf{C}([t_0,\infty),\mathbf{R})$ such that

$$\int_{T}^{\infty}\frac{\varphi_+^2(s)}{\rho(s)r(s)}\mathrm{d}s=\infty,\quad \varphi_+(t)=\max\{\varphi(t),0\},\tag{23}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\Big[H(t,s)\rho(s)Q(s)-\frac{1}{4}\rho(s)r(s)h_1^2(t,s)\Big]\mathrm{d}s\geqslant\varphi(T).\tag{24}$$

Then every solution $x(t)$ of equation (E) is oscillatory or $\lim\limits_{t\to\infty}x(t)=0$.

Proof It is the same with Theorem 2. When that $y(t)$ is (A) type, we have (2.15) holds. Thus for every $T>t_1$, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\Big[H(t,s)\rho(s)Q(s)-\frac{1}{4}\rho(s)r(s)h_1^2(t,s)\Big]\mathrm{d}s\\&\leqslant w(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\Big[\sqrt{H(t,s)B(s)}w(s)+\frac{h_1(t,s)}{2\sqrt{B(s)}}\Big]^2\mathrm{d}s.\end{aligned}$$

Using (24), we obtain

$$w(T) \geqslant \varphi(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[\sqrt{H(t,s)B(s)}w(s) + \frac{h_1(t,s)}{2\sqrt{B(s)}}\Big]^2 \mathrm{d}s,$$

thus

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[\sqrt{H(t,s)B(s)}w(s) + \frac{h_1(t,s)}{2\sqrt{B(s)}}\Big]^2 \mathrm{d}s < \infty. \tag{25}$$

Define

$$u(t) = \frac{1}{H(t,T)} \int_T^t H(t,s)B(s)w^2(s)\mathrm{d}s, \quad v(t) = \frac{1}{H(t,T)} \int_T^t \sqrt{H(t,s)}h_1(t,s)w(s)\mathrm{d}s.$$

From (25), we get $\liminf\limits_{t\to\infty}[u(t)+v(t)] < \infty$.

When $y(t)$ is (A) type, the rest of the proof is similar to that of Theorem the respective one in [9,10]. When $y(t)$ is (B) type. From Lemma 4 we have $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} y(t) = 0$. The proof is complete.

Example 1　Consider the third order nonlinear neutral differential equation

$$\begin{aligned}&\Big((t+1)^{-r}\Big[x(t)+\frac{1}{3}x\Big(t-\frac{1}{2}\Big)\Big]''\Big)' + t^\lambda\Big(\lambda\frac{2-\cos t}{t}\\ &+(2+\sin t)\Big)x(t-1)(1+x^2(t-1))(1+(x'(t))^2)=0, \text{ for } t\geqslant 1.\end{aligned} \tag{E_0}$$

Where r and λ are positive constants. Here $r(t)=(t+1)^{-r}$, $p(t)=\frac{1}{3}$, $\tau(t)=t-\frac{1}{2}$, $f(u)=u(1+u^2)$, $q(t)=t^\lambda(\lambda\frac{2-\cos t}{t}+(2+\sin t))$, with $K=1$, $\sigma(t)=t-1$, $g(v)=1+v^2$, with $L=1$. We have

$$\int_1^\infty \frac{1}{r(s)}\mathrm{d}s = \infty, \tag{26}$$

also,

$$\begin{aligned}\int_{t_o}^t q(s)\mathrm{d}s &= \int_{t_o}^t s^\lambda\Big(\lambda\frac{2-\cos s}{s}+(2+\sin s)\Big)\mathrm{d}s \geqslant \int_{t_o}^t s^\lambda\Big(\lambda\frac{2-\cos s}{s}+\sin s\Big)\mathrm{d}s\\ &= \int_{t_o}^t \mathrm{d}[s^\lambda(2-\cos s)] = t^\lambda(2-\cos t) - t_0^\lambda(2-\cos t_0) \geqslant t^\lambda - k_0 \to \infty, as\ t\to\infty.\end{aligned} \tag{27}$$

From (26) and (27) we see that $(H_1)-(H_5)$ and (5) hold. To apply Theorem 2, it remains to satisfy the condition (17). Taking $H(t,s)=(t-s)^2$, $\rho(t)=1$, then $h_1(t,s)=h(t,s)=2$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \rho(s)r(s)h_1^2(t,s)\mathrm{d}s < \infty.$$

From (13) we have $Q(t)=\alpha\beta KL(1-p)\frac{\sigma^2(t)}{t}q(t) = \frac{2\alpha\beta}{3}\frac{(t-1)^2}{t}q(t)$, then there is a $T>1$ sufficient large such that $Q(t)\geqslant q(t)$ for $t\geqslant T$. For $t\geqslant s\geqslant T$, we have

$$\begin{aligned}\frac{1}{t^2}\int_T^t (t-s)^2Q(s)\mathrm{d}s &\geqslant \frac{1}{t^2}\int_T^t (t-s)^2q(s)\mathrm{d}s = \frac{1}{t^2}\int_T^t \Big[2(t-s)\Big(\int_{t_0}^s q(u)\mathrm{d}u\Big)\Big]\mathrm{d}s\\ &\geqslant \frac{2}{t^2}\int_T^t (t-s)(s^\lambda - k_0)\mathrm{d}s = \frac{2}{(\lambda+1)(\lambda+2)}t^\lambda + \frac{K_1}{t^2} + \frac{K_2}{t} - K_0,\end{aligned}$$

where K_i $(i = 0, 1, 2)$ are constants. Consequently, condition (17) is satisfied. It is easy to see that the condition (5) is satisfied. Hence, every solution of equation (E_0) oscillates or converges to zero.

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