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Conformal transformation between some Finsler Einstein spaces

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Abstract: Liouville's Theorem proved that the Euclidean space can be mapped conformally on itself only by a composition of Möbius transformations. For Riemann spaces, Brinkmann obtained general results. Little work has been done on Finsler spaces. This paper, by navigation idea and properties of conformal map, proved that the conformal transformation between Einstein Randers (or Kropina) spaces must be homothetic.

Key words: Einstein spaces; conformal maps; Randers metrics; Kropina metrics;

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某类 Finsler-Einstein 空间之间的共形映射

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摘要: Liouville 定理证明了欧氏空间到自身的共形变换是莫比乌斯变换. 关于 Riemann 空间，Brinkmann 首先得到了一般的结论. 但对 Finsler 空间的研究乏人问津. 本文运用导航术和共形映射的性质证明了 Randers 空间(或 Kropina 空间)之间保Einstein度量的共形变换必是相似变换.

关键词: Einstein空间; 共形映射; Randers 度量; Kropina 度量

0 Introduction

In Riemann's thesis of 1854, he introduced a metric structure in a general space based on the element of arc

$$\mathrm{d}s = F(x^1, \cdots, x^n; \mathrm{d}x^1, \cdots, \mathrm{d}x^n),$$

which is known as Finsler geometry. An important special case is

$$F^2 = g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j.$$

Historical developments have conferred the name Riemannian geometry to this case.

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An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor is proportional to the metric. The study of Riemannian spaces conformally related to Einstein spaces is a problem which has been addressed since the 1920 s[1-10]. It is well known that an n-dimensional Euclidean space, that is, a Riemann space whose squared line element is

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + \cdots + (\mathrm{d}x^n)^2,$$

can, for $n \geqslant 3$, be mapped conformally on itself only by inversions and similarity transformations (Liouville's Conformality Theorem). When n-dimensional Euclidean space is replaced by Riemann Einstein spaces, Brinkmann[1] firstly obtained general result. He studied the necessary and sufficient conditions for Riemannian spaces to be conformally related to Einstein spaces in n dimensions. Until now, the problem have not been solved completely.

Let F be a Finsler metric on an n-dimensional manifold M. F is called an Einstein metric with Einstein scalar σ if $\mathrm{Ric} = \sigma F^2$, where $\sigma = \sigma(x)$ is a scalar function on M. It generalizes the definition of Einstein metrics in Riemannian geometry.

As is known that every Riemann surface is Einstein. However, Finsler surfaces are typically not Einstein, with counterexamples provided by Numata metrics. Thus Einstein metrics in Finsler geometry are more complicated than those in Riemann's. Recently, some progress has been made on Finsler Einstein metrics of (α, β) type. An (α, β)-metric on M is expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a positive definite Riemannian metric and β a 1-form. The (α, β)-metrics form an important class of Finsler metrics appearing iteratively in formulating physics, mechanics, seismology, biology, control theory, etc. Bao and Robles[11] have investigated Randers Einstein metrics in 2003. They have obtained necessary and sufficient conditions for Randers metrics to be Einstein. For every non-Randers (α, β)-metric with a polynomial function of degree greater than 2, it was proved that it is an Einstein metric if and only if it is Ricci-flat[12]. Zhang and Shen[13] obtained that a Kropina metric F is an Einstein metric if and only if h is a Riemann Einstein metric and W is a unit Killing vector field with respect to h, where the pair (h, W) is the navigation data of F.

Randers metrics, named by Ingarden, were introduced by Randers in the context of general relativity. They arise naturally as the geometry of light rays in stationary spacetimes. They can be written as $F = \alpha + \beta$, the simplest (α, β)-metrics. The Kropina metric $F = \frac{\alpha^2}{\beta}$ is also an (α, β)-metric, which was considered by V. K. Kropina firstly. Such a metric is of physical interest in the sense that it describes the general dynamical system represented by a Lagrangian function, although it has the singularity.

Little work has been done on conformally related Einstein Finsler spaces. In this paper we solved such problem in two special Finsler spaces, i.e., Randers spaces and Kropina spaces.

Theorem 0.1 *Conformal transformation between Einstein Randers spaces must be homothetic.*

We obtain the corresponding result for Kropina metrics.

Theorem 0.2 *Conformal transformation between Einstein Kropina spaces must be homothetic.*

For a Minkowski metric, a special Einstein metric, we have the following result.

Corollary 0.3 *Every conformally flat Einstein Randers metric* (*or Kropina metric*) *must be a Minkowskian.*

1 Preliminaries

Let x denote a point on the manifold M and $y \in T_xM$ a tangent vector based at x. Tangent space coordinates y^i can then arise from the expansion $y = y^i \frac{\partial}{\partial x^i}$ in terms of a local coordinate basis. A Finsler metric F satisfies positively homogeneous of degree one in y, i.e., $F(x, cy) = cF(x, y)$ for all positive c. The fundamental tensor, formally analogous to the metric tensor in Riemannian geometry, is defined as

$$g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^i y^j},$$

where $()_{y^i}$ denotes the partial derivatives with respect to y^i. F is called a Minkowski metric if $F(x, y) = F(y)$. We call F conformally flat if F is conformal to a Minkowski metric.

Let G^i be the geodesic coefficients of F, which are defined by

$$G^i := \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}.$$

For any $x \in M$ and $y \in T_xM \backslash \{0\}$, the Riemann curvature $\mathbf{R}_y$ is defined by $\mathbf{R}_y := R^i{}_k \frac{\partial}{\partial x^i} \bigotimes \mathrm{d}x^k$ with

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Ricci curvature is the trace of the Riemann curvature, which is defined by

$$\mathrm{Ric} := R^k{}_k.$$

F is called an Einstein metric with Einstein scalar σ if

$$\mathrm{Ric} = \sigma F^2, \tag{1}$$

where $\sigma = \sigma(x)$ is a scalar function on M. It generalizes the definition of Einstein metric in Riemannian geometry. In particular, F is said to be Ricci constant (resp. Ricci flat) if F satisfies (1) with σ=constant (resp. $\sigma = 0$). For more basic knowledge of Finsler geometry, one can refer to [14].

2 The rigidity theorem for certain conformal map

Let (M, h) be a Riemann metric on n-dimensional manifold, $n \geqslant 3$. Let W be a vector field on M. Taking the nature coordinates $(U; x^i)$, we can write $W = W^i(x)\frac{\partial}{\partial x^i}$. Denote $W_i := h_{ij}W^j$ and $W_0 := W_i y^i$. Accordingly, we have a Riemann manifold $(M, \widetilde{h})$ and a vector field $\widetilde{W}$ on M. In the following, we shall denote the quantities for $\widetilde{h}$ by the same letter with the tildes and corresponding indices. Let "|" and "," denote the covariant differentiation with respect to h and $\widetilde{h}$, respectively. For convenience, if $\widetilde{h} = \psi^{-1}h$ and $\widetilde{W}_i = \psi^{-1}W_i$ hold for some positive C^∞ function $\psi=\psi(x)$ on M, then we call (h,W) and $(\widetilde{h},\widetilde{W})$ a conformal pair and ψ conformal factor. Such a pair is denoted by $\{(h,W);(\widetilde{h},\widetilde{W});\psi\}$.

Let us recall the properties of conformal map between Riemannian metrics.

Lemma 2.1 Let (M,h) and $(M,\widetilde{h}=\psi^{-1}h)$ be two n-dimensional Riemannian spaces, where $h=\sqrt{h_{ij}(x)y^iy^j}$ and $\widetilde{h}=\sqrt{\widetilde{h}_{ij}(x)y^iy^j}$, respectively. Suppose ψ is a positive C^∞ function on M. Then

$$\begin{cases}\widetilde{\gamma}^i_{jk}=\gamma^i_{jk}-\psi^{-1}\delta^i_j\psi_k-\psi^{-1}\delta^i_k\psi_j+\psi^{-1}\psi^ih_{jk},\\ \widetilde{R}_{ij}=R_{ij}+(n-2)\psi^{-1}\psi_{i|j}+\{\psi^{-1}\psi^k{}_{|k}-(n-1)\psi^{-2}\psi^k\psi_k\}h_{ij},\end{cases}$$

where $\psi_k:=\frac{\partial\psi}{\partial x^k}$ and $\psi^k:=h^{ik}\psi_i$.

A vector field $W=W^i(x)\frac{\partial}{\partial x^i}$ on a Riemann manifold (M,h) is called a conformal vector field with a conformal factor $c=c(x)$ if it satisfies

$$W_{i|j}+W_{j|i}=ch_{ij},$$

where $W_i:=h_{ij}W^j$ and "|" denotes the covariant derivative with respect h.

Theorem 2.2 (Rigidity Theorem) For a conformal pair, the conformal transformation, which preserves Einstein metrics and non-zero conformal vector fields, must be homothetic.

Proof Let M be n-dimensional manifold and $\{(h,W);(\widetilde{h},\widetilde{W});\psi\}$ a conformal pair. Assume that h and $\widetilde{h}$ are both Einstein metrics on M with Einstein scalars δ and $\widetilde{\delta}$, respectively. W and $\widetilde{W}$ are conformal vector fields with respect to h and $\widetilde{h}$, respectively. That is $W_{j|k}+W_{k|j}=ch_{jk}$ and $\widetilde{W}_{j,k}+\widetilde{W}_{k,j}=\widetilde{c}\widetilde{h}_{jk}$ hold for some functions $c=c(x)$ and $\widetilde{c}=\widetilde{c}(x)$ on M.

By Lemma 2.1, we have

$$\widetilde{W}_{j,k}=\frac{\partial\widetilde{W}_j}{\partial x^k}-\widetilde{W}_i\widetilde{\gamma}^i_{jk}=\psi^{-1}W_{j|k}+\psi^{-2}\psi_jW_k-\psi^{-2}W_i\psi^ih_{jk}.$$

So we have

$$\widetilde{W}_{j,k}+\widetilde{W}_{k,j}=\psi^{-1}(W_{j|k}+W_{k|j})+\psi^{-2}(\psi_jW_k+\psi_kW_j)-2\psi^{-2}W_i\psi^ih_{jk}.$$

Assume $\widetilde{W}_{j,k}+\widetilde{W}_{k,j}=\widetilde{c}\widetilde{h}_{jk}$. We obtain

$$\begin{aligned}0&=\widetilde{W}_{j,k}+\widetilde{W}_{k,j}-\widetilde{c}\widetilde{h}_{jk}\\&=\psi^{-1}(W_{j|k}+W_{k|j})+\psi^{-2}(W_j\psi_k+W_k\psi_j)-2\psi^{-2}W^i\psi_ih_{jk}-\widetilde{c}\psi^{-2}h_{jk},\end{aligned}$$

which is equivalent to

$$0=(c\psi-\widetilde{c}-2W^i\psi_i)h_{jk}+W_j\psi_k+W_k\psi_j. \tag{2}$$

Contracting (2) with h^{jk} yields $c\psi-\widetilde{c}=\frac{2(n-1)}{n}W^i\psi_i$. So we can simplify (2) to

$$0=-\frac{2}{n}W^i\psi_ih_{jk}+W_j\psi_k+W_k\psi_j. \tag{3}$$

Contracting (3) with W^jW^k yields $W^i\psi_i = 0$ for $W^iW_i = \|W\|_h^2 \neq 0$. Plugging $W^i\psi_i = 0$ into (3) gives

$$0 = W_j\psi_k + W_k\psi_j. \tag{4}$$

Contracting (4) with W^j, we have $\psi_k = 0$, which means that ψ is constant. It completes the proof of Theorem 2.2.

3 The proof of main theorems

An (α, β)-metric on M is expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a positive definite Riemannian metric and $\beta = b_i(x)y^i$ a 1-form.

Let $\phi = 1 + s$. Then (α, β)-metric defined by ϕ is given by $F = \alpha + \beta$. It is easy to verify that F is a Finsler metric if and only if $||\beta_x||_\alpha < 1$ for all $x \in M$, where $||\beta_x||_\alpha := a^{ij}b_ib_j$. Such a metric is called a Randers metric, which is the simplest (α, β)-metrics. Thus more intensive study has done on Randers metrics. If $\phi = s^{-1}$, then (α, β)-metric defined by ϕ is given by $F = \frac{\alpha^2}{\beta}$, which we call a Kropina metric. Throughout the paper, we shall restrict our consideration to the domain where $\beta = b_i(x)y^i \neq 0$ for Kropina metrics.

First we focus on Randers metric. It is known that a Finsler metric is a Randers metric if and only if it is a solution to the navigation problem on a Riemannian manifolds[15]. Put $b^i := a^{ij}b_j$ and $\epsilon := 1 - ||\beta_x||_\alpha^2$. Define Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and $W = W^i(x)\frac{\partial}{\partial x^i}$ with

$$h_{ij}(x) := \epsilon(a_{ij} - b_ib_j), \quad W^i := -\frac{b^i}{\epsilon}.$$

Then the solution determined by h and W yields the Randers metric

$$F = \frac{\sqrt{\epsilon h^2 + W_0^2}}{\epsilon} - \frac{W_0}{\epsilon}, \quad W_0 = W_iy^i,$$

where $W_i := h_{ij}W^j$, $1 - W^iW_i = 1 - h(x, W)^2 = \epsilon$. The pair (h, W) is called the navigation data of F. Navigation is an efficient method to characterize and study Randers metrics.

Proof of Theorem 0.1 Let $F = \alpha + \beta$ and $\widetilde{F} = \psi^{-1}F$. Assume $\widetilde{F} = \widetilde{\alpha} + \widetilde{\beta}$. Let (h, W) and $(\widetilde{h}, \widetilde{W})$ be the navigation data of F and $\widetilde{F}$, respectively. Then $\widetilde{a}_{ij} = \psi^{-2}a_{ij}$ and $\widetilde{b}_i = \psi^{-1}b_i$. So we have

$$\begin{cases} \widetilde{\epsilon} = 1 - \widetilde{b}^2 = 1 - \widetilde{a}^{ij}\widetilde{b}_i\widetilde{b}_j = 1 - a^{ij}b_ib_j = \epsilon, \\ \widetilde{h}_{ij} = \widetilde{\epsilon}(\widetilde{a}_{ij} - \widetilde{b}_i\widetilde{b}_j) = \epsilon\psi^{-2}(a_{ij} - b_ib_j) = \psi^{-2}h_{ij}, \\ \widetilde{W}_i = -\widetilde{\epsilon}\widetilde{b}_i = -\epsilon\psi^{-1}b_i = \psi^{-1}W_i. \end{cases}$$

This means that $\{(h, W); (\widetilde{h}, \widetilde{W}); \psi\}$ is a conformal pair. It is known that a Randers metric F is an Einstein metric if and only if h is an Einstein metric and W is a conformal vector, where (h, W) is the navigation data of F[11]. Then by Theorem 2.2, we prove Theorem 0.1.

Secondly, using the similar method in proving Theorem 0.1, We can get Theorem 0.2.

A Finsler metric $F=\frac{\alpha^2}{\beta}$ is of Kropina type if and only if it solves the navigation problem on some Riemannian manifold (M,h), under the influence of a wind W with $h(W,W)=1$[13]. The pair (h,W) is called the navigation data of F. We can define Riemannian metric $h=\sqrt{h_{ij}(x)y^iy^j}$ and $W=W^i\frac{\partial}{\partial x^i}$ with

$$h_{ij}=e^{2\rho}a_{ij},\quad 2W_i=e^{2\rho}b_i \quad\text{and}\quad e^{2\rho}b^2=4. \tag{5}$$

Then we obtain that $F=\frac{h^2}{2W_0}$.

Proof of Theorem 0.2 Let $\widetilde{F}=\psi^{-1}F$ and $\widetilde{F}=\frac{\widetilde{\alpha}^2}{\widetilde{\beta}}$. Let (h,W) and $(\widetilde{h},\widetilde{W})$ be the navigation data of F and $\widetilde{F}$, respectively. Suppose $\widetilde{h}_{ij}=e^{2\widetilde{\rho}}\widetilde{a}_{ij}$ and $h_{ij}=e^{2\rho}a_{ij}$. Then $\widetilde{a}_{ij}=\psi^{-2}a_{ij}$ and $\widetilde{b_i}=\psi^{-1}b_i$. So we have

$$\begin{cases}\widetilde{b}^2=\widetilde{a}^{ij}\widetilde{b}_i\widetilde{b}_j=a^{ij}b_ib_j=b^2,\\ \widetilde{h}_{ij}=e^{2\widetilde{\rho}}\widetilde{a}_{ij}=e^{2\widetilde{\rho}}\psi^{-2}a_{ij}=e^{2(\widetilde{\rho}-\rho)}\psi^{-2}h_{ij},\\ 2\widetilde{W}_i=e^{2\widetilde{\rho}}\widetilde{b}_i=e^{2\widetilde{\rho}}\psi^{-1}b_i=2e^{2(\widetilde{\rho}-\rho)}\psi^{-1}W_i.\end{cases} \tag{6}$$

From (5) and the first equation of (6), we get $\widetilde{\rho}=\rho$. So the last two equations of (6) can be simplified as

$$\begin{cases}\widetilde{h}_{ij}=\psi^{-2}h_{ij},\\ \widetilde{W}_i=\psi^{-1}W_i,\end{cases}$$

which means that $\{(h,W);(\widetilde{h},\widetilde{W});\psi\}$ is a conformal pair.

As is known that a Kropina metric F is an Einstein metric if and only if h is an Einstein metric and W is a unit Killing vector, where (h,W) is the navigation data of F[13]. Then by Theorem 2.2, we prove Theorem 0.2.

By Theorem 0.1 and Theorem 0.2, we can prove Corollary 0.3.

Proof of Corollary 0.3 Let F be a conformally flat Einstein Randers metric (or Kropina metric) on manifold M. It is known that a Randers metric (or a Kropina metric) F is conformally flat if and only if there exists a local coordinate system in which F can be expressed as $F=\psi^{-1}(\sqrt{a_{ij}y^iy^j}+b_iy^i)$ (or $F=\psi^{-1}\frac{a_{ij}y^iy^j}{b_iy^i}$), where ψ is a positive function on M, a_{ij} and b_i are constants. Let $\widetilde{F}=\sqrt{a_{ij}y^iy^j}+b_iy^i$ (or $\widetilde{F}=\frac{a_{ij}y^iy^j}{b_iy^i}$). Obviously, $\widetilde{F}$ is an Einstein metric. So by Theorem 0.1 (or Theorem 0.2), we conclude that ψ must be constant. Thus F is also a Minkowskian.

Remark Theorem 2.2 is essential to proofs of Theorem 0.1 and Theorem 0.2. Thus a nature question arises: whether every conformal transformation, which preserves Riemann Einstein metrics and certain conditions (similar to conformal vector fields), must be homothetic?

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