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Some minimax inequalities for vector valued mappings

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Abstract: By using a linear scalarization function and minimax equalities in scalar case, some types of minimax inequalities for vector-valued functions were established under natural quasi cone convex and properly quasi cone convex assumptions. An example was given to illustrate that the result is a generalization of the corresponding one in reference.

Key words: minimax inequality; vector valued function; natural quasi cone convex ; properly quasi cone convex

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几类向量值极大极小不等式

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摘要: 利用线性标量化函数和实值的极大极小定理，在自然拟凸和真拟凸假设下，证明了几类向量值函数极大极小不等式. 并给出了一个例子说明定理结论是是相关文献结果的推广.

关键词: 极大极小不等式; 向量值函数; 自然拟凸; 真拟凸

0 Introduction

It is well known that minimax problems play important roles in various fields of mathematics. For example, the minimax problems of real functions have been discussed, such as in [1–3].

Vector optimization theory has been widely developed in recent years, and the minimax problems for vector-valued fuctions attracted a lot considerable attention. By introducing reasonable definition for the minimal and maximal point of the vector valued function in an ordered vector space, a number of papers established minimax inequalities under suitable hypotheses of compactness, convexity, and continuity. In [4], Nieuwenhuis obtained a minimax theorem of the

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vector valued function satisfying $f(x,y)=x+y$, but no general result is given. In [5], Tanaka proved the minimax theorem of a separated vector valued fuction. In [6], Tanaka gave some existence theorem for the weak saddle point and established another type of minimax theorem. In [7], Ferro proved a general minimax theorem when the minimal set of a fuction is a single point. In [8], Li et al. obtained some minimax inequalities for set valued mappings by using a section theorem and a linear scalarization function. Recently, in [9,10], Zhang and Li proved a general minimax theorem for set valued mappings.

Motivated by the work of Ferro, the aim of this paper is to study minimax theorems for vector-valued mapping with the weak convex assumption. First, we introduce the natural quasi cone convexity and we obtain the minimax inequality

$$\mathrm{Max}\bigcup_{x\in X_0}\mathrm{Min_w}f(x,Y_0)\subset \mathrm{Min}\Big\{\mathrm{co}\big(\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)\big)\Big\}+C$$

by using a linear scalarization fuction and Sion's minimax theorem. Then we show that $\mathrm{Min}\big\{\mathrm{co}\big(\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)\big)\big\}+C$ is equal to $\mathrm{Min}\big(\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)\big)+C$ under suitable assumption. Our result is a generalization of Ferro's minimax theorem.

The organization of the rest is as follow. In Section 1, some preliminary results are stated, mostly about convexity properties, and existence of minimal point and maximal point. In Section 2, we obtain some minimax inequalities by using sepration theorem and minimax equality in scalar case.

1 Preliminaries

In all that follows, let V be $\mathbf{R}^n$ and X and Y be two real Hausdorff topological vector spaces. Assume that C is a pointed closed convex cone in V with nonempty interior $\mathrm{int}\,C$. Let V^* be the topological dual space of V and $C^*:=\{f\in V^*|f(v)\geqslant 0,\forall v\in C\}$ be the dual cone of C. We define the binary ralation:

$$x\leqslant_C y\Leftrightarrow x\in y-C,\quad \forall x,y\in V$$

Definition 1.1[6] Let X_0 be a convex subset of X. A vector valued function $f:X_0\to V$ is said to be

(i) properly quasi C-convex on X_0 if for every $x_1,x_2\in X_0$ and $\lambda\in[0,1]$

$$\text{either}\quad f(\lambda x_1+(1-\lambda)x_2)\leqslant_C f(x_1)\quad\text{or}\quad f(\lambda x_1+(1-\lambda)x_2)\leqslant_C f(x_2);$$

(ii) natural quasi C-convex on X_0 if for every $x_1,x_2\in X_0$ and $\lambda\in[0,1]$, there exists $\mu\in[0,1]$ such that

$$f(\lambda x_1+(1-\lambda)x_2)\leqslant_C \mu f(x_1)+(1-\mu)f(x_2);$$

(iii) C-convexlike on X_0 if for every $x_1,x_2\in X_0$ and $\lambda\in[0,1]$, there exist $x\in X_0$

$$f(x)\leqslant_C \lambda f(x_1)+(1-\lambda)f(x_2).$$

It is clear that f is C-convexlike on X_0 if and only if $f(X_0)+C$ is convex.

Lemma 1.1[6] Let X_0 be a convex subset of X. Then the following statements hold:

(i) every properly quasi C-convex function is natural quasi C-convex;

(ii) every natural quasi C-convex is C-convex like when f is continuous.

A C-convex function is not always properly quasi C-convex, and vice versa.

Definition 1.2[11] Let $F : X \to 2^V$ be a set-valued mapping.

(i) F is said to be upper semicontinuous (u.s.c.) at $x_0 \in X$ if, for any neighborhood $N(F(x_0))$, there exists a neighborhood $N(x_0)$ of x_0 such that $F(x) \subset N(F(x_0)), \forall x \in N(x_0)$.

(ii) F is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$ if, for any sequence $\{x_n\} \subset X$, $x_n \to x_0$, $v_0 \in F(x_0)$ implies that there exists a sequence $v_n \in F(x_n)$, $\forall n \in N$, such that $v_n \to v_0$.

(iii) F is said to be continuous at $x_0 \in X$ if F is both u.s.c. and l.s.c. at x_0.

Definition 1.3 Let $A \subset V$ be a nonempty subset.

(i) A point $z \in A$ is called a (weakly) minimal point of A if $A \cap (z - C) = \{z\} (A \cap (z - \mathrm{int} C) = \emptyset)$; we denote the set of all (weakly) minimal points of A by $\mathrm{Min}A(\mathrm{Min_w}A)$;

(ii) A point $z \in A$ is called a (weakly) maximal point of A if $A \cap (z + C) = \{z\} (A \cap (z + \mathrm{int} C) = \emptyset)$; we denote the set of all (weakly) maximal points of A by $\mathrm{Max}A(\mathrm{Max_w}A)$;

Lemma 1.2[7] Let $A \subset V$ be a nonempty compact subset. Then (i) $\mathrm{Min}A \neq \emptyset$, $\mathrm{Min_w}A \neq \emptyset$; (ii) $A \subset \mathrm{Min}A + C$, $A \subset \mathrm{Min_w}A + C$; (iii) $\mathrm{Max}A \neq \emptyset$, $\mathrm{Max_w}A \neq \emptyset$; (iv) $A \subset \mathrm{Max}A - C$, $A \subset \mathrm{Max_w}A - C$.

Lemma 1.3[7] Let X_0 and Y_0 be compact subset of X and Y respectively. Let $f : X_0 \times Y_0 \to V$ be a continuous vector-valued mapping, then set-valued mapping

$$\Gamma(x) = \mathrm{Min_w} f(x, Y_0) \quad \text{and} \quad L(y) = \mathrm{Max_w} f(X_0, y)$$

are compact valued and u.s.c on X_0 and Y_0, respectively.

Lemma 1.4[11] Let X_0 be a subset of X and $F : X_0 \to 2^V$ be a set valued mapping. If X_0 is compact and F is u.s.c. with compact values, then $F(X_0)$ is compact.

Lemma 1.5[3] Let X_0 and Y_0 be compact convex subsets of topological vector spaces X and Y, respectively. And let $f : X_0 \times Y_0 \to \mathbf{R}$ be a real function such that

(i) for each $x \in X_0$, $f(x, \cdot)$ is l.s.c. and quasiconvex on Y_0;

(ii) for each $y \in Y_0$, $f(\cdot, y)$ is u.s.c. and quasiconcave on X_0;

Then $\min\limits_{y \in Y_0} \max\limits_{x \in X_0} f(x, y) = \max\limits_{x \in X_0} \min\limits_{y \in Y_0} f(x, y)$.

Lemma 1.6 Let X_0 be convex subset of X, for any linear function $k \in C^*$,$f : X_0 \to V$

(i) if f is natural quasi C-convex on X_0, then $k \circ f : X_0 \to \mathbf{R}$ is quasi convex on X_0;

(ii) if $-f$ is natural quasi C-convex on X_0, then $k \circ f : X_0 \to \mathbf{R}$ is quasi concave on X_0.

Proof We only prove the first assertion. For any $r \in \mathbf{R}$, consider the level set $lev_{\leqslant r} = \{x \in X_0 \mid k \circ f(x) \leqslant r\}$. Suppose $x_1, x_2 \in lev_{\leqslant r}$, $t \in [0, 1]$. Then, $tx_1 + (1 - t)x_2 \in X_0$ and by the natural quasi C-convexity of f, there exist $\mu \in [0, 1]$ and $c \in C$ such that $f(tx_1 + (1 - t)x_2) =$

$\mu f(x_1) + (1-\mu)f(x_2) - c$. We have

$$\begin{aligned} k \circ f(tx_1 + (1-t)x_2) &= k(\mu f(x_1) + (1-\mu)f(x_2) - c) \\ &= \mu k f(x_1) + (1-\mu)k f(x_2) - k(c) \\ &\leqslant \mu r + (1-\mu)r - k(c) \\ &\leqslant r. \end{aligned}$$

Hence $tx_1 + (1-t)x_2 \in lev_{\leqslant r}$. This completes the proof.

2 Minimax theorem

In this section, we present some types of minimax theorems for vector-valued functions.

Theorem 2.1 Let X_0, Y_0 be compact convex sets of X, Y, respectively. If $f : X_0 \times Y_0 \to V$ is a continuous function satisfying

(i) $f(x, \cdot)$ is natural quasi C-convex for every $x \in X_0$,

(ii) $-f(\cdot, y)$ is natural quasi C-convex for every $y \in Y_0$, then

$$\mathrm{Min_w}\Big\{\mathrm{co}\Big(\bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y)\Big)\Big\} \subset \mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min_w} f(x, Y_0) + V \backslash \mathrm{int}\, C.$$

Proof By Lemma 1.3, $\mathrm{Min_w} f(x, Y_0)$ is compact valued and u.s.c. on X_0. Then by Lemma 1.4, $\bigcup\limits_{x\in X_0} \mathrm{Min_w} f(x, Y_0)$ is compact, so

$$\mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min_w} f(x, Y_0) \neq \emptyset.$$

Similarly, $\bigcup\limits_{y\in Y_0} \mathrm{Max_w} f(X_0, y))$ is compact, and then $\mathrm{co}(\bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y))$ is also compact(In finite space, the convex hull of compact set is also compact). So

$$\mathrm{Min_w}\Big\{\mathrm{co}\Big(\bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y)\Big)\Big\} \neq \emptyset.$$

Let $L(y) = \mathrm{Max_w} f(X_0, y)$, $y \in Y_0$. For arbitrary $\alpha \in \mathrm{Min_w}\{\mathrm{co}(\bigcup\limits_{y\in Y_0} L(y))\}$, we have

$$(\alpha - \mathrm{int} C) \bigcap \mathrm{co}(L(y)) = \emptyset.$$

By separation theorem, there exist $\delta \in \mathbf{R}$, $k \in V^*$ such that

$$\begin{aligned} &k(\alpha - c_1) \leqslant \delta \leqslant k(z), \quad \forall z \in \mathrm{co}\Big(\bigcup_{y\in Y_0} L(y)\Big), \quad \forall c_1 \in C \\ &k(\alpha - c_2) < \delta, \quad \forall c_2 \in \mathrm{int}\, C. \end{aligned}$$

It is clearly that $k \in C^*$. Consider the continuous function $g = k \circ f : X_0 \times Y_0 \to \mathbf{R}$. By Lemma 1.6, all the conditions of Lemma 1.5 are satisfied. It follows that

$$\min_{y\in Y_0} \max_{x\in X_0} g(x, y) = \max_{x\in X_0} \min_{y\in Y_0} g(x, y).$$

Since for every $y_0 \in Y_0$, there exist $x_0 \in X_0$ such that

$$f(x_0, y_0) \in \mathrm{Max_w} f(X_0, y_0) \subseteq \mathrm{co}\Big(\bigcup_{y\in Y_0} L(y)\Big),$$

which implies that $k \circ f(x_0, y_0) \geqslant \delta$. Thus we have

$$\max_{x\in X_0} k \circ f(x, y_0)) \geqslant \delta, \quad \forall y_0 \in Y_0,$$

that is, $\min\limits_{y\in Y_0} \max\limits_{x\in X_0} k \circ f(x, y) \geqslant \delta$. By the minimax equality in the scalar case, we obtain that

$$\max_{x\in X_0} \min_{y\in Y_0} k \circ f(x, y) \geqslant \delta.$$

So there exists $x_\alpha \in X_0$ such that $\min\limits_{y\in Y_0} k \circ f(x_\alpha, y) \geqslant \delta$, that is

$$k \circ f(x_\alpha, y) \geqslant \delta \geqslant k(\alpha), \quad \forall y \in Y_0,$$

which means that $\alpha \notin f(x_\alpha, y) + \mathrm{int}C$, $\forall y \in Y_0$; i.e.

$$\alpha \in f(x_\alpha, y) + V\backslash \mathrm{int}C, \quad \forall y \in Y_0.$$

So, by Lemma 1.2

$$\begin{aligned}\mathrm{Min_w}\Big\{\mathrm{co}\Big(\bigcup_{y\in Y_0} L(y)\Big)\Big\} &\subset \bigcup_{x\in X_0} \mathrm{Min_w} f(x, Y_0) + V\backslash \mathrm{int}C \\ &\subset \mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min_w} f(x, Y_0) - C + V\backslash \mathrm{int}C \\ &= \mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min_w} f(x, Y_0) + V\backslash \mathrm{int}C.\end{aligned}$$

Remark 2.1 In scalar case, we always have

$$\mathrm{Min_w}\Big\{\mathrm{co}\Big(\bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y)\Big)\Big\} = \mathrm{Min_w} \bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y)$$

Theorem 2.2 Let X_0, Y_0 be compact convex sets in X, Y, respectively. Suppose f : $X_0 \times Y_0 \to V$ is a continuous function such that

(i) $f(x, \cdot)$ is natural quasi C-convex for every $x \in X_0$;

(ii) $-f(\cdot, y)$ is natural quasi C-convex for every $y \in Y_0$;

(iii) $\mathrm{Max} \bigcup\limits_{x\in X_0} \mathrm{Min_w} f(x, Y_0) \subset \mathrm{Min_w} f(x, Y_0) + C$, for every $x \in X_0$.

Then,

$$\mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min_w} f(x, Y_0) \subset \mathrm{Min}\Big\{\mathrm{co}\Big(\bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y)\Big)\Big\} + C$$

Proof Let $L(y) = \mathrm{Max_w} f(X_0, y)$, $y \in Y_0$. Since $\mathrm{co}(\bigcup\limits_{y\in Y_0} L(y))$ is compact, $\mathrm{co}(\bigcup\limits_{y\in Y_0} L(y))+ C$ is closed and convex. Let $\alpha \in V$ and suppose that $\alpha \notin \mathrm{co}(\bigcup\limits_{y\in Y_0} L(y))+C$. Then, by separation theorems, there exists $\delta, \epsilon \in \mathbf{R}$, $k \in V^*$ such that

$$k(\alpha) \leqslant \delta - \epsilon < \delta \leqslant k(z + c), \quad \forall z \in \mathrm{co}\Big(\bigcup_{y\in Y_0} L(y)\Big), \quad \forall c \in C.$$

It is clearly that $k \in C^*$. let $c = 0 \in C$, and we have

$$k(\alpha) \leqslant \delta - \epsilon < \delta \leqslant k(z), \quad \forall z \in \mathrm{co}\Big(\bigcup_{y \in Y_0} L(y) \Big).$$

Consider the continuous function $g = k \circ f : X_0 \times Y_0 \to \mathbf{R}$. By Lemma 1.6, all the conditions of Lemma 1.5 are satisfied. It follows that

$$\min_{y \in Y_0} \max_{x \in X_0} g(x, y) = \max_{x \in X_0} \min_{y \in Y_0} g(x, y).$$

For every $y \in Y_0$, there exists $x_y \in X_0$ such that

$$f(x_y, y) = z_y \in L(y) \subset \mathrm{co}\Big(\bigcup_{y \in Y_0} L(y) \Big).$$

Hence, for every $y \in Y_0$

$$\max_{x \in X_0} g(x, y) \geqslant g(x_y, y) = k \circ f(x_y, y) \geqslant \delta > \delta - \epsilon \geqslant k(\alpha).$$

That is, $\min\limits_{y \in Y_0} \max\limits_{x \in X_0} g(x, y) > k(\alpha)$. By the minimax equality of scalar value,

$$\max_{x \in X_0} \min_{y \in Y_0} g(x, y) > k(\alpha).$$

So there exist $x_\alpha \in X_0$, such that $\min\limits_{y \in Y_0} g(x_\alpha, y) > k(\alpha)$. That is, for every $y \in Y_0$

$$g(x_\alpha, y) = k \circ f(x_\alpha, y) > k(\alpha).$$

Hence, $k(f(x_\alpha, y) - \alpha) > 0$. This means $f(x_\alpha, y) - \alpha \notin -C$, for every $y \in Y_0$; i.e., $\alpha \notin f(x_\alpha, y) + C$. So we have

$$\alpha \notin f(x_\alpha, Y_0) + C = \mathrm{Min}_{\mathrm{w}} f(x_\alpha, Y_0) + C.$$

Thus, $\mathrm{Min}_{\mathrm{w}} f(x_\alpha, Y_0) + C \subset \mathrm{Min}\{\mathrm{co}(\bigcup\limits_{y \in Y_0} \mathrm{Max}_{\mathrm{w}} f(X_0, y))\} + C$. Clearly, if (iii) holds, then

$$\mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_{\mathrm{w}} f(x, Y_0) \subset \mathrm{Min}\Big\{ \mathrm{co}\Big(\bigcup_{y \in Y_0} \mathrm{Max}_{\mathrm{w}} f(X_0, y) \Big) \Big\} + C.$$

Remark 2.2 Assumption(iii) is raised by Ferro in [7]. This condition always holds in scalar case.

Theorem 2.3 Let X_0, Y_0 be compact convex sets in X, Y, respectively. Suppose $f : X_0 \times Y_0 \to V$ is a continuous vector-valued function such that

(i) $f(x, \cdot)$ is natural quasi C-convex for every $x \in X_0$;

(ii) $-f(\cdot, y)$ is properly quasi C-convex for every $y \in Y_0$;

(iii) $\mathrm{Max} \bigcup\limits_{x \in X_0} \mathrm{Min}_{\mathrm{w}} f(x, Y_0) \subset \mathrm{Min}_{\mathrm{w}} f(x, Y_0) + C$, for every $x \in X_0$.

Then, $\mathrm{Max} \bigcup\limits_{x \in X_0} \mathrm{Min}_{\mathrm{w}} f(x, Y_0) \subset \mathrm{Min} \bigcup\limits_{y \in Y_0} \mathrm{Max}_{\mathrm{w}} f(X_0, y) + C$.

Proof By Lemma 2.3 in [7], $L(y)=\mathrm{Max_w}f(X_0,y)$ is continuous and single valued. Now let $x\in X_0$ be arbitrary and $y_1,y_2\in Y_0$. By the natural quasi convexity of $f(x,\cdot)$, there exist $\mu\in[0,1]$ such that

$$f(x,\lambda y_1+(1-\lambda)y_2)\leqslant_C \mu f(x,y_1)+(1-\mu)f(x,y_2).$$

Since $L(y)=\mathrm{Max_w}f(X_0,y)$ is single valued, by Lemma 1.2

$$\begin{aligned}f(x,\lambda y_1+(1-\lambda)y_2) &\leqslant_C \mu f(x,y_1)+(1-\mu)f(x,y_2)\\ &\leqslant_C \mu\mathrm{Max_w}f(X_0,y_1)+(1-\mu)\mathrm{Max_w}f(X_0,y_2)\end{aligned}$$

Since $x\in X_0$ is arbitrary, then

$$\mathrm{Max_w}f(X_0,\lambda y_1+(1-\lambda)y_2)\leqslant_C \mu\mathrm{Max_w}f(X_0,y_1)+(1-\mu)\mathrm{Max_w}f(X_0,y_2).$$

So $L(y)=\mathrm{Max_w}f(X_0,y)$ is natural quasi convex on Y_0 and continuous. Then by lemma 1.1, $L(y)=\mathrm{Max_w}f(X_0,y)$ is convexlike on Y_0. Hence,

$$\begin{aligned}\mathrm{Min}\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)+C &= \bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)+C\\ &= \mathrm{co}\Big\{\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)+C\Big\}\\ &= \mathrm{co}\Big(\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)\Big)+C\\ &= \mathrm{Min}\Big\{\mathrm{co}\Big(\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)\Big)\Big\}+C\end{aligned}$$

The conclusion holds by Theorem 2.3.

Remark 2.3 In [7], the same result is proved when condition (i) is replaced by $f(x,\cdot)$ is C-convex for every $x\in X_0$. Since every C-convex function is natural quasi C-convex, the Theorem 3.1 in [7] is a special case of Theorem 2.3. The following example explains the case.

Example 2.1 Let $X_0=Y_0=[0,1]$, $V=\mathbf{R}^2$, $C=\mathbf{R}^2_+=\{(v_1,v_2)\in\mathbf{R}^2|v_1\geqslant 0,v_2\geqslant 0\}$ defined a function $f:X_0\times Y_0\to V$ by $f(x,y)=(y^2x,(1-y^2)x)$. Then, $f(x,\cdot)$ is natural quasi C-convex for every $x\in X_0$ and $-f(\cdot,y)$ is properly quasi C-convex for every $y\in Y_0$. Since $f(x,\cdot)$ is neither C-convex or properly quasi C-convex, we cannot claim the conclusion by Theorem 3.1 in [7]. However, for every $x\in X_0$

$$\mathrm{Min_w}f(x,Y_0)=f(x,Y_0)=\{(y^2x,(1-y^2)x)\mid y\in[0,1]\};$$

$$\mathrm{Max}\bigcup_{x\in X_0}\mathrm{Min_w}f(x,Y_0)=\{(y^2,1-y^2)\mid y\in[0,1]\}.$$

Hence the assumption (iii) in Theorem 2.3 holds, then

$$\mathrm{Max}\bigcup_{x\in X_0}\mathrm{Min_w}f(x,Y_0)\subset\mathrm{Min}\bigcup_{y\in Y_0}\mathrm{Max_w}f(X_0,y)+C.$$

In fact, $\mathrm{Max_w} f(X_0, y) = (y^2, 1-y^2)$; $\mathrm{Min} \bigcup_{y\in Y_0} \mathrm{Max_w} f(X_0, y) = \{(y^2, 1-y^2) \mid y \in [0,1]\}$.

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如果 g 不依赖于 y, 由式(15)和引理3 (平移不变性定理)、引理5 (比较定理)得

$$\begin{aligned}\varepsilon_{g,T}[Y+b] &\leqslant \varepsilon_{g,T}[1_{(Y>0)}+b] = \varepsilon_{g,T}[1_{(Y>0)}]+b \\ &= \varepsilon_{g,T}[1_{(Y>0)}]+\varepsilon_{g,T}[b].\end{aligned}$$

证毕.

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