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Multidimensional credibility models with random common effects

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Abstract: Multidimensional credibility models with a random common effect were built, and the inhomogeneous and homogeneous credibility estimators were derived. Further, some properties of these estimators were presented. Analogy to the classical credibility theory, the multidimensional credibility estimators with the common effect can also be expressed as the weighted sum of individual mean, collective mean and collective premium, where these weights are so-called credibility factor matrices. Finally, a numerical example was given to show the calculations of multidimensional credibility estimators.

Key words: credibility premium; random common effect; multidimensional credibility; orthogonal projection.

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具有共同效应的多维信度模型

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摘要: 建立了具有随机共同效应的多维信度模型, 得到了该模型中的非齐次与齐次信度估计, 并讨论了这些估计的性质. 类似于经典的信度理论, 具有共同效应的多维信度仍然可以表示为个体数据、聚合数据与聚合保费的加权和, 其权重被称为信度因子矩阵. 最后, 给出一个数值例子说明了多维信度估计的计算方法.

关键词: 信度保费; 共同效应; 多维信度; 正交投影

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0 Introduction

In business such as motor insurance, tariffs have become very refined since the premiums depend on many rating factors and co-variables. To calculate such tariffs a regular method is using multidimensional credibility. Multidimensional credibility models were first proposed in [1], and subsequently were applied to many other situations. The essential feature of multidimensional credibility is to consider the observations of different categories and to use this information in a methodologically consistent way. More detail about multidimensional credibility can be found in [2].

In multidimensional credibility models, we consider a claim sequence of p-dimension random variable $X_{i1}, X_{i2}, \cdots X_{in_i}$ over n_i time periods for the ith contract, which can be characterized by a random risk parameter Θ_i whose (prior) distribution is $\pi(\theta)$. Under certain assumptions, the standard paradigms using empirical Bayes method can be applied to all the historical data to estimate the prior distribution $\pi(\theta)$, and in turn to predict the future loss of each X_i, $i = 1, 2, \cdots, K$, at the next period. In the multidimensional credibility, Jewell (in [1]) assumed that the random vectors $(X_{i1}, X_{i2}, \cdots X_{in_i}, \Theta_i)$, $i = 1, 2, \cdots, K$, are independent across individuals (independence over risks) for each i.

Such independence assumptions may be appropriate in some practical situations. However, it is far from being a universal structure. In fact, it has been recognized that there many important insurance scenarios where these assumptions are violated. For example, there exist many common factors which affect the claims of the insurance portfolios. Fortunately, in recent years, more and more remarkable efforts are put in the existing actuarial literature to study the impacts of dependent risks in various aspects; see e.g., [3-6], and the references therein. In credibility theory, a special dependence induced by the common effect was proposed in [7], and they derived the credibility estimators of risk premium in Normal-Normal case. Then this models were subsequently investigated in [8] who generalized the results of [7] to distribution-free credibility model.

In this paper, we aim at studying multidimensional credibility with a dependence structure characterized by a random common effect. The rest of the paper is arranged as follows. In Section 1, models and assumptions are introduced and some preliminaries are prepared. Section 2 derives the credibility formulae for the multidimensional credibility model with a common effect and some remarks are presented. The homogeneous credibility estimators are discussed in section 3. In Section 4, a numerical example is presented to show the calculation of multidimensional credibility with a common effect and some conclusions are made.

1 Model formulations and preliminaries

Consider a portfolio of K insured individuals. Firstly, the risk quality of an individual i is characterized by a risk parameter Θ_i and the common effect which is represented as a random variable β. The available claim data is $\{X_{ij}, i = 1, 2, \cdots K, j = 1, 2, \cdots, n_i\}$, where X_{ij} is a p-dimension random vector. In this paper, in order to simplify calculations, the balanced credibility models are considered, i.e, $n_1 = n_2 = \cdots = n_K = n$. However, with some slightly revisions, it is easily to extend it to the unbalanced cases. Formally, the assumptions of the models are stated as the following.

Assumption 1.1 The common effect random variable β has known expectation $\mathrm{E}(\beta)=\mu_\beta$ and variance $\mathrm{Var}(\beta)=\sigma_\beta^2$.

Assumption 1.2 Given β, the random risk parameter vectors Θ_i, $i=1,\cdots,K$, are mutually independent and identically distributed, with the same structure distribution $\pi(\theta|\beta)$;

Assumption 1.3 For a fixed contract i, given the common effect β and the structure parameter Θ_i, the claims random vector $X_{i1},X_{i2},\cdots,X_{in}$ are conditionally independent and identically distributed with conditional expectation $\mathrm{E}(X_{ij}|\Theta_i,\beta)=\mu(\Theta_i,\beta)$ and conditional variance $\mathrm{Var}(X_{ij}|\Theta_i,\beta)=\sum(\Theta_i,\beta)$, where $X_{ij}=\begin{pmatrix} X_{ij}^{(1)} \\ \vdots \\ X_{ij}^{(p)} \end{pmatrix}$ is a p-dimensional random vector, and $\sum(\Theta_i,\beta)$ is a $p\times p$ covariance matrix. We also assume that

$$\begin{aligned}&\mathrm{E}\left[\mu(\Theta_i,\beta)|\beta\right]=\mu_1(\beta),\quad \mathrm{Var}\left[\mu(\Theta_i,\beta)|\beta\right]=S(\beta),\quad \mathrm{E}\left[\sum(\Theta_i,\beta)|\beta\right]=\sum(\beta),\\ &\mathrm{Var}\left[\mu_1(\beta)\right]=T_0,\quad \mathrm{E}\left[\mu_1(\beta)\right]=\mu_0,\quad \mathrm{E}(S(\beta))=S_0,\quad \mathrm{E}[\sum(\beta)]={\sum}_0.\end{aligned}\tag{1.1}$$

Write $\overline{X_i}=\frac{1}{n}\sum_{j=1}^n X_{ij}$ for the average of the claim experience of individual i and $\overline{\overline{X}}=\frac{1}{K}\sum_{i=1}^K\overline{X_i}$ for the overall average claim experience of all individuals. Our goals are to predict/estimate the future claim vector $X_{i,n+1}$ based on the linear combinations of the overall sample $\{X_{ij},i=1,2,\cdots,K,j=1,2,\cdots,n\}$ in credibility theory. Firstly, we denote inhomogeneous linear function classes of the samples as

$$\begin{aligned}L(X,1)=\Big\{\widehat{X_{i,n+1}}=\underset{p\times1}{A}+\sum_{s=1}^K\sum_{t=1}^n\underset{p\times p}{B_{st}}X_{st},\\ \text{where } A,B_{st} \text{ are non-random vector or matrices}\Big\},\end{aligned}\tag{1.2}$$

and the homogeneous class as

$$Le(X)=\left\{\widehat{X_{i,n+1}}=\sum_{s=1}^K\sum_{t=1}^n\underset{p\times p}{B_{st}}X_{st},\ \text{with } \mathrm{E}\left(\widehat{X_{i,n+1}}\right)=\mathrm{E}(X_{i,n+1})\right\}\tag{1.3}$$

respectively. Therefore, the estimators of $X_{i,n+1}$ which are optimal in the classes $L(X,1)$ and $Le(X)$ are called inhomogeneous and homogeneous credibility estimators which are denoted by $\widehat{X_{i,n+1}}^*$ and $\widehat{X_{i,n+1}}^H$ respectively. Here, "Optimal" means that the mean square error matrix of the the estimator arrive at the minimal value in the matrix sense, i.e., for any other inhomogeneous (or homogeneous) linear estimator $\widehat{X_{i,n+1}}$, $\mathrm{MSE}(\widehat{X_{i,n+1}})-\mathrm{MSE}(\widehat{X_{i,n+1}}^*)$ (or $\mathrm{MSE}(\widehat{X_{i,n+1}})-\mathrm{MSE}(\widehat{X_{i,n+1}}^H)$) is a non-negative definite matrix, where the mean square error matrix of $\widehat{X}$ is defined as

$$\mathrm{MSE}(\widehat{X})=\mathrm{E}\left(\widehat{X}-X\right)\left(\widehat{X}-X\right)'.\tag{1.4}$$

So, in order to find $\widehat{X_{i,n+1}}^*$ and $\widehat{X_{i,n+1}}$, we must solve the following optimization problems

$$\min_{A,B_{st}} \mathrm{E}\left[\left(X_{i,n+1} - A - \sum_{s=1}^{K}\sum_{t=1}^{n} B_{st}X_{st}\right)\left(X_{i,n+1} - A - \sum_{s=1}^{K}\sum_{t=1}^{n} B_{st}X_{st}\right)'\right], \tag{1.5}$$

and

$$\begin{cases} \min\limits_{A,B_{st}} \mathrm{E}\left[\left(X_{i,n+1} - \sum_{s=1}^{K}\sum_{t=1}^{n} B_{st}X_{st}\right)\left(X_{i,n+1} - \sum_{s=1}^{K}\sum_{t=1}^{n} B_{st}X_{st}\right)'\right] \\ \text{with } \sum_{s=1}^{K}\sum_{t=1}^{n} B_{st}\mathrm{E}\left(X_{st}\right) = \mathrm{E}(X_{i,n+1}) \end{cases} \tag{1.6}$$

respectively. For the convenience of calculation, we give the following lemma. Its proof can be found in [8].

Lemma 1.1 Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be a random vector with expectation $\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$ and covariance matrix $\begin{pmatrix} \sum_{XX} & \sum_{XY} \\ \sum_{YX} & \sum_{YY} \end{pmatrix}$. Then

(1) $\mathrm{E}(Y - A - BX)(Y - A - BX)'$ can be minimized by

$$A = \mu_Y - \sum_{YX}\sum_{XX}^{-1}\mu_X \quad \text{and} \quad B = \sum_{YX}\sum_{XX}^{-1};$$

(2) Under the constraint $\mu_Y = B\mu_X$, $\mathrm{E}\left(Y - BX\right)\left(Y - BX\right)'$ can be minimized by

$$B = \left(\sum_{YX} + \frac{\left(\mu_Y - \sum_{YX}\sum_{XX}^{-1}\mu_X\right)\mu_X'}{\mu_X'\sum_{XX}^{-1}\mu_X}\right)\sum_{XX}^{-1}.$$

Consequently, Y can be optimally predicted in the class of inhomogeneous linear functions of X by

$$\mathrm{proj}(Y|L(X,1)) = \mu_Y + \sum_{YX}\sum_{XX}^{-1}(X - \mu_X), \tag{1.7}$$

and in the class of homogeneous linear functions of X by

$$\mathrm{proj}(Y|Le(X)) = \left(\sum_{YX} + \frac{\left(\mu_Y - \sum_{YX}\Sigma_{XX}^{-1}\mu_X\right)\mu_X'}{\mu_X'\sum_{XX}^{-1}\mu_X}\right)\sum_{XX}^{-1}X, \tag{1.8}$$

where $\mathrm{proj}(Y|M)$ represents the projection of Y on the space M. Expressions (1.7) and (1.8) are referred to as respectively the inhomogeneous and homogeneous projections (estimators) of Y onto X. For a degenerated random variable $Y = \mu_Y$, $\Sigma_{YX} = 0$ and so (1.8) can be simplified to

$$\mathrm{proj}(\mu_Y|X) = \frac{\mu_Y\mu_X'}{\mu_X'\sum_{XX}^{-1}\mu_X}\sum_{XX}^{-1}X. \tag{1.9}$$

Therefore, from the Lemma 1.1, the inhomogeneous credibility estimator can be found by the following formula.

$$\begin{aligned}\widehat{X_{i,n+1}}^{*} &= \operatorname{proj}\left(X_{i,n+1}\,|L\left(X,1\right)\right)\\ &= \mathrm{E}\left[X_{i,n+1}\right]+\operatorname{Cov}\left(X_{i,n+1},X\right)\operatorname{Cov}\left(X,X\right)^{-1}\left(X-\mathrm{E}\left(X\right)\right),\end{aligned} \tag{1.10}$$

where $X=(X_1',\cdots,X_K')'$ and $X_i=(X_{i1}',\cdots,X_{in}')'$. For the projection operator "proj", we have the following results.

Lemma 1.2 For the two closed subspace $M'\subset M\subset L^2$ and $Y\in L^2$, then

$$\operatorname{proj}(X|M')=\operatorname{proj}\left(\operatorname{proj}\left(X|M\right)|M'\right), \tag{1.11}$$

where $L^2=\{Y: Y \text{ is random vector with the covariance matrix } \operatorname{Cov}\left(Y,Y\right)\}$. The Eqn. (1.11) are so-called the iterativity of projections. The proof can be found in [2].

Since $Le\left(X\right)\subseteq L\left(X,1\right)$, from (1.11) the homogeneous credibility can be derived as

$$\begin{aligned}\widehat{X_{i,n+1}}^{H} &= \operatorname{proj}\left(X_{i,n+1}\,|Le\left(X\right)\right)\\ &= \operatorname{proj}\left(\operatorname{proj}\left(X_{i,n+1}\,|L\left(X,1\right)\right)|Le\left(X\right)\right)\\ &= \operatorname{proj}\left(\widehat{X_{i,n+1}}^{*}\,|Le\left(X\right)\right).\end{aligned} \tag{1.12}$$

The (1.12) is very useful to derive the homogeneous credibility estimators, which is discussed in the next section. The following lemma gives a convenient calculation for matrix converse. The proofs can be found in [9].

Lemma 1.3 If the A, B, C and D are matrices with adaptive orders, then the following formula holds true,

$$(A+BCD)^{-1}=A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}. \tag{1.13}$$

2 Multidimensional credibility model

In order to derive the credibility estimator of $X_{i,n+1}$, from (1.6), we should calculate the covariance matrix $\operatorname{Cov}\left(X_{i,n+1},X\right)$ and $\operatorname{Cov}\left(X,X\right)$. The lemma below states some simple but fundamental features of the dependence structure just specified.

Theorem 2.4 Under the assumptions of 1.1, 1.2, and 1.3, we get following conclusions:
(1) The means of $X_{i,n+1}$ and X_i are

$$\mathrm{E}\left[X_{i,n+1}\right]=\mu_0,\ \mathrm{E}\left(X_i\right)=\mathbf{1}_n\otimes\mu_0,\quad i=1,2,\cdots,K, \tag{2.14}$$

respectively, where $\mathbf{1}_n$ is an n-vector with 1 in all the n entries.
(2) The covariance of X is given by

$$\sum_{XX}\triangleq\operatorname{Cov}\left(X,X\right)=I_K\otimes\left(I_n\otimes\sum_0+\mathbf{1}_n\mathbf{1}_n'\otimes S_0\right)+\mathbf{1}_{nK}\mathbf{1}_{nK}'\otimes T_0, \tag{2.15}$$

where $\otimes$ indicates the Kronecker product of matrices.

(3) The covariance between $X_{i,n+1}$ and X is

$$\sum_{X_{i,n+1},X} \triangleq \mathrm{Cov}\left(X_{i,n+1}, X\right) = \mathbf{1}'_{nK} \otimes T_0 + \left(e'_i \otimes \mathbf{1}'_n\right) \otimes S_0, \tag{2.16}$$

where e_i is a vector with 1 in the ith entry and 0 in the other entries.
(4) The inverse of the variance matrix of X is given by

$$\sum_{XX}^{-1} = I_K \otimes \Phi - \begin{pmatrix} \Phi\left(\mathbf{1}_n \otimes I_p\right) \\ \vdots \\ \Phi\left(\mathbf{1}_n \otimes I_p\right) \end{pmatrix} \Omega^{-1} \begin{pmatrix} \left(\mathbf{1}'_n \otimes I_p\right)\Phi & \cdots & \left(\mathbf{1}'_n \otimes I_p\right)\Phi \end{pmatrix}, \tag{2.17}$$

where

$$\Phi = I_n \otimes \sum_0^{-1} - \left(\mathbf{1}_n \otimes \sum_0^{-1}\right)\left(n\sum_0^{-1} + S_0^{-1}\right)^{-1}\left(\mathbf{1}_n \otimes \sum_0^{-1}\right)' \quad \text{and}$$

$$\Omega = T_0^{-1} + nK\left(\sum_0 + nS_0\right)^{-1}. \tag{2.18}$$

Proof (1) Write $\Theta = (\Theta_1, \Theta_2, \cdots, \Theta_K)'$. From the dual conditional expectation theorem, one can get

$$\mathrm{E}\left[X_{i,n+1}\right] = \mathrm{E}\left[\mathrm{E}\left(X_{i,n+1}|\Theta, \beta\right)\right] = \mathrm{E}\left[\mathrm{E}\left(\mu\left(\Theta_i, \beta\right)|\beta\right)\right] = \mathrm{E}\left[\mu_1\left(\beta\right)\right] = \mu_0,$$

and $\mathrm{E}\left(X_i\right) = \mathrm{E}\left[\mathrm{E}\left(X_i|\Theta_i, \beta\right)\right] = \mathbf{1}_n \otimes \mu_0$.

(2) Secondly, from the assumptions of 1.1, 1.2 and 1.3, we have

$$\begin{aligned}\mathrm{Cov}(X_{ij}, X_{st}) &= \mathrm{Cov}\left(\mu\left(\Theta_i, \beta\right), \mu\left(\Theta_i, \beta\right)\right) + \mathrm{E}\left[\mathrm{Cov}\left(X_{ij}, X_{st}|\Theta, \beta\right)\right] \\ &= \mathrm{Cov}\left(\mu_1\left(\beta\right), \mu_1\left(\beta\right)\right) + \mathrm{E}\left[\mathrm{Cov}(\mu\left(\Theta_i, \beta\right), \mu\left(\Theta_i, \beta\right)|\beta)\right] + \mathrm{E}\left[\mathrm{Cov}(X_{ij}, X_{st}|\Theta, \beta)\right] \\ &= \begin{cases} T_0, & i \neq s, \\ S_0 + T_0, & i = s,\ j \neq t, \\ S_0 + T_0 + \sum_0, & i = s,\ j = t. \end{cases}\end{aligned}$$

Then

$$\mathrm{Cov}(X_i, X_s) = \begin{cases} \mathbf{1}_n\mathbf{1}'_n \otimes T_0, & i \neq s, \\ \mathbf{1}_n\mathbf{1}'_n \otimes \left(T_0 + S_0\right) + I_n \otimes \sum_0, & i = s. \end{cases}$$

Therefore, the covariance matrix of X is given by

$$\sum_{XX} = \mathrm{Cov}\left(X, X\right) = I_K \otimes \left(I_n \otimes \sum_0 + \mathbf{1}_n\mathbf{1}'_n \otimes S_0\right) + \mathbf{1}_{nK}\mathbf{1}'_{nK} \otimes T_0,$$

which gives (2.15).

(3) To check (2.16), notice that $\mathrm{Cov}\left(X_{i,n+1}, X_{st}|\Theta, \beta\right) = 0$, then

$$\mathrm{Cov}\left(X_{i,n+1}, X_{st}\right) = \mathrm{Cov}\left(\mu\left(\Theta_i, \beta\right), \mu\left(\Theta_s, \beta\right)\right) = \begin{cases} T_0 + S_0, & i = s, \\ T_0, & i \neq s, \end{cases}$$

$$\mathrm{Cov}\left(X_{i,n+1}, X_s\right) = \begin{cases} \mathbf{1}'_n \otimes \left(T_0 + S_0\right), & i = s, \\ \mathbf{1}'_n \otimes T_0, & i \neq s. \end{cases}$$

Thus $\sum_{X_{i,n+1},X} = \mathbf{1}'_{nK} \otimes T_0 + (e'_i \otimes \mathbf{1}'_n) \otimes S_0$.

(4) Finally, from (1.13) and notice $\mathbf{1}_n\mathbf{1}'_n \otimes S_0 = \begin{pmatrix} I_p \\ \vdots \\ I_p \end{pmatrix} S_0 \begin{pmatrix} I_p & \cdots & I_p \end{pmatrix}$, we have

$$\left(I_n \otimes \sum_0 + \mathbf{1}_n\mathbf{1}'_n \otimes S_0\right)^{-1} = I_n \otimes \sum_0^{-1} - \left(\mathbf{1}_n \otimes \sum_0^{-1}\right) \left(n\sum_0^{-1} + S_0^{-1}\right)^{-1} \left(\mathbf{1}_n \otimes \sum_0^{-1}\right)' = \Phi. \tag{2.19}$$

Hence,

$$\Phi\left(\mathbf{1}_n \otimes I_p\right) = \begin{pmatrix} \sum_0^{-1} \\ \vdots \\ \sum_0^{-1} \end{pmatrix} - \begin{pmatrix} \sum_0^{-1} \\ \vdots \\ \sum_0^{-1} \end{pmatrix} \left(n\sum_0^{-1} + S_0^{-1}\right)^{-1} n\sum_0^{-1}, \tag{2.20}$$

$$\left(\mathbf{1}_n \otimes I_p\right)'\Phi = \begin{pmatrix} \sum_0^{-1} & \cdots & \sum_0^{-1} \end{pmatrix} - \begin{pmatrix} \sum_0^{-1} & \cdots & \sum_0^{-1} \end{pmatrix} \left(n\sum_0^{-1} + S_0^{-1}\right)^{-1} n\sum_0^{-1}, \tag{2.21}$$

and

$$\left(\mathbf{1}_{nK} \otimes I_p\right)'\left(\mathbf{1}_K \otimes \Phi\right)\left(\mathbf{1}_{nK} \otimes I_p\right) = Kn\sum_0^{-1}\left(n\sum_0^{-1} + S_0^{-1}\right)^{-1} S_0^{-1} = Kn\left(\sum_0 + nS_0\right)^{-1}.$$

Observe that $\mathbf{1}_{nK} = \mathbf{1}_K \otimes \mathbf{1}_n$. Then using(1.13) again, we have

$$\sum_{XX}^{-1} = \left[I_K \otimes \left(I_n \otimes \sum_0 + \mathbf{1}_n\mathbf{1}'_n \otimes S_0\right) + \mathbf{1}_{nK}\mathbf{1}'_{nK} \otimes T_0\right]^{-1}$$
$$= I_K \otimes \Phi - \begin{pmatrix} \Phi\left(\mathbf{1}_n \otimes I_p\right) \\ \vdots \\ \Phi\left(\mathbf{1}_n \otimes I_p\right) \end{pmatrix} \Omega^{-1} \begin{pmatrix} \left(\mathbf{1}'_n \otimes I_p\right)\Phi & \cdots & \left(\mathbf{1}'_n \otimes I_p\right)\Phi \end{pmatrix},$$

which gives (2.17). The theorem is thus proved.

At this point, having revealed the structural features of the dependence in the models, we can derive the inhomogeneous credibility estimator in the following theorem.

Theorem 2.5 Under the assumptions 1.1, 1.2, and 1.3, the optimal linear inhomogeneous unbiased estimator for $X_{i,n+1}$ by solving the optimization problem (1.5) is given by

$$\widehat{X_{i,n+1}}^{*} = Z_1\overline{X_i} + Z_2\overline{\overline{X}} + \left(I_p - Z_1 - Z_2\right)\mu_0, \tag{2.22}$$

where the credibility factors are

$$Z_1 = nS_0\left(\sum_0 + nS_0\right)^{-1}, \quad Z_2 = nK\sum_0\left(\sum_0 + nS_0\right)^{-1} T_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1}. \tag{2.23}$$

Proof From Lemma 1.1, we have $\widehat{X_{i,n+1}}^{*} = \mathrm{proj}\left(X_{i,n+1}\left|L\left(X,1\right)\right.\right)$. We thus prove the theorem by computing $\mathrm{proj}\left(X_{i,n+1}\left|L\left(X,1\right)\right.\right) = \mathrm{E}\left[X_{i,n+1}\right] + \sum_{X_{i,n+1},X}\sum_{XX}^{-1}\left(X-\mathrm{E}\left(X\right)\right)$. From the theorem 2.4, we have known that

$$\mathrm{E}\left[X_{i,n+1}\right]=\mu_0,\ \mathrm{E}\left(X\right)=\mathbf{1}_{nK}\otimes\mu_0,\quad \sum_{\mu(\Theta_i,\beta),X}=\mathbf{1}'_{nK}\otimes T_0+\left(e'_i\otimes\mathbf{1}'_n\right)\otimes S_0,$$

and

$$\sum_{XX}^{-1}=I_K\otimes\Phi-\begin{pmatrix}\Phi\left(\mathbf{1}_n\otimes I_p\right)\\ \vdots\\ \Phi\left(\mathbf{1}_n\otimes I_p\right)\end{pmatrix}\Omega^{-1}\begin{pmatrix}\left(\mathbf{1}'_n\otimes I_p\right)\Phi & \cdots & \left(\mathbf{1}'_n\otimes I_p\right)\Phi\end{pmatrix}.$$

Firstly, from some matrix computations, we can derive

$$\begin{aligned}
&\left(e'_i\otimes\mathbf{1}'_n\right)\otimes S_0\sum_{XX}^{-1}\left(X-EX\right)\\
&=nS_0\left(\sum_0+nS_0\right)^{-1}\left(X_i-\overrightarrow{\mu_0}\right)-n^2KS_0\left(\sum_0+nS_0\right)^{-1}T_0\left(\sum_0+nS_0+nKT_0\right)^{-1}\left(\overline{\overline{X}}-\overrightarrow{\mu_0}\right)\\
&=nK\left[I_p-nS_0\left(\sum_0+nS_0\right)^{-1}\right]T_0\left(\sum_0+nS_0+nKT_0\right)^{-1}\left(\overline{\overline{X}}-\overrightarrow{\mu_0}\right)\\
&=nK\sum_0\left(\sum_0+nS_0\right)^{-1}T_0\left(\sum_0+nS_0+nKT_0\right)^{-1}\left(\overline{\overline{X}}-\overrightarrow{\mu_0}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{1}'_{nK}\otimes T_0\sum_{XX}^{-1}\left(X-EX\right)\\
&=nKT_0\left(\sum_0+nS_0\right)^{-1}\left(T_0^{-1}+Kn\left(\sum_0+nS_0\right)^{-1}\right)^{-1}T_0^{-1}\left(\overline{\overline{X}}-\overrightarrow{\mu_0}\right)\\
&=nKT_0\left(\sum_0+nS_0+nKT_0\right)^{-1}\left(\overline{\overline{X}}-\overrightarrow{\mu_0}\right).
\end{aligned}$$

Therefore, the theorem follows from the following computations.

$$\begin{aligned}
&\mathrm{proj}\left(X_{i,n+1}\left|L\left(X,1\right)\right.\right)\\
&=\mathrm{E}\left[X_{i,n+1}\right]+\sum_{X_{i,n+1},X}\sum_{XX}^{-1}\left(X-\mathrm{E}\left(X\right)\right)\\
&=\mu_0+\left(e'_i\otimes\mathbf{1}'_n\right)\otimes S_0\sum_{XX}^{-1}\left(X-EX\right)+\mathbf{1}'_{nK}\otimes T_0\sum_{XX}^{-1}\left(X-\mathrm{E}\left(X\right)\right)
\end{aligned}$$

$$= \mu_0 + nS_0\left(\sum_0 + nS_0\right)^{-1}(X_i - \mu_0) + nK\sum_0\left(\sum_0 + nS_0\right)^{-1} T_0$$
$$\left(\sum_0 + nS_0 + nKT_0\right)^{-1}\left(\overline{\overline{X}} - \mu_0\right)$$
$$= Z_1\overline{X_i} + Z_2\overline{\overline{X}} + (I_p - Z_1 - Z_2)\,\mu_0.$$

which give the results of the theorem.

From Theorem 2.5, we can see that the credibility estimators of $X_{i,n+1}$ is the weighted sums of individual sample mean $\overline{X_i}$, the overall sample mean $\overline{\overline{X}}$ and the collective mean μ_0. The Z_1 and Z_2 in the Theorem 2.5 are credibility factor matrices. In the univariate case, as we known, the credibility factor must satisfy $0 \leqslant Z_i \leqslant 1$ (see [8]). In our models, we derived the following proposition.

Proposition 2.6 $Z_1 \geqslant 0$, $I_p - Z_1 \geqslant 0$, $Z_2 \geqslant 0$, $I_p - Z_2 \geqslant 0$, $I_p - Z_1 - Z_2 \geqslant 0$, where "$\geqslant 0$" means the matrix is non-negative definite.

Proof Obviously, the matrices $S_0, \sum_0, T_0$ are all non-positive definite since they are covariance matries, then we have

$$Z_1 = nS_0\left(\sum_0 + nS_0\right)^{-1} \geqslant 0,\ Z_2 = nK\sum_0\left(\sum_0 + nS_0\right)^{-1} T_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1} \geqslant 0,$$

and $I_p - Z_1 = \sum_0\left(\sum_0 + nS_0\right)^{-1} \geqslant 0$.

We can easily see from some matrix calculations that

$$I_p - Z_2 = I_p - nK\sum_0\left(\sum_0 + nS_0\right)^{-1} T_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1}$$
$$= \left[\sum_0 + nS_0 + nKT_0 - \sum_0\left(\sum_0 + nS_0\right)^{-1} nKT_0\right]\left(\sum_0 + nS_0 + nKT_0\right)^{-1}$$
$$= \left[\sum_0 + nS_0 + \left(I_p - \sum_0\left(\sum_0 + nS_0\right)^{-1}\right) nKT_0\right]\left(\sum_0 + nS_0 + nKT_0\right)^{-1}$$
$$= \left[\sum_0 + nS_0 + nS_0\left(\sum_0 + nS_0\right)^{-1} nKT_0\right]\left(\sum_0 + nS_0 + nKT_0\right)^{-1}$$
$$\geqslant 0,$$

and

$$
\begin{aligned}
& I_p - Z_1 - Z_2 \\
& = I_p - nS_0\left(\sum_0 + nS_0\right)^{-1} - nK\sum_0\left(\sum_0 + nS_0\right)^{-1} T_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1} \\
& = \left[\sum_0 + nS_0 + nS_0\left(\sum_0 + nS_0\right)^{-1} nKT_0\right]\left(\sum_0 + nS_0 + nKT_0\right)^{-1} - nS_0\left(\sum_0 + nS_0\right)^{-1} \\
& = \left(\sum_0 + nS_0\right)\left(\sum_0 + nS_0 + nKT_0\right)^{-1} + nS_0\left(\sum_0 + nS_0\right)^{-1} \\
& \quad \left[nKT_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1} - I_p\right] \\
& = \left(\sum_0 + nS_0\right)\left(\sum_0 + nS_0 + nKT_0\right)^{-1} - nS_0\left(\sum_0 + nS_0\right)^{-1} \\
& \quad \left(\sum_0 + nS_0\right)\left(\sum_0 + nS_0 + nKT_0\right)^{-1} \\
& = \left(\sum_0 + nS_0\right)\left(\sum_0 + nS_0 + nKT_0\right)^{-1} - nS_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1} \\
& = \sum_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1} \\
& \geqslant 0.
\end{aligned}
$$

Remark 1　We note from Theorem 2.5 that

$$
\widehat{X_{i,n+1}}^{*} = \widehat{X_{i,n+1}}^{c} + nK\sum_0\left(\sum_0 + nS_0\right)^{-1} T_0\left(\sum_0 + nS_0 + nKT_0\right)^{-1}(\overline{\overline{X}} - \mu_0), \qquad (2.24)
$$

where

$$
\widehat{X_{i,n+1}}^{c} = nS_0\left(\sum_0 + nS_0\right)^{-1}\overline{X_i} + \sum_0\left(\sum_0 + nS_0\right)^{-1}\mu_0 \qquad (2.25)
$$

is the classical credibility premium (see [2]). The second term of (2.24) reflects the contribution of the common effect to the credibility premium. Obviously, from (2.24), if $T_0 = 0$, then

$$
\widehat{X_{i,n+1}}^{*} \overset{\text{reduce to}}{=} \widehat{X_{i,n+1}}^{c}, \qquad (2.26)
$$

i.e, the common effect does not exist, then the risks in different contracts are independent of each other.

Remark 2　If the claims X_{ij} are univariate random variable, under assumptions 1.1–1.3, while some structure parameters are denoted by $\sum_0 = \sum^2$, $S_0 = \tau^2$, $T_0 = a$ and $\mu_0 = \mu$, then the credibility estimator $X_{i,n+1}$ of (2.22) (denoted by $\widehat{X_{i,n+1}}^{*}$ too) becomes

$$
\widehat{X_{i,n+1}}^{*} = \frac{n\tau^2}{n\tau^2 + \sum^2}\overline{X_i} + \frac{nKa\sum^2}{(n\tau^2 + \sum^2)(nKa + n\tau^2 + \sum^2)}\overline{\overline{X}} + \frac{\sum^2}{\left(\sum^2 + n\tau^2 + nKa\right)}\mu, \quad (2.27)
$$

where

$$\overline{X_i}=\frac{1}{n}\sum_{j=1}^{n}X_{ij}\quad \text{and } \overline{\overline{X}}=\frac{1}{K}\sum_{i=1}^{K}\overline{X_i}. \tag{2.28}$$

are individual sample mean and the overall sample mean respectively. The (2.27) is the univariate credibility estimators with common effects considered in [7,8].

3 Homogeneous credibility estimator

When μ_0 is unknown, we resort to the second part of Lemma 1.1 (or equivalently, equation (1.8)) to establish the optimal unbiased homogeneous credibility estimator of $X_{i,n+1}$.

Theorem 3.7 Under assumptions 1.1–1.3, the optimal linear homogeneous unbiased estimator of $X_{i,n+1}$ by solving (1.6) is

$$\widehat{X_{i,n+1}}^{H}=Z_1\overline{X_i}+Z_2\overline{\overline{X}}+(I_p-Z_1-Z_2)\widetilde{\mu_0}, \tag{3 .29}$$

where $\widetilde{\mu_0}=\frac{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mu_0}\mu_0$, and Z_1, Z_2 are defined as in (2.23).

Proof From (1.12), we have known that

$$\widehat{X_{i,n+1}}=\text{proj}\left(\widehat{X_{i,n+1}}^{*}\middle| Le(X)\right). \tag{3 .30}$$

Since

$$\widehat{X_{i,n+1}}^{*}=Z_1\overline{X_i}+Z_2\overline{\overline{X}}+\left(I_p-Z_1-Z_2\right)\mu_0, \tag{3 .31}$$

and $\overline{X_i},\overline{\overline{X}}\in Le(X)$, then from (1.11) we have

$$\widehat{X_{i,n+1}}=Z_1\overline{X_i}+Z_2\overline{\overline{X}}+(1-Z_1-Z_2)\text{proj}\left(\mu_0|Le(X)\right).$$

In addition, the (1.9) gives that $\text{proj}\left(\mu_0|Le(X)\right)=\frac{\mu_0\mu_X'\sum_{XX}^{-1}X}{\mu_X'\sum_{XX}^{-1}\mu_X}$, and $\sum_{XX}^{-1}$ is given by (2.17). So we should calculate the $\mu_X'\sum_{XX}^{-1}X$ and $\mu_X'\sum_{XX}^{-1}\mu_X$. By some matrix calculations, we have

$$\mu_X'\sum_{XX}^{-1}\mu_X=(\mathbf{1}_{nK}\otimes\mu_0)'\sum_{XX}^{-1}(\mathbf{1}_{nK}\otimes\mu_0)=Kn\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mu_0,$$

$$\mu_X'\sum_{XX}^{-1}X=(\mathbf{1}_{nK}\otimes\mu_0)'\sum_{XX}^{-1}X=Kn\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}.$$

Then we can get

$$\text{proj}\left(\mu_0|Le(X)\right)=\frac{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mu_0}\mu_0=\widetilde{\mu_0}. \tag{3 .32}$$

This completes the proof of the theorem.

Remark 3 We observe that the homogeneous estimator $\widehat{X_{i,n+1}}^{H}$ are not an estimator in general, because the estimator $\widetilde{\mu_0}=\frac{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mu_0}\mu_0$ still depends on μ_0 which is unknown in homogeneous estimator. However, it can be considered as a pseudo estimator. By the iterative method, for instance,

$$\widehat{\mu_0}^{(m+1)}=\frac{\widehat{\mu_0}^{(m)\prime}\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\widehat{\mu_0}^{(m)\prime}\left(\sum_0+nS_0+nKT_0\right)^{-1}\widehat{\mu_0}^{(m)}}\widehat{\mu_0}^{(m)}.$$

We can derive the desired estimator of $\widehat{\mu_0}$. Generally, we expect that the $\frac{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\mu_0'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mu_0}\mu_0$ is independent of μ_0. Fortunately, if $\mu_0=a\mathbf{1}_p$, we have

$$\widetilde{\mu_0}=\frac{\mathbf{1}_p'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\mathbf{1}_p'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mathbf{1}_p}\mathbf{1}_p, \tag{3 .33}$$

which is independent of μ_0. Then

$$\widehat{X_{i,n+1}}=Z_1\overline{X_i}+Z_2\overline{\overline{X}}+(I_p-Z_1-Z_2)\frac{\mathbf{1}_p'\left(\sum_0+nS_0+nKT_0\right)^{-1}\overline{\overline{X}}}{\mathbf{1}_p'\left(\sum_0+nS_0+nKT_0\right)^{-1}\mathbf{1}_p}\mathbf{1}_p. \tag{3 .34}$$

4 A numerical example

If the claim X_{ij} is a p-dimensional random variable, and its distribution is dependent risk parameter Θ_i and common effect which are also random variables, for $i=1,2,\cdots,K, j=1,2,\cdots,n$. We assume that $X_{ij}\overset{i.i.d}{\sim}N\left(\Theta_i+\beta,\sum\right), j=1,2,\cdots,n+1$, $\Theta_i\overset{i.i.d}{\sim}N\left(0,S\right), i=1,2,\cdots,K$, and $\beta\sim N\left(\mu,T\right)$. Then according to the notations in Section 2, we get

$$\mu\left(\Theta_i,\beta\right)=\Theta_i+\beta,\ \text{and}\ T_0=T,\ S_0=S,\ \sum_0=\sum,\ \mu_0=\mu.$$

Then the inhomogeneous credibility estimator of $X_{i,n+1}$ is given by

$$\widehat{X_{i,n+1}}^*=Z_1\overline{X_i}+Z_2\overline{\overline{X}}+\left(I_p-Z_1-Z_2\right)\mu_0, \tag{4 .35}$$

where credibility factors matrices are

$$Z_1=nS\left(\sum+nS\right)^{-1},\ \text{and}\ Z_2=nK\sum\left(\sum+nS\right)^{-1}T\left(\sum+nS+nKT\right)^{-1}.$$

We further assume that $\sum=\sum^2 I_p$, $S=\tau^2 I_p$ $T=\gamma^2 I_p$ and $\mu=a\mathbf{1}_p$. Then

$$\begin{aligned}\widetilde{\mu_0} &= \frac{\mu'\left(\sum+nS+nKT\right)^{-1}\overline{\overline{X}}}{\mu'\left(\sum+nS+nKT\right)^{-1}\mu'}\mu' \\ &= \frac{a\mathbf{1}_p'\left(\sum^2 I_p+n\tau^2 I_p+nK\gamma^2 I_p\right)^{-1}\overline{\overline{X}}}{a\mathbf{1}_p'\left(\sum^2 I_p+n\tau^2 I_p+nK\gamma^2 I_p\right)^{-1}a\mathbf{1}_p}a\mathbf{1}_p \\ &= \left(\frac{1}{nKp}\sum_{i=1}^{K}\sum_{j=1}^{n}\sum_{t=1}^{p}X_{ij}^{(t)}\right)\mathbf{1}_p.\end{aligned}$$

Thus the parameter a can be estimated by

$$\widehat{a}=\frac{1}{nKp}\sum_{i=1}^{K}\sum_{j=1}^{n}\sum_{t=1}^{p}X_{ij}^{(t)}. \tag{4 .36}$$

In this case, the homogeneous estimator of $X_{i,n+1}$ is

$$\widehat{X_{i,n+1}}=Z_1\overline{X_i}+Z_2\overline{\overline{X}}+\widehat{a}(I_p-Z_1-Z_2)\mathbf{1}_p, \tag{4 .37}$$

where $\widehat{a}$ is given by (4.36).

In the multidimensional credibility model, the posterior predictor of $X_{i,n+1}$ is given by

$$\widetilde{X_{i,n+1}} = \mathrm{E}\,(X_{i,n+1} |\, X_1, X_2, \cdots, X_K)\,, \tag{4 .38}$$

which are called Bayes premium of $X_{i,n+1}$. According to the Bayes theorem, $\widetilde{X_{i,n+1}}$ is the optimal estimator in the all measurable function of the sample $\{X_1, X_2, \cdots, X_K\}$. However, the calculation of Bayes premium is very difficult since $\widetilde{X_{i,n+1}}$ is dependent of the joint distribution of $(X_1, X_2, \cdots, X_K, \Theta)$, which are generally unknown in practice. Since the credibility estimators $\widehat{X_{i,n+1}}^{*}$ (or $\widehat{X_{i,n+1}}^{H}$) depend only on some moments, as is shown in Theorem 2.5 (or Theorem 2.6). Therefore, the credibility estimator can be directly used in practice.

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