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Expression of the Moore-Penrose inverse of $\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$

DU Fa-peng[1,2], XUE Yi-feng[1]

(1. *Department of Mathematics, East China Normal University, Shanghai* 200241, *China;*
2. *College of Mathematics and Physical Sciences, Xuzhou Institute of Technology, Xuzhou Jiangsu* 221008, *China*)

Abstract: By using theory of bounded linear operators on Hilbert spaces, the Shermen-Morrison-Woodbury (SMW) formula's Moore-Penrose inverse was presented. The formula obtained can be used to compute certain perturbation of $\boldsymbol{A}^+$ and the Moore–Penrose inverses of some operator matrices.

Key words: Hilbert space; Moore-Penrose inverse; idempotent operator

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$\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$ 的 Moore-Penrose 逆的表示

杜法鹏[1,2], 薛以锋[1]

(1. 华东师范大学 数学系, 上海 200241; 2. 徐州工程学院 数理学院, 江苏 徐州 221008)

摘 要: 在 Hilbert 空间上有界线性算子的条件下, 进一步推广了 Shermen-Morrison-Woodbury(SMW) 公式的 Moor-Penrose 逆的表示. 这个公式可以用来计算 $\boldsymbol{A}^+$ 的某些扰动和某些算子矩阵的 Moore–Penrose 逆.

关键词: Hilbert 空间; Moore-Penrose 逆; 幂等算子

0 Introduction

Let $\boldsymbol{A}$ be a nonsingular $m\times m$ matrix and $\boldsymbol{X}, \boldsymbol{Y}$ be two $m\times n$ matrices. It is known that $\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$ is nonsingular iff $I_n-\boldsymbol{Y}^*\boldsymbol{A}^{-1}\boldsymbol{X}$ is nonsingular, and in that case the well-known Shermen–Morrison–Woodbury formula (SMW) can be expressed as

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^{-1}=\boldsymbol{A}^{-1}+\boldsymbol{A}^{-1}\boldsymbol{X}(I_n-\boldsymbol{Y}^*\boldsymbol{A}^{-1}\boldsymbol{X})^{-1}\boldsymbol{Y}^*\boldsymbol{A}^{-1}. \tag{0.1}$$

This formula and some related formula have a lot of applications in statistics, networks, optimization and partial differential equations. Please see [1–3] for details. Clearly, Eq. (0.1)

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第一作者: 杜法鹏, 男, 讲师, 博士研究生, 研究方向为应用泛函分析, 算子代数.
E-mail: jsdfp@163.com.

fails when $\boldsymbol{A}$ or $\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$ is singular. Steerneman and Kleij in [4] proved that when $\boldsymbol{A}$ is singular and $\boldsymbol{I}_n-\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}$ is nonsingular, then

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=\boldsymbol{A}^++\boldsymbol{A}^+\boldsymbol{X}(I_n-\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X})^{-1}\boldsymbol{Y}^*\boldsymbol{A}^+$$

under conditions that

$$\operatorname{rank}(\boldsymbol{A},\boldsymbol{X})=\operatorname{rank}\boldsymbol{A},\quad \operatorname{rank}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{Y}^*\end{pmatrix}=\operatorname{rank}\boldsymbol{A}.$$

He also showed that if $\boldsymbol{A}$ is nonsingular and $\boldsymbol{Y}^*\boldsymbol{A}^{-1}\boldsymbol{X}=\boldsymbol{I}_n$, then

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=(I_m-\boldsymbol{X}_1\boldsymbol{X}_1^+)\boldsymbol{A}^{-1}(I_m-\boldsymbol{Y}_1\boldsymbol{Y}_1^+),\tag{0.2}$$

where $\boldsymbol{X}_1=\boldsymbol{A}^{-1}\boldsymbol{X},\ \ \boldsymbol{Y}_1=(\boldsymbol{A}^{-1})^*\boldsymbol{Y}$ (cf. [4, Theorem 3]).

Let H, K be Hilbert spaces and let $L(H,K)$ denote the set of all bounded linear operators from H to K. Recently Chen, Hu and Xu studied the Moore-Penrose inverse of $\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$ when $\boldsymbol{A}\in L(H)\triangleq L(H,H)$ and $\boldsymbol{X},\boldsymbol{Y}\in L(K,H)$ in [5]. They prove that if $\boldsymbol{A}$ is invertible and $\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$, $\boldsymbol{X}$, $\boldsymbol{Y}$ have closed ranges, then

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=(\boldsymbol{I}-\boldsymbol{X}_1\boldsymbol{X}_1^+)\boldsymbol{A}^{-1}(\boldsymbol{I}-\boldsymbol{Y}_1\boldsymbol{Y}_1^+)$$

iff $\boldsymbol{Y}_1^*\boldsymbol{X}\boldsymbol{Y}_1^*=\boldsymbol{Y}_1^*,\ \boldsymbol{X}\boldsymbol{Y}_1^*\boldsymbol{X}=\boldsymbol{X}$, where $\boldsymbol{X}_1=\boldsymbol{A}^{-1}\boldsymbol{X},\ \ \boldsymbol{Y}_1=(\boldsymbol{A}^{-1})^*\boldsymbol{Y}$. This result generalizes Theorem 3 of [4].

In this paper we assume that $\boldsymbol{A}\in L(H)$ and $\boldsymbol{X},\boldsymbol{Y}\in L(K,H)$ with $R(\boldsymbol{A})$ closed and $R(\boldsymbol{X})\subseteq R(\boldsymbol{A}), R(\boldsymbol{Y})\subseteq R(\boldsymbol{A}^*)$. We prove that

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=(\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+))$$

if $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*=\boldsymbol{X}\boldsymbol{Y}^*$; and

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=(\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X})(\boldsymbol{A}^+\boldsymbol{X})^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{Y}^*\boldsymbol{A}^+))$$

if $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}=\boldsymbol{X}$ and $\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*=\boldsymbol{Y}^*$. These expressions generalize corresponding expressions of $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+$ given in [4,5].

1 Preliminaries

Let $T\in L(K,H)$, denote by $R(T)$ (resp. $N(T)$) the range (resp. kernel) of T. Let $\boldsymbol{A}\in L(H)$. Recall from [6] that $\boldsymbol{B}\in L(H)$ is the Moore-Penrose inverse of $\boldsymbol{A}$, if $\boldsymbol{B}$ satisfies the following equations:

$$\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}=\boldsymbol{A},\ \ \boldsymbol{B}\boldsymbol{A}\boldsymbol{B}=\boldsymbol{B},\ \ (\boldsymbol{A}\boldsymbol{B})^*=\boldsymbol{A}\boldsymbol{B},\ \ (\boldsymbol{B}\boldsymbol{A})^*=\boldsymbol{B}\boldsymbol{A}.$$

In this case $\boldsymbol{B}$ is denote by that $\boldsymbol{A}^+$. It is well-known that $\boldsymbol{A}$ has the Moore-Penrose inverse iff $R(\boldsymbol{A})$ is closed in H. When $\boldsymbol{A}^+$ exists, $R(\boldsymbol{A}^+)=R(\boldsymbol{A}^*)$, $N(\boldsymbol{A}^+)=N(\boldsymbol{A}^*)$ and $(\boldsymbol{A}^+)^*=(\boldsymbol{A}^*)^+$.

Lemma 1.1 Let $\boldsymbol{A}\in L(H)$ *with* $R(\boldsymbol{A})$ *closed and* $\boldsymbol{X},\boldsymbol{Y}\in L(K,H)$.

(1) $R(\boldsymbol{X}) \subseteq R(\boldsymbol{A})$ iff $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{X} = \boldsymbol{X}$, $R(\boldsymbol{Y}) \subseteq R(\boldsymbol{A}^*)$ iff $\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{Y}^*$.

(2) Suppose that $R(\boldsymbol{X}) \subseteq R(\boldsymbol{A})$ and $R(\boldsymbol{Y}) \subseteq R(\boldsymbol{A}^*)$ then

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*) = \boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$$

iff $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{X}\boldsymbol{Y}^*$.

Proof (1) Since $R(\boldsymbol{A}) = R(\boldsymbol{A}\boldsymbol{A}^+)$ and $R(\boldsymbol{A}^*) = R(\boldsymbol{A}^+\boldsymbol{A})$, the assertion follows.

(2) Using (1), we can check directly that $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*) = \boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$ if and only if $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{X}\boldsymbol{Y}^*$.

In order to compute $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+$, we need the following two lemmas which come from [7].

Lemma 1.2 Let $S \in L(H)$ be an idempotent operator. Denote by $O(S)$ the orthogonal projection of H onto $R(S)$. Then $I-S-S^*$ is invertible in L(H) and $O(S) = -S(I-S-S^*)^{-1}$.

Lemma 1.3 Let T, $B \in L(H)$ with $TBT = T$, Then $T^+ = (I - O(I-BT))BO(TB)$.

Lemma 1.4 Let $S \in L(H)$ be an idempotent operator. Then $O(S) = SS^+$ and $O(I-S) = I - S^+S$.

Proof $S^2 = S$ implies that $R(S)$ is closed and $R(I-S) = N(S) = R(S^*)^{\perp}$. Thus S^+ exists and $O(S) = SS^+$, $O(I-S) = I - S^+S$.

2 Main results

In this section, we will generalize Eq (0.1) and Eq (0.2). Firstly, we have

Proposition 2.1 *Let* $\boldsymbol{A} \in L(H)$ *with* $R(\boldsymbol{A})$ *closed and* $\boldsymbol{X}, \boldsymbol{Y} \in L(K,H)$ *with* $R(\boldsymbol{X}) \subseteq R(\boldsymbol{A})$ *and* $R(\boldsymbol{Y}) \subseteq R(\boldsymbol{A}^*)$. *Assume that* $\boldsymbol{I}-\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}$ *is invertible in* $L(H)$. Then $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+$ *exists and*

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+ = \boldsymbol{A}^+ + \boldsymbol{A}^+\boldsymbol{X}(\boldsymbol{I}-\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X})^{-1}\boldsymbol{Y}^*\boldsymbol{A}^+. \tag{2.1}$$

Proof Put $\boldsymbol{B} = \boldsymbol{A}^+ + \boldsymbol{A}^+\boldsymbol{X}(\boldsymbol{I}-\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X})^{-1}\boldsymbol{Y}^*\boldsymbol{A}^+$. Simple computation shows that $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{B} = \boldsymbol{A}\boldsymbol{A}^+$ and $\boldsymbol{B}(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*) = \boldsymbol{A}^+\boldsymbol{A}$ by Lemma 1.1 (1). Thus,

$$\begin{aligned} &(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{B}(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*) = \boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*, \quad \boldsymbol{B}(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{B} = \boldsymbol{B}, \\ &((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{B})^* = (\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{B}, \quad (\boldsymbol{B}(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*))^* = \boldsymbol{B}(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*), \end{aligned}$$

that is, $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+ = \boldsymbol{B}$.

Now we consider the case that $\boldsymbol{I}-\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}$ is not invertible, we have

Theorem 2.2 Let $\boldsymbol{A} \in L(H)$ with $R(\boldsymbol{A})$ closed and $\boldsymbol{X}, \boldsymbol{Y} \in L(K,H)$ with $R(\boldsymbol{X}) \subseteq R(\boldsymbol{A})$ and $R(\boldsymbol{Y}) \subseteq R(\boldsymbol{A}^*)$.

(1) *If* $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{X}\boldsymbol{Y}^*$, *then* $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+$ *exists and*

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+ = (\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)). \tag{2.2}$$

Especially, if $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X} = \boldsymbol{X}$ *and* $\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{Y}^*$, *then*

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+ = (\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X})(\boldsymbol{A}^+\boldsymbol{X})^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{Y}^*\boldsymbol{A}^+)). \tag{2.3}$$

(2) *Assume that* $R(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)$, $R(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)$ *and* $R(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)$ *are closed in* H. *Then Eq.* (2.2) *implies that* $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*=\boldsymbol{X}\boldsymbol{Y}^*$.

(3) *Assume that* $R(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)$, $R(\boldsymbol{A}^+\boldsymbol{X})$ *and* $R(\boldsymbol{Y}^*\boldsymbol{A}^+)$ *are closed. Then Eq.* (2.3) *indicates that* $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}=\boldsymbol{X}$ *and* $\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*=\boldsymbol{Y}^*$.

Proof (1) In this case, $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)=\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*$. Thus $R(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)$ is closed, i.e., $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+$ exists and hence

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=(\boldsymbol{I}-O(\boldsymbol{I}-\boldsymbol{A}^+(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)))\boldsymbol{A}^+O((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+)$$

by Lemma 1.1 (2). Since $(\boldsymbol{I}-2\boldsymbol{A}^+\boldsymbol{A})^2=\boldsymbol{I},\ (\boldsymbol{I}-2\boldsymbol{A}^+\boldsymbol{A})\boldsymbol{A}^+=-\boldsymbol{A}^+$,

$$\begin{aligned}\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*+(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^*&=(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*+(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^*)(2\boldsymbol{A}^+\boldsymbol{A}-\boldsymbol{I}),\\ (\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A})(\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^*)&=\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}.\end{aligned}$$

It follows that

$$\begin{aligned}O(\boldsymbol{I}-\boldsymbol{A}^+(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*))&=O(\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}+\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)\\ &=-(\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}+\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(2\boldsymbol{A}^+\boldsymbol{A}-\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^*)^{-1}\\ &=(\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}+\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(\boldsymbol{I}-2\boldsymbol{A}^+\boldsymbol{A})(\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^*)^{-1}\\ &=\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}+O(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}O((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+)&=-(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{A}\boldsymbol{A}^+-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)-(\boldsymbol{A}\boldsymbol{A}^+-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^*)^{-1}\\ &=(-\boldsymbol{A}\boldsymbol{A}^++\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)(\boldsymbol{I}-2\boldsymbol{A}\boldsymbol{A}^++\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^++(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^*)^{-1}\\ &=(\boldsymbol{A}\boldsymbol{A}^+-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)(\boldsymbol{I}-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+-(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^*)^{-1}\\ &=\boldsymbol{A}\boldsymbol{A}^+-\boldsymbol{I}+O(\boldsymbol{I}-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+).\end{aligned}$$

Therefore, we have

$$\begin{aligned}(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+&=(\boldsymbol{I}-O(\boldsymbol{I}-\boldsymbol{A}^+(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)))\boldsymbol{A}^+O((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)\boldsymbol{A}^+)\\ &=(\boldsymbol{A}^+\boldsymbol{A}-O(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*))\boldsymbol{A}^+O(\boldsymbol{I}-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)\\ &=\boldsymbol{I}-O(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*))\boldsymbol{A}^+O(\boldsymbol{I}-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+).\end{aligned}$$

From $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*=\boldsymbol{X}\boldsymbol{Y}^*$, we get that $\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*$ and $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+$ are all idempotent operators. It follow from Lemma 1.4 that

$$O(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)=(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^+,\quad O(\boldsymbol{I}-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)=\boldsymbol{I}-(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+).$$

Therefore, we have

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+=(\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)).$$

When $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}=\boldsymbol{X}$ and $\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*=\boldsymbol{Y}^*$, we have $R(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)=R(\boldsymbol{A}^+\boldsymbol{X})$ and $R(I-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)=N(\boldsymbol{Y}^*\boldsymbol{A}^+)$ so that

$$O(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)=(\boldsymbol{A}^+\boldsymbol{X})(\boldsymbol{A}^+\boldsymbol{X})^+,\quad O(\boldsymbol{I}-\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)=\boldsymbol{I}-(\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{Y}^*\boldsymbol{A}^+).$$

and consequently, we get (2.3).

(2) In this case,

$$R((\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^*) = R((\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^+) \subseteq N((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+) = N((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^*),$$

that is, $[N(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)]^\perp \subseteq [R(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)]^\perp$. So $R(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*) \subseteq N(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)$ and consequently, $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{X}\boldsymbol{Y}^*$.

(3) When Eq (2.3) holds,

$$R((\boldsymbol{Y}^*\boldsymbol{A}^+)^*) = R((\boldsymbol{Y}^*\boldsymbol{A}^+)^+) \subseteq N((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+) = N((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^*),$$
$$R((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^*) = R((\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+) \subseteq N((\boldsymbol{A}^+\boldsymbol{X})^+) = N((\boldsymbol{A}^+\boldsymbol{X})^*).$$

Then $R(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*) \subseteq N(\boldsymbol{Y}^*\boldsymbol{A}^+)$ and $R(\boldsymbol{A}^+\boldsymbol{X}) \subseteq N(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)$. So

$$\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{Y}^*, \quad \boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X} = \boldsymbol{X}.$$

Suppose $H = \mathbb{C}^m$ and $K = \mathbb{C}^n$. Let $\boldsymbol{A} \in L(H)$ and $\boldsymbol{X}, \boldsymbol{Y} \in L(K, H)$. Since

$$\operatorname{rank}(\boldsymbol{A}, \boldsymbol{X}) = \operatorname{rank}\boldsymbol{A} \Leftrightarrow R(\boldsymbol{X}) \subseteq R(\boldsymbol{A}),$$
$$\operatorname{rank}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{Y}^*\end{pmatrix} = \operatorname{rank}\boldsymbol{A} \Leftrightarrow R(\boldsymbol{Y}) \subseteq R(\boldsymbol{A}^*),$$

we can express Theorem 2.2 (1) as follows.

Corollary 2.3 *Let* $\boldsymbol{A}$ *be an* $m \times m$ *matrix and* $\boldsymbol{X}, \boldsymbol{Y}$ *be two* $m \times n$ *matrices. Suppose that* $\operatorname{rank}(\boldsymbol{A}, \boldsymbol{X}) = \operatorname{rank}\boldsymbol{A}$ *and* $\operatorname{rank}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{Y}^*\end{pmatrix} = \operatorname{rank}\boldsymbol{A}$. *Then*

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+ = (\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)(\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^*)^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+))$$

if $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{X}\boldsymbol{Y}^*$ *and*

$$(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^*)^+ = (\boldsymbol{I}-(\boldsymbol{A}^+\boldsymbol{X})(\boldsymbol{A}^+\boldsymbol{X})^+)\boldsymbol{A}^+(\boldsymbol{I}-(\boldsymbol{Y}^*\boldsymbol{A}^+)^+(\boldsymbol{Y}^*\boldsymbol{A}^+))$$

when $\boldsymbol{X}\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X} = \boldsymbol{X}$ and $\boldsymbol{Y}^*\boldsymbol{A}^+\boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{Y}^*$.

Before ending this note, we give an example as follows.

Example 2.4 *Put* $\boldsymbol{A} = \begin{pmatrix}1&1&1&1\\0&0&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix}1&1&0\\1&0&0\\1&0&0\\1&0&0\end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\\1&0&1\end{pmatrix}$. *Then*

$$\boldsymbol{A}^+ = \begin{pmatrix}\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}&0\\ \frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}&0\\ 0&\frac{1}{2}&\frac{1}{2}&-1\\ 0&0&0&1\end{pmatrix}, \quad \boldsymbol{X}\boldsymbol{Y}^* = \begin{pmatrix}0&0&1&1\\0&0&1&1\\0&0&1&1\\0&0&1&1\end{pmatrix},$$

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$$(\boldsymbol{A}^{+}\boldsymbol{X}\boldsymbol{Y}^{*})^{+}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\frac{1}{2}\\0&0&0&\frac{1}{2}\end{pmatrix},\quad (\boldsymbol{X}\boldsymbol{Y}^{*}\boldsymbol{A}^{+})^{+}=\begin{pmatrix}0&0&0&0\\\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\\0&0&0&0\end{pmatrix}.$$

It is easy to verify that $R(\boldsymbol{X})\subseteq R(\boldsymbol{A})$, $R(\boldsymbol{Y})\subseteq R(\boldsymbol{A}^{*})$ and $\boldsymbol{X}\boldsymbol{Y}^{*}\boldsymbol{A}^{+}\boldsymbol{X}\boldsymbol{Y}^{*}=\boldsymbol{X}\boldsymbol{Y}^{*}$. So by Corollary 2.3, $(\boldsymbol{A}-\boldsymbol{X}\boldsymbol{Y}^{*})^{+}=\begin{pmatrix}\frac{1}{2}&0&0&0\\\frac{1}{2}&0&0&0\\0&0&0&-1\\0&0&0&0\end{pmatrix}$.

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