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双模量矩形板的大挠度弯曲计算分析

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摘 要：双模量矩形板在外载荷作用下，会形成各向同性的拉伸区和压缩区，把双模量矩形板看成两种各向同性材料组成的层合板，采用弹性力学理论建立了双模量矩形板在外载荷作用下的静力平衡方程，利用静力平衡方程确定了双模量矩形板的中性面位置，推导出了双模量矩形板的大挠度弯曲变形微分方程。用加权残值法求得了双模量矩形板的大挠度弯曲变形时板中点挠度，把该方法计算结果与有限元计算结果进行了比较，说明了该计算方法是可靠的，并讨论分析了双模量对矩形板大挠度弯曲变形的影响。

关键词：双模量；矩形板；层合板；中性面；大挠度
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LARGE DEFLECTION BENDING CALCULATION AND ANALYSIS OF BIMODULOUS RECTANGULAR PLATE

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Abstract: A bimodulous rectangular plate could form an isotropic compression and a tensile area under external loads. Thusly, a bimodulous rectangular plate was regarded as a laminated plate composited of two kind of isotropic material. The static equilibrium equation of the bimodulous rectangular plate under the condition of external loads was established by using elastic mechanics theory. The location of the neutral plane in the bimodulous rectangular plate was determined by the utilization of static equilibrium equations. The large deflection bending deformation differential equations of the bimodulous rectangular plate was derived, and the middle point deflection of the bimodulous rectangular plate was gained with method of weighted residuals. Then the calculation results were compared with that obtained by finite element method, and it show that the method above is reliable. Meanwhile the effect of the bimodulous on the large deflection bending deformation of a rectangular plate was discussed and analyzed.

Key words: bimodulous; rectangular plate; laminated plate; neutral plane, large deflection

大量研究表明，土木工程中的混凝土、航天航空及机械工程中的石墨、塑料、合金等许多材料都具有拉压弹性模量不同的性质。尤其是近几年发展起来的复合材料更具有明显的拉压弹性模量不同的特性，如美国研制的具有世界一流水平的“超黑粉”纳米吸波材料中的黑粉，就是把石墨作为吸收剂制成石墨/环氧树脂复合材料，其中石墨材料的拉压弹性模量之比高达 4 倍。所以，用不同弹性模量

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本构关系对这些材料制成的结构进行计算分析已备受关注[1−2]。对于拉压不同弹性模量材料，弹性系数不仅依赖于结构材料，还根据结构各点位移或应力状态的不同而不同，亦即与结构材料、形状、边界条件及外载荷有关。在梁、弹性平面等问题的结构中，已开始考虑材料的不同弹性模量特性[3−8]。但是，对薄板及薄壳弯曲问题的分析与计算，仍延用传统的同弹性模量弹性理论和默认中性面位置与中心面位置重合[9−10]，且至今未见到有关双模量矩形板大挠度弯曲变形计算分析的有关文献。本文则把双模量矩形板看成两种各向同性材料组成的层合板，利用静力平衡方程确定了双模量矩形板的中性面位置，建立了双模量薄板弯曲变形的微分方程，研究了双模量矩形薄板的大挠度弯曲变形问题，讨论分析了拉压弹性模量对矩形薄板大挠度弯曲变形的影响。

1　双模量矩形板中性面的确定

对于双模量矩形板的纯弯曲，其中性面不再位于板厚的正中央，而是形成了各向同性的拉伸区和压缩区。假设双模量矩形板变形前垂直于中面的直线，变形后仍为垂直于中曲面的直线段，且长度不变，即直法线假定。由弹性力学理论可知双模量矩形板纯弯曲时的应力表达式为：

$$\begin{cases}\sigma_x=-\dfrac{E_1 z}{1-\mu_1^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_1\dfrac{\partial^2 w}{\partial y^2}\right)\\ \sigma_y=-\dfrac{E_1 z}{1-\mu_1^2}\left(\dfrac{\partial^2 w}{\partial y^2}+\mu_1\dfrac{\partial^2 w}{\partial x^2}\right),\quad z>0\\ \tau_{xy}=-\dfrac{E_1 z}{1+\mu_1}\dfrac{\partial^2 w}{\partial x\partial y}\end{cases}\tag{1}$$

$$\begin{cases}\sigma_x=-\dfrac{E_2 z}{1-\mu_2^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_2\dfrac{\partial^2 w}{\partial y^2}\right)\\ \sigma_y=-\dfrac{E_2 z}{1-\mu_2^2}\left(\dfrac{\partial^2 w}{\partial y^2}+\mu_2\dfrac{\partial^2 w}{\partial x^2}\right),\quad z<0\\ \tau_{xy}=-\dfrac{E_2 z}{1+\mu_2}\dfrac{\partial^2 w}{\partial x\partial y}\end{cases}\tag{2}$$

式中：E_1、μ_1分别为拉伸弹性模量及泊松比；E_2、μ_2分别为压缩弹性模量及泊松比。

设h为板厚，受拉区高度为h_1，双模量矩形板弯曲时横截面内力应满足以下关系：

$$\begin{cases}\dfrac{E_1}{1-\mu_1^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_1\dfrac{\partial^2 w}{\partial y^2}\right)\displaystyle\int_0^{h_1}z\mathrm{d}z+\\ \quad\dfrac{E_2}{1-\mu_2^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_2\dfrac{\partial^2 w}{\partial y^2}\right)\displaystyle\int_{h_1-h}^{0}z\mathrm{d}z=0\\ \dfrac{E_1}{1-\mu_1^2}\left(\dfrac{\partial^2 w}{\partial y^2}+\mu_1\dfrac{\partial^2 w}{\partial x^2}\right)\displaystyle\int_0^{h_1}z\mathrm{d}z+\\ \quad\dfrac{E_2}{1-\mu_2^2}\left(\dfrac{\partial^2 w}{\partial y^2}+\mu_2\dfrac{\partial^2 w}{\partial x^2}\right)\displaystyle\int_{h_1-h}^{0}z\mathrm{d}z=0\end{cases}\tag{3}$$

把式(3)中两式相加可得：

$$\frac{E_1}{1-\mu_1}\int_0^{h_1}z\mathrm{d}z+\frac{E_2}{1-\mu_2}\int_{h_1-h}^{0}z\mathrm{d}z=0\tag{4}$$

由式(4)可以求得中性面的位置为：

$$h_1=\frac{h\sqrt{E_2(1-\mu_1)}}{\sqrt{E_1(1-\mu_2)}+\sqrt{E_2(1-\mu_1)}}\tag{5}$$

2　薄板大挠度弯曲微分方程

由弹性力学得薄板微单元的平衡微分方程为：

$$\begin{cases}\dfrac{\partial\tau_{xz}}{\partial z}=-\dfrac{\partial\sigma_x}{\partial x}-\dfrac{\partial\tau_{xy}}{\partial y}\\ \dfrac{\partial\tau_{yz}}{\partial z}=-\dfrac{\partial\sigma_y}{\partial y}-\dfrac{\partial\tau_{xy}}{\partial x}\end{cases}\tag{6}$$

对于双模量矩形板可知其上下面边界条件为：

$$(\tau_{xz})\big|_{h_1-h}^{h_1}=0,\quad(\tau_{yz})\big|_{h_1-h}^{h_1}=0\tag{7}$$

把式(6)代入式(7)中可得τ_{xz}、τ_{yz}的表达式为：

$$\begin{cases}\tau_{xz}=\dfrac{E_1}{2(1-\mu_1^2)}(z^2-h_1^2)\dfrac{\partial}{\partial x}\nabla^2 w, & 0\leqslant z\leqslant h_1\\ \tau_{xz}=\dfrac{E_2}{2(1-\mu_2^2)}[z^2-(h-h_1)^2]\dfrac{\partial}{\partial x}\nabla^2 w, & h_1-h\leqslant z\leqslant 0\end{cases}\tag{8}$$

$$\begin{cases}\tau_{yz}=\dfrac{E_1}{2(1-\mu_1^2)}(z^2-h_1^2)\dfrac{\partial}{\partial y}\nabla^2 w, & 0\leqslant z\leqslant h_1\\ \tau_{yz}=\dfrac{E_2}{2(1-\mu_2^2)}[z^2-(h-h_1)^2]\dfrac{\partial}{\partial y}\nabla^2 w, & h_1-h\leqslant z\leqslant 0\end{cases}\tag{9}$$

利用式(1)、式(2)、式(8)、式(9)可以得到下式：

$$\begin{cases} M_x = \dfrac{E_1}{1-\mu_1^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_1\dfrac{\partial^2 w}{\partial y^2}\right)\int_0^{h_1} z^2 \mathrm{d}z - \\ \quad \dfrac{E_2}{1-\mu_2^2}\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_2\dfrac{\partial^2 w}{\partial y^2}\right)\int_{h_1-h}^{0} z^2 \mathrm{d}z = \\ \quad \left[\dfrac{E_2(h_1-h)^3}{3(1-\mu_2^2)}-\dfrac{E_1h_1^3}{3(1-\mu_1^2)}\right]\dfrac{\partial^2 w}{\partial x^2}+ \\ \quad \left[\dfrac{E_2(h_1-h)^3\mu_2}{3(1-\mu_2^2)\mu_1}-\dfrac{E_1h_1^3}{3(1-\mu_1^2)}\right]\mu_1\dfrac{\partial^2 w}{\partial y^2} \\ M_y = -\dfrac{E_1}{1-\mu_1^2}\left(\dfrac{\partial^2 w}{\partial y^2}+\mu_1\dfrac{\partial^2 w}{\partial x^2}\right)\int_0^{h_1} z^2 \mathrm{d}z - \\ \quad \dfrac{E_2}{1-\mu_2^2}\left(\dfrac{\partial^2 w}{\partial y^2}+\mu_2\dfrac{\partial^2 w}{\partial x^2}\right)\int_{h_1-h}^{0} z^2 \mathrm{d}z = \\ \quad \left[\dfrac{E_2(h_1-h)^3}{3(1-\mu_2^2)}-\dfrac{E_1h_1^3}{3(1-\mu_1^2)}\right]\dfrac{\partial^2 w}{\partial y^2}+ \\ \quad \left[\dfrac{E_2(h_1-h)^3\mu_2}{3(1-\mu_2^2)\mu_1}-\dfrac{E_1h_1^3}{3(1-\mu_1^2)}\right]\mu_1\dfrac{\partial^2 w}{\partial x^2} \\ M_{xy} = -\dfrac{E_1}{1+\mu_1}\dfrac{\partial^2 w}{\partial x\partial y}\int_0^{h_1} z^2 \mathrm{d}z - \\ \quad \dfrac{E_2}{1+\mu_2}\dfrac{\partial^2 w}{\partial x\partial y}\int_{h_1-h}^{0} z^2 \mathrm{d}z = \\ \quad \dfrac{E_2(h_1-h)^3}{3(1+\mu_2)}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{E_1h_1^3}{3(1+\mu_1)}\dfrac{\partial^2 w}{\partial x\partial y} \end{cases} \qquad (10)$$

$$\begin{cases} Q_x = \dfrac{\partial}{\partial x}\nabla^2 w\int_0^{h_1}\dfrac{E_1}{2(1-\mu_1^2)}(z^2-h_1^2)\mathrm{d}z + \\ \quad \dfrac{\partial}{\partial x}\nabla^2 w\int_{h_1-h}^{0}\dfrac{E_2}{2(1-\mu_2^2)}[z^2-(h-h_1)^2]\mathrm{d}z = \\ \quad \left[\dfrac{E_2(h_1-h)^3}{3(1-\mu_2^2)}-\dfrac{E_1h_1^3}{3(1-\mu_1^2)}\right]\dfrac{\partial}{\partial x}\nabla^2 w \\ Q_y = \dfrac{\partial}{\partial y}\nabla^2 w\int_0^{h_1}\dfrac{E_1}{2(1-\mu_2^2)}(z^2-h_1^2)\mathrm{d}z + \\ \quad \dfrac{\partial}{\partial y}\nabla^2 w\int_{h_1-h}^{0}\dfrac{E_2}{2(1-\mu_2^2)}[z^2-(h-h_1)^2]\mathrm{d}z = \\ \quad \left[\dfrac{E_2(h_1-h)^3}{3(1-\mu_2^2)}-\dfrac{E_1h_1^3}{3(1-\mu_1^2)}\right]\dfrac{\partial}{\partial y}\nabla^2 w \end{cases} \qquad (11)$$

由弹性力学理论可以知道，当双模量矩形板在垂直于板面的外分布载荷 $q(x,y)$ 作用下，外分布载荷与双模量矩形板的内力应满足以下二式：

$$\begin{cases} \dfrac{\partial M_x}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}=Q_x \\ \dfrac{\partial M_{xy}}{\partial x}+\dfrac{\partial M_y}{\partial y}=Q_y \end{cases} \qquad (12)$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+ N_y\frac{\partial^2 w}{\partial y^2}+q(x,y)=0 \qquad (13)$$

式中：N_x、N_y、N_{xy} 是由横向分布载荷 $q(x,y)$ 引起的中面拉力。

由弹性力学理论可知还可以得到相容方程为：

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \qquad (14)$$

因为 N_x、N_y、N_{xy} 均为由横向分布载荷 $q(x,y)$ 引起的中面拉力，所以应变可表示为：

$$\begin{cases} \varepsilon_x=\dfrac{1}{E_1h}(N_x-\mu_1N_y) \\ \varepsilon_y=\dfrac{1}{E_1h}(N_y-\mu_1N_x) \\ \gamma_{xy}=\dfrac{2(1+\mu_1)}{E_1h}N_{xy} \end{cases} \qquad (15)$$

令 $N_x=h\sigma_x=h\dfrac{\partial^2\varphi}{\partial^2 y}$，$N_y=h\sigma_y=h\dfrac{\partial^2\varphi}{\partial^2 x}$，

$$N_{xy}=h\tau_{xy}=-h\frac{\partial^2\varphi}{\partial x\partial y} \qquad (16)$$

把式(10)－式(12)代入式(13)中，把式(15)－式 (16)代入式(14)中，即可得到双模量薄板的大挠度微分方程组为：

$$\begin{cases} D\nabla^4 w = h\left(\dfrac{\partial^2\varphi}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2\varphi}{\partial y^2}\dfrac{\partial^2 w}{\partial x^2}-2\dfrac{\partial^2\varphi}{\partial x\partial y}\dfrac{\partial^2 w}{\partial x\partial y}\right)+ \\ \quad q(x,y) \\ \nabla^4\varphi = E_1\left[\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2-\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2}\right] \end{cases} \qquad (17)$$

式中：

$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4},$$

$$D=\frac{E_1h_1^3}{3(1-\mu_1^2)}-\frac{E_2(h_1-h)^3}{3(1-\mu_2^2)}。$$

3　双模量矩形板大挠度的求解

当横向分布载荷作用在板上时，设挠度函数 w 及平面力函数 φ 分别用梁函数表达式为：

$$w(x,y)=F_1X_1Y_1\,,\quad \varphi(x,y)=F_2X_2Y_2 \tag{18}$$

式中：X_1、Y_1、X_2、Y_2 皆为梁函数；F_1、F_2 为伽辽金常数。

把式(18)代入式(17)中利用伽辽金原理可得：

$$\begin{cases}DF_1A_{11}=hF_2A_{12}+A_{13}\\ F_2A_{21}=E_1F_1^2A_{22}\end{cases} \tag{19}$$

式中：

$$A_{11}=\int_0^a\int_0^b\left(\frac{\mathrm{d}^4X_1}{\mathrm{d}x^4}X_1Y_1^2+2\frac{\mathrm{d}^2X_1}{\mathrm{d}x^2}X_1\frac{\mathrm{d}^2Y_1}{\mathrm{d}y^2}Y_1+X_1^2Y_1\frac{\mathrm{d}^4Y_1}{\mathrm{d}y^4}\right)\mathrm{d}x\mathrm{d}y,$$

$$A_{13}=\int_0^a\int_0^b q(x,y)X_1Y_1\mathrm{d}x\mathrm{d}y\,,$$

$$A_{12}=\int_0^a\int_0^b\left(\frac{\mathrm{d}^2X_1}{\mathrm{d}x^2}X_1Y_1^2X_2\frac{\mathrm{d}^2Y_2}{\mathrm{d}y^2}+X_1^2\frac{\mathrm{d}^2Y_1}{\mathrm{d}y^2}Y_1Y_2\frac{\mathrm{d}^2X_2}{\mathrm{d}x^2}-2\frac{\mathrm{d}X_1}{\mathrm{d}x}\frac{\mathrm{d}Y_1}{\mathrm{d}y}X_1Y_1\frac{\mathrm{d}X_2}{\mathrm{d}x}\frac{\mathrm{d}Y_2}{\mathrm{d}y}\right)\mathrm{d}x\mathrm{d}y,$$

$$A_{21}=\int_0^a\int_0^b\left(\frac{\mathrm{d}^4X_2}{\mathrm{d}x^4}X_2Y_2^2+2\frac{\mathrm{d}^2X_2}{\mathrm{d}x^2}X_2\frac{\mathrm{d}^2Y_2}{\mathrm{d}y^2}Y_2+X_2^2Y_2\frac{\mathrm{d}^4Y_2}{\mathrm{d}y^4}\right)\mathrm{d}x\mathrm{d}y,$$

$$A_{22}=\int_0^a\int_0^b\left[\left(\frac{\mathrm{d}X_1}{\mathrm{d}x}\frac{\mathrm{d}Y_1}{\mathrm{d}y}\right)^2X_2Y_2-2\frac{\mathrm{d}^2X_1}{\mathrm{d}x^2}X_1\frac{\mathrm{d}^2Y_1}{\mathrm{d}y^2}Y_1X_2Y_2\right]\mathrm{d}x\mathrm{d}y\text{。}$$

由式(19)即可以得到方程：

$$DA_{11}F_1=A_{13}+\frac{E_1hA_{22}A_{12}}{A_{21}}F_1^3 \tag{20}$$

当双模量矩形板在集中载荷 P 作用时，由于梁函数为三次多项式，有奇异性不能利用卡门方程，但是可以利用泛函数：

$$U=2(1+\mu_1)\iint\left[\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2-\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}\right]\mathrm{d}x\mathrm{d}y-\frac{1}{2E_1h}\iint\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)^2\mathrm{d}x\mathrm{d}y+$$

$$\frac{D}{2}\iint\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y+$$

$$\left[\frac{E_1h_1^3}{3(1+\mu_1)}-\frac{E_2(h_1-h)^3}{3(1+\mu_2)}\right]\iint\left(\frac{\partial^2w}{\partial x\partial y}\right)^2\mathrm{d}x\mathrm{d}y+$$

$$2\mu_1\left[\frac{E_1h_1^3}{3(1-\mu_1^2)}-\frac{E_2(h_1-h)^3\mu_2}{3(1-\mu_1^2)\mu_1}\right]\iint\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}\mathrm{d}x\mathrm{d}y+$$

$$\frac{1}{2}\iint\left[\frac{\partial^2\varphi}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\mathrm{d}x\mathrm{d}y-$$

$$P\iint w\delta(x-\xi)\delta(y-\eta)\mathrm{d}x\mathrm{d}y \tag{21}$$

式中 ξ、η 分别为集中载荷的横坐标和纵坐标。

把式(18)代入式(21)中，利用最小势能原理 $\dfrac{\partial U}{\partial F_1}=0$、$\dfrac{\partial U}{\partial F_2}=0$ 可得：

$$\begin{cases}B_{11}F_1+B_{12}F_1F_2=B_{13}\\ B_{21}F_2=B_{22}F_1^2\end{cases} \tag{22}$$

$$B_{11}=D\int_0^a\int_0^b\left[\left(\frac{\mathrm{d}X_1}{\mathrm{d}x}Y_1\right)^2+\left(X_1\frac{\mathrm{d}Y_1}{\mathrm{d}y}\right)^2\right]\mathrm{d}x\mathrm{d}y+$$

$$2\left[\frac{E_1h_1^3}{3(1+\mu_1)}-\frac{E_2(h_1-h)^3}{3(1+\mu_2)}\right]\int_0^a\int_0^b\left(\frac{\mathrm{d}X_1}{\mathrm{d}x}\frac{\mathrm{d}Y_1}{\mathrm{d}y}\right)^2\mathrm{d}x\mathrm{d}y+$$

$$4\mu_1\left[\frac{E_1h_1^3}{3(1-\mu_1^2)}-\frac{E_2(h_1-h)^3\mu_2}{3(1-\mu_2^2)\mu_1}\right]\cdot$$

$$\int_0^a\int_0^b\frac{\mathrm{d}^2X_1}{\mathrm{d}x^2}X_1Y_1\frac{\mathrm{d}^2Y_1}{\mathrm{d}y^2}\mathrm{d}x\mathrm{d}y,$$

$$B_{12}=\int_0^a\int_0^b\left[\frac{\mathrm{d}^2X_2}{\mathrm{d}x^2}Y_2\left(X_1\frac{\mathrm{d}Y_1}{\mathrm{d}y}\right)^2+X_2\frac{\mathrm{d}^2Y_2}{\mathrm{d}y^2}\left(\frac{\mathrm{d}X_1}{\mathrm{d}x}Y_1\right)^2-2\frac{\mathrm{d}X_2}{\mathrm{d}x}\frac{\mathrm{d}Y_2}{\mathrm{d}y}\frac{\mathrm{d}X_1}{\mathrm{d}x}X_1Y_1\frac{\mathrm{d}Y_1}{\mathrm{d}y}\right]\mathrm{d}x\mathrm{d}y,$$

$$B_{13}=P\int_0^a\int_0^b[X_1Y_1\delta(x-\xi)\delta(y-\eta)\mathrm{d}x\mathrm{d}y,$$

$$B_{21}=\frac{1}{E_1h}\int_0^a\int_0^b\left[\frac{\mathrm{d}^2X_2}{\mathrm{d}x^2}Y_2+X_2\frac{\mathrm{d}^2Y_2}{\mathrm{d}y^2}\right]\mathrm{d}x\mathrm{d}y-$$

$$4(1+\mu_1)\int_0^a\int_0^b\left[\left(\frac{\mathrm{d}X_2}{\mathrm{d}x}\frac{\mathrm{d}Y_2}{\mathrm{d}y}\right)^2-\frac{\mathrm{d}^2X_2}{\mathrm{d}x^2}X_2Y_2\frac{\mathrm{d}^2Y_2}{\mathrm{d}y^2}\right]\mathrm{d}x\mathrm{d}y,$$

$$B_{22}=\frac{1}{2}\int_0^a\int_0^b\left[\frac{\mathrm{d}^2X_2}{\mathrm{d}x^2}Y_2\left(X_1\frac{\mathrm{d}Y_1}{\mathrm{d}y}\right)^2+X_2\frac{\mathrm{d}^2Y_2}{\mathrm{d}y^2}\left(\frac{dX_1}{dx}Y_1\right)^2-2\frac{\mathrm{d}X_2}{\mathrm{d}x}\frac{\mathrm{d}Y_2}{\mathrm{d}y}\frac{\mathrm{d}X_1}{\mathrm{d}x}X_1Y_1\frac{\mathrm{d}Y_1}{\mathrm{d}y_1}\right]\mathrm{d}x\mathrm{d}y$$

由式(22)可以得到方程：

$$B_{11}F_1 + \frac{B_{12}B_{22}}{B_{21}}F_1^3 = B_{13} \tag{23}$$

求解式(20)、式(23)即可得到双模量矩形板在外载荷作用下大挠度变形时板的中点挠度。

对于在均布载荷 q 作用下的四边简支方薄板可设 w 及 φ 的表达式为：

$$\begin{cases} w = F_1\left(\dfrac{x^4}{a^4} - 2\dfrac{x^3}{a^3} + \dfrac{x}{a}\right)\left(\dfrac{y^4}{a^4} - 2\dfrac{y^3}{a^3} + \dfrac{y}{a}\right) \\ \varphi = F_2\left(\dfrac{x^4}{a^4} - 2\dfrac{x^3}{a^3} + \dfrac{x^2}{a}\right)\left(\dfrac{y^4}{a^4} - 2\dfrac{y^3}{a^3} + \dfrac{y^2}{a}\right) \end{cases} \tag{24}$$

把式(24)代入式(20)中且设方薄板中点挠度为 w_0 可以得到：

$$qa^4 = 241.7Dw_0 + 3.6502E_1hw_0^3 \tag{25}$$

当 $E_1 = E_2 = E$，$\mu_1 = \mu_2 = 0.316$ 时可把式(25)化为：

$$\frac{qa^4}{Eh^4} = 22.3759\frac{w_0}{h} + \frac{3.6502w_0^3}{h^3} \tag{26}$$

对于在均布载荷 q 作用下的薄方板，当 $E_1 = E_2 = E$，$\mu_1 = \mu_2 = 0.316$ 时，贾春元则给出了载荷与板中点挠度的关系式为[11]：

$$\frac{qa^4}{Eh^4} = 22.25\frac{w_0}{h} + \frac{3.9008w_0^3}{h^3} \tag{27}$$

$\dfrac{w_0}{h}$ 分别取 1、1.2、1.4、1.6，式(26)与式(27)的计算结果误差分别为 0.4768%、0.8432%、1.2218%、1.5996%，说明本文方法是可靠的。

4　算例分析

为了具体分析拉压弹性模量对双模量矩形板大挠度弯曲变形的影响，以表 1 所示材料四边简支方板作用均布载荷为例进行计算。其中材料 1 为玻璃丝织物，材料 2 为硅铝合金[10]。材料 3－材料 6 为假想的分别与材料 1 和材料 2 拉区材料参数(或压区材料参数)相同的单模量材料。

表 1　材料特性参数

Table1　Material characteristic parameters

材料	E_1/GPa	E_2/GPa	μ_1	μ_2
材料 1	30.38	16.17	0.35	0.19
材料 2	66.93	73.42	0.34	0.39
材料 3	30.38	30.38	0.35	0.35
材料 4	16.17	16.17	0.19	0.19
材料 5	66.93	66.93	0.34	0.34
材料 6	73.42	73.42	0.39	0.39

为了验证本文计算方法正确性，分别用 ANSYS7.0 和本文方法(即式(25))计算了由材料 1 或材料 3 组成的均布载荷作用下四边简支方板中点挠度(w_0)。板边长 a=2000mm，h=100mm。由材料 1 组成的板有限元计算时采用 8 节点 SOLID185 单元，该单元具有大变形，大应变能力。由式(5)计算出受拉区高度 h_1= 60mm，上层 60mm 材料参数为 E=16.17GPa，μ=0.19，下层 40mm 材料参数为 E=30.38，μ=0.35，厚度方向单元尺寸为 20mm，边长方向单元尺寸为 40mm，采用 Large Displacement static analysis 求解。由材料 3 组成的板有限元计算时采用采用 8 节点 SOLID185 单元，该单元具有大变形，大应变能力。材料参数为 E=30.38，μ=0.35，厚度方向单元尺寸为 20mm，边长方向单元尺寸为 40mm。采用 Large Displacement static analysis 求解。qa^4/h^4 =100GPa 时材料 1 和材料 3 组成四边简支方板时节点平面外位移如图 1 和图 2 所示。计算结果及结果比较如表 2 所示。结果比较数值为本文结果与 ANSYS7.0 计算结果差值除以 ANSYS7.0 计算结果。

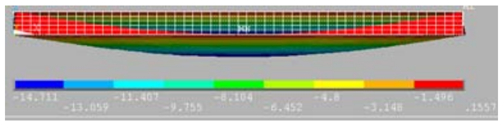

图 1　节点平面外位移(材料 1)

Fig.1　The nodes displacement out of plane (material 1)

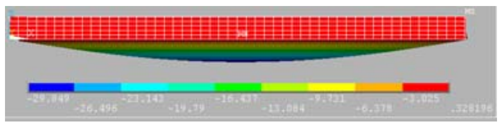

图 2　节点平面外位移(材料 3)

Fig.2　The nodes displacement out of plane (material 3)

表 2　本文结果与 ANSYS 7.0 结果比较

Table 2　The results comparison between this paper and ANSYS 7.0

材料		$qa^4/h^4/(\times10^{10})$	10	20	30	40	50	60	70
材料 1	w_0/mm	本文方法	21.0	40.8	59.8	75.0	89.3	102.2	113.8
		ANSYS 7.0	20.4	40.2	58.7	75.3	90.6	104.5	116.4
	误差/(%)		2.94	1.49	1.87	−0.40	−1.43	−2.20	−2.23
材料 3	w_0/mm	本文方法	14.3	28.3	41.9	54.8	66.9	78.4	89.1
		ANSYS 7.0	14.1	27.6	40.9	54.3	65.7	78.1	88.5
	误差/(%)		1.42	2.54	2.44	0.92	1.83	0.38	0.68

为了分析均布载荷作用下四边简支方板大挠度弯曲时载荷与挠度的非线性关系，采用式(25)计算了均布载荷作用下四边简支方板 $\dfrac{w_0}{h}$ 分别取 0.2，

0.4，…，1.6 时 $\frac{qa^4}{h^4}$ 的计算结果，如表 3 所示。

表 3　载荷与挠度的非线性关系　/(×10^10)

Table 3　The nonlinear relation between load andn deflection

w_0/h		0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$\frac{qa^4}{h^4}$ / (N/m²)	材料 1	9.51	19.56	30.66	43.37	58.21	75.70	96.39	120.81
	材料 2	32.78	66.73	103.02	142.83	187.34	237.70	295.10	360.72
	材料 3	14.04	28.60	44.23	61.46	80.82	102.84	128.05	156.99
	材料 4	6.80	13.89	21.55	30.05	39.69	50.75	63.50	78.24
	材料 5	30.68	62.54	96.73	134.45	176.86	225.13	280.44	343.95
	材料 6	35.10	71.48	110.43	153.25	201.21	255.60	317.71	388.82

为了分析双模量对均布载荷作用下四边简支方板中点挠度影响，比较了双模量四边简支方板和分别与该双模量材料受拉区材料或受压区材料相同的单模量四边简支方板的中点挠度相差大小，比较时将单模量四边简支方板中点挠度与双模量四边简支方板中点挠度除以双模量四边简支方板中点挠度，计算结果如表 4 所示。

表 4　不同材料 qa^4/h^4 计算结果比较　/(%)

Table 4　The results comparison of qa^4/h^4 between different materials

w_0/h	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
材料 1 和材料 3	47.63	46.26	44.25	41.72	38.86	35.85	32.85	29.95
材料 1 和材料 4	–28.50	–28.96	–29.73	–30.71	–31.81	–32.97	–34.12	–35.24
材料 2 和材料 5	–6.41	–6.28	–6.10	–5.87	–5.59	–5.29	–4.97	–4.65
材料 2 和材料 6	7.08	7.12	7.20	7.29	7.40	7.53	7.66	7.79

由表 2 可以看出，采用 ANSYS7.0 和本文方法计算的结果比较相近，二者相差都小于工程上所允许的误差(5%)，这验证了本文方法的可靠性。

对表 3、表 4 的计算结果进行分析可知，对于拉压弹性模量不同材料的薄板大挠度弯曲变形，不考虑拉压弹性模量相异时其计算结果与实际情况相差较大，超过了工程上所允许的计算误差。所以，对于拉压弹性模量相差较大的薄板大挠度弯曲变形，其中点挠度的计算不宜采用相同弹性模量经典薄板理论，而应该采用双模量薄板大挠度理论。

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