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Two-Dimensional Weakly Interacting Bose Gas in the Fluctuation Region

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We study the crossover between the mean-field and critical behavior of the two-dimensional Bose gas throughout the fluctuation region of the Berezinskii–Kosterlitz–Thouless phase transition point. We argue that this crossover is described by universal (for all weakly interacting $|\psi|^4$ models) relations between thermodynamic parameters of the system, including superfluid and quasi-condensate densities. We establish these relations with high-precision Monte Carlo simulations of the classical $|\psi|^4$ model on a lattice, and check their asymptotic forms against analytic expressions derived on the basis of the mean-field theory.

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I. INTRODUCTION

The mean-field (MF) approach to the weakly interacting Bose gas (BG) is a well established theoretical tool [1, 2, 3]. However, it is not adequate in the fluctuation region of the superfluid phase transition. The situation is most dramatic in the two-dimensional (2D) case, where the size of the fluctuation region, ΔT, is almost insensitive to the smallness of interaction [4] ($\hbar = 1$):

$$\Delta T/T_c \sim 1/\ln(1/mU) .\qquad (1)$$

Here T_c is the critical temperature, m is the particle mass, and U is the effective long-wavelength interaction constant. The regime of weak interaction corresponds to a small dimensionless parameter

$$mU \ll 1 ,\qquad (2)$$

which close to the transition point is equivalent to the condition $nU \ll T$ (n is the particle density).

For the purposes of the present paper the microscopic origin of the effective interaction U is not important. However, to make connection between our results and realistic experimental systems we briefly review how U relates to the interatomic interaction potential $V(r)$. The value of U corresponds to the pair vertex (the sum of ladder diagrams) with typical external momenta $\sim n^{1/2}$. It can be written in a generic form as

$$U = \frac{V_0}{1 + (mV_0/4\pi)\ln(1/nd^2)} ,\qquad (3)$$

where microscopic parameters V_0 and d depend on the case. The simplest one is that of a weak short-ranged potential satisfying the condition $V(r \sim r_0) \ll 1/mr_0^2$, where r_0 is the potential radius. In this case, $V_0 = \int V(r) d^2r$, and with logarithmic accuracy assumed in Eq. (3) one may put $d = r_0$ provided $nr_0^2 \ll 1$ (when $nr_0^2 \gtrsim 1$ the logarithmic term in the denominator can be neglected). In the *quasi*-2D system [5, 6], when the localization length of 3D atoms in the direction perpendicular to the 2D plane (axis $\hat{\mathbf{z}}$) is much larger than r_0, one has

first to average the pair interaction over a wavefunction in the $\hat{\mathbf{z}}$-direction, $\phi_0(z)$. Now, $V_0 = (4\pi a/m)\int |\phi_0(z)|^4 dz$, where a is the 3D scattering length. Introducing the localization length by $l_z^{-1} = \int |\phi_0(z)|^4 dz$, we have $V_0 = 4\pi a/ml_z$. With the same logarithmic accuracy, in Eq. (3) $d \approx l_z$ for $nl_z^2 \ll 1$ (cf. Refs. 5, 6). Finally, the case of strong short-ranged potential [which in the context of weakly interacting gas implies $\ln(1/nr_0^2) \gg 1$] formally corresponds to the limit $V_0 \to \infty$ in Eq. (3). In this case the effective interaction depends only on the parameter $\ln(1/nd^2)$ with $d \approx r_0$ (see Ref. 7).

Because of weak log-dependence of the fluctuation region on interaction one may wonder whether the MF theory makes sense at all in 2D (apart from the academic limit of exponentially small mU), and, if it does, then when. As a characteristic example of how problematic it is to reach the proper asymptotic limit, consider a dilute gas with very small nd^2 when $mU \sim 4\pi/\ln(1/nd^2) \ll 1$. With the same logarithmic accuracy it follows then that the critical point can be found without even resorting to the Berezinskii–Kosterlitz–Thouless physics [3, 4], $T_c \approx (2\pi n/m)\ln^{-1}(1/mU)$. Meanwhile, a more accurate result for the critical point is (for future reference and convenience we write the answer for critical density as a function of temperature) [8] :

$$n_c = \frac{mT}{2\pi}\ln\left(\frac{\xi}{mU}\right) ,\quad \xi = 380 \pm 3 .\qquad (4)$$

Obviously, an enormous value of ξ makes it virtually impossible to reach the limit of small U when ξ can be ignored.

There is, however, a very important point about the fluctuation region of a weakly interacting BG: In the limit of small U all $|\psi|^4$ models—quantum or classical, continuous or discrete—allow a universal description [3, 9]. This observation follows from a simple fact that interactions are important only for long-wavelength components of the order parameter field, $\psi(\mathbf{r})$, with momenta $k \lesssim k_c = m\sqrt{UT} \ll k_T = \sqrt{mT}$, and in this limit the

effective Hamiltonian is given by the $|\psi|^4$ model

$$H[\psi] = \int \left\{ \frac{1}{2m} |\nabla \psi|^2 + \frac{U}{2} |\psi|^4 - \mu |\psi|^2 \right\} d\mathbf{r} , \qquad (5)$$

where μ is the effective chemical potential. The microscopic physics of the model is important only at much higher momenta, $k \gg k_c$, where the system behavior is ideal (in linear in U approximation) and thus may be easily accounted for analytically.

This observation was successfully used in Refs. 8, 10, 11 (both in 3D and 2D) in the study of the critical point dependence on interaction. Same considerations apply, though, not only to the critical point itself, but to the whole fluctuation region around it, and one thus expects that, e.g., the superfluid density dependence on density, $n_s(n - n_c)$, or chemical potential, $n_s(\mu - \mu_c)$, is also universal close to the transition point and into the region where the MF theory takes over.

The study of this universal behavior is the subject of this paper. We found that even for very weak interaction, say, $mU \sim 0.01$, the conventional MF theory result $n_s/n = 1 - T/T_c$ may not be used since the fluctuation region is still of order T_c itself. However, if T_c is related to the density by Eq. (4), the modified version of the MF theory developed in this paper works remarkably well. In particular, the Equation of state and the quasicondensate density may be predicted very accurately up to T_c (this is not true for the superfluid density, though).

In Sec. II we establish the universal form of the Equation of state and the dependence of the superfluid density and quasicondensate density on chemical potential along with their asymptotic behavior away from the critical point. In Sec. III we describe the numeric model and the simulation procedure. Our results are presented and compared to the MF and Kosterlitz–Thouless theories in Sec. IV. We conclude in Sec. V with discussing the obtained universal results in the context of quantum Bose gases.

II. UNIVERSAL RELATIONS FOR WEAKLY INTERACTING $|\psi|^4$ MODELS

The critical point of the BG is defined by Eq. (4) and the corresponding relation for the chemical potential [8]

$$\mu_c = \frac{mTU}{\pi} \ln \left(\frac{\xi_\mu}{mU} \right) , \qquad \xi_\mu = 13.2 \pm 0.4 \qquad (6)$$

which can be rewritten in the form

$$\mu_c = 2n_c U + \frac{mTU}{\pi} \ln \left(\frac{\xi_\mu}{\xi} \right) . \qquad (7)$$

Both n_c and μ_c are model specific, and their values depend on the ultra-violet cutoff, k_*, through the logarithm $\ln(k_*/k_c)$; for the quantum gas $k_* \sim k_T$. However, if the dominant MF type contribution to the chemical potential, $2nU$, is subtracted, the difference is an ultra-violet-cutoff-independent quantity. The same is true for the

difference $\mu - \mu_c$ or $n - n_c$. It seems natural then to introduce a dimensionless variable

$$X = (\mu - \mu_c)/mTU , \qquad (8)$$

as a universal control parameter with the typical variation across the fluctuation region of order unity. The Equation of state may then be written in the universal form as

$$\frac{2nU - \mu}{mTU} = \theta(X) , \qquad (9)$$

where θ is a dimensionless function. By subtracting critical values from n and μ we can restate it as

$$n - n_c = mT\lambda(X) , \qquad \lambda(X) = [\theta(X) - \theta_0 + X]/2 , \quad (10)$$

with

$$\theta_0 \equiv \theta(0) = \frac{1}{\pi} \ln(\xi/\xi_\mu) . \qquad (11)$$

From previous results [8] we have $\theta_0 = 1.07 \pm 0.01$, with the error bar being largely determined by the uncertainty in ξ_μ. The $\lambda(X)$ function describes the so-called adsorption isotherm which is relevant to the situation where the 2D system is formed by atoms adsorbed on a surface. In the case of a trapped gas, the function $\lambda(X)$ describes the density profile of the gas in the hydrostatic regime (see the discussion in Sec. V).

The behavior of the superfluid density is described by a dimensionless function f:

$$n_s = (2mT/\pi) f(X) . \qquad (12)$$

According to the Kosterlitz-Thouless theory [12] (see also below),

$$f(X \to +0) \longrightarrow 1 + \sqrt{2\kappa' X} , \qquad (13)$$

where κ' is some constant to be defined numerically.

Finally, in the superfluid region an important quantity is the quasicondensate density n_0, which we define by the relation

$$n_0 = \sqrt{Q} , \quad Q = 2\langle |\psi|^2 \rangle^2 - \langle |\psi|^4 \rangle . \qquad (14)$$

The idea behind this definition is as follows. The notion of the quasicondensate [5] implies that the field ψ has the following structure

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \psi_1(\mathbf{r}) , \qquad (15)$$

$$\psi_0(\mathbf{r}) \approx \sqrt{n_0}\, e^{i\Phi(\mathbf{r})} , \qquad (16)$$

where n_0 is called the quasicondensate density, and ψ_1 is the Gaussian field independent of ψ_0. Under these conditions, $Q \equiv n_0^2$. [The Gaussian field ψ_1 obeys the Wick's theorem and thus does not contribute to Q.] Away from the superfluid region the notion of quasicondensate gradually becomes ill-defined, but the quantity Q is still of interest as a measure of local non-Gaussian correlations.

We will jargonically use the term 'quasicondensate density' even well inside the fluctuation region, understanding by n_0 the quantity $\sqrt{Q}$.

Since the MF theory result predicts $n_0 \approx n_s$, it is appropriate to characterize the dependence of Q on μ in close analogy to Eq. (12):

$$\sqrt{Q} = (2mT/\pi)\, g(X) \,. \tag{17}$$

The three functions — $\theta(X)$, $f(X)$, and $g(X)$ — completely characterize system properties in the vicinity of the critical point

We now turn to the MF and Kosterlitz–Thouless theories to establish asymptotic behavior of functions $\theta(X)$, $f(X)$, and $g(X)$.

Asymptotic behavior at $X \to \infty$. The notion of quasicondensate is well-defined in this region and its density n_0 obeys a typical MF relation [5]

$$(n_0 + 2n')U = \mu \,, \tag{18}$$

where

$$n' = n - n_0 \tag{19}$$

is the non-quasicondensate part of the particle density. [One may worry how far the analogy between the genuine condensate and quasicondensate goes; the answer is that apart from the long-range order problem they are indistinguishable at the MF level, and our simulations confirm this assertion.] Comparing (18)-(19) with (9), we see that

$$n_0(X \to \infty) \longrightarrow mT\,\theta(X) \,. \tag{20}$$

or

$$g(X \to \infty) \longrightarrow (\pi/2)\,\theta(X) \,. \tag{21}$$

An explicit expression for the non-quasicondensate part is given by [13]:

$$n' = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{\epsilon(k) + n_0 U - E(k)}{2E(k)} + \frac{\epsilon(k)\,\nu_E}{E(k)} \right] \,, \tag{22}$$

where $\epsilon(k) = k^2/2m$ is the free particle dispersion law, $E(k) = \sqrt{\epsilon(k)[\epsilon(k) + 2n_0 U]}$ is the Bogoliubov quasiparticle spectrum, and $\nu_E = [\exp[E(k)/T] - 1]^{-1}$ is the Bose distribution function. In the region of interest, $\theta \sim 1$, the first term in the integral is smaller than the second one by the gas parameter $mU \ll 1$, and should be omitted [16]. With the same accuracy, the second term yields

$$n' \approx -(mT/2\pi)\ln(2n_0 U/T) \,. \tag{23}$$

With the help of Eq. (20) the total density $n = n' + n_0$ may be now written as

$$
\begin{aligned}
n &\approx \frac{mT}{2\pi}\ln(1/mU) + \frac{mT}{2}\left[2\theta - \frac{1}{\pi}\ln(2\theta) \right] \\
&\equiv n_c + \frac{mT}{2}\left[2\theta - \frac{1}{\pi}\ln(2\xi\theta) \right] \,.
\end{aligned}
\tag{24}
$$

Substituting this relation into (10) we find the asymptotic behavior of $\theta(X)$:

$$\theta - \pi^{-1}\ln\theta \to X + \pi^{-1}\ln(2\xi_\mu) \quad \text{at} \quad X \to \infty \,. \tag{25}$$

To find $f(X \to \infty)$, we consider the standard expression for the normal component density [2]

$$n_n = -\frac{1}{2m}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{\mathrm{d}\nu_E}{\mathrm{d}E} \right] k^2 \,, \tag{26}$$

and compare it to the expression (22) for the non-quasicondensate part of the particle density. After integration by parts and a straightforward algebra we find that up to higher-order in mU terms, $n_n - n' = mT/2\pi$, which means that

$$n_s = n_0 - mT/2\pi \,, \tag{27}$$

and, accordingly,

$$f(X \to \infty) \longrightarrow g(X) - 1/4 \longrightarrow (\pi/2)\,\theta(X) - 1/4 \,. \tag{28}$$

It is important to note that while for obtaining asymptotic relations (25) and (27)-(28) we employed the theory of the weakly interacting *quantum Bose gas*, the final results are valid for *any* weakly interacting 2D system of the $|\psi|^4$ universality class, since these pertain to the universal long-wave behavior of the system.

Asymptotic behavior at $X \to -\infty$. In the region $X < 0$ the f-function is identically zero; the quasicondensate density is of no special interest in the normal phase [$g(X) \to 0$ in this limit]. Hence, the only quantity we have to look at is the Equation of state, $\theta(X)$. Once again, we resort to the MF equation for the effective chemical potential $\mu' = \mu - 2nU = -\theta mUT$, and calculate the total density from the integral:

$$n \approx \int \frac{\mathrm{d}^2k}{(2\pi)^2}\left[\exp(\epsilon/T + \theta mU) - 1 \right]^{-1} \tag{29}$$

$$\approx -(mT/2\pi)\ln(\theta mU) \equiv n_c - \frac{mT}{2\pi}\ln(\theta\xi) \,. \tag{30}$$

This expression may be immediately related back to the Equation of state (9) and leads to the relation

$$\theta + \pi^{-1}\ln\theta \to |X| - \pi^{-1}\ln\xi_\mu \quad \text{at} \quad X \to -\infty \,. \tag{31}$$

One has to understand the limit $|X| \to \infty$ in the following sense: it describes the system behavior close to the transition point but outside the fluctuation region. For the quantum gas ($mU \ll 1$) it means $1 \ll |X| \ll 1/mU$. Of course, one may easily calculate system properties for any $|X| \gg 1$ using MF theory presented above, and, take care of the phonon contribution to the non-quasicondensate and superfluid densities at $n \sim T/U$ [contained in the integrals of Eqs. (22) and (26)], or, instead of Eq. (30), consider a more accurate expression for the dilute density limit $n \approx -(mT/2\pi)\ln\left[1 - e^{-\theta mU}\right]$ to include the Boltzmann gas into the picture. Such obvious generalizations are not considered in this paper.

The vicinity of the Kosterlitz-Thouless point. The thermodynamic limit close to the point of the Berezinskii-Kosterlitz-Thouless transition requires simulations of exponentially large systems, and thus can hardly be treated numerically without renormalization group (RG) analysis of finite size corrections. Fortunately, one can take advantage of the Kosterlitz-Thouless RG equations that describe the flow of the superfluid density $n_s(L)$ with increasing the system size L. In terms of dimensionless function $f_L = (\pi/2mT)\, n_s(L)$ these equations are [12]

$$\frac{\mathrm{d}f_L}{\mathrm{d}\ln L} = -y^2 f_L^2 \,, \tag{32}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\ln L} = 2(1 - f_L)\, y \,, \tag{33}$$

where $y(L)$ is the vortex-pair fugacity. By excluding variable y and integrating the remaining RG equation, one obtains the following relation

$$F(f_{L_2}, f_{L_1}, \kappa) = 4\ln(L_2/L_1) \,, \tag{34}$$

where F is defined as an integral

$$F(a, b, \kappa) = \int_a^b \frac{\mathrm{d}t}{t^2(\ln t - \kappa) + t} \,, \tag{35}$$

and $\kappa(X)$ is a size-independent (at $k_c L \gg 1$) parameter.

By performing large-scale simulations of systems with different sizes $L_1 < L_2 < L_3 < ...$, one may solve Eq. (34) for parameter $\kappa(X)$, verify that it is system size independent, and then determine the thermodynamic value $f(X) = f_{L=\infty}(X)$ from the relation

$$1/f + \ln f = \kappa \,, \tag{36}$$

that immediately follows from (34)-(35) at $L \to \infty$. Since Eq. (36) has a root only at $\kappa \geq 1$, we conclude that $\kappa = 1$ corresponds to the critical point. In contrast to the superfluid density, $\kappa(X)$ has no singularities at the critical point $X = 0$ and may be expanded into Taylor series

$$\kappa(X) \approx 1 + \kappa' X + ... \,. \tag{37}$$

The solution of Eq. (36) for small X is then given by formula (13).

III. NUMERICAL PROCEDURE

Although all derivations presented in the previous section were done for the quantum BG, we expect them to be universal and apply for any model with effective long wave-length Hamiltonian (5) with small mU. Classical lattice algorithms are much more efficient than quantum ones and allow high-accuracy simulations of very large system sizes. Also, simulations of the classical lattice

FIG. 1: The Equation of state $\theta(X)$ compared to its asymptotic large $|X|$ expressions, see Eqs. (25) and (31)

model directly test the idea of universality, since they have to agree with all Eqs. (11), (21), (25), (28), (31).

Our simulations were done for the simple square lattice Hamiltonian

$$H = \sum_{\mathbf{k}\in BZ} [E(\mathbf{k}) - \mu]|\psi_k|^2 + \frac{U}{2}\sum_i |\psi_i|^4 \,, \tag{38}$$

where ψ_k is the Fourier transform of the lattice field ψ_i, and $E(\mathbf{k}) = [2 - \cos(k_x a) - \cos(k_y a)]/ma^2$ is the tight-binding dispersion law; momentum $\mathbf{k}$ being defined within the first Brillouin zone (BZ). We employed the recently developed Worm algorithm for classical statistical systems [17] which has direct Monte Carlo estimators for all quantities of interest here and does not suffer from critical slowing down.

We performed simulations for system sizes $L = 64$, 128, 256, 512 and two values of interaction strength $U = 1/4$ and $U = 1/16$ to eliminate finite-size and finite-U corrections to the results. Each quantity for each point in X was calculated with relative accuracy better then 10^{-3} (down to 10^{-4} for smaller system sizes). Our final results for $\theta(X)$, $f(X)$, and $g(X)$ presented in Figs. 1, 3, and 5 below are thus obtained with accuracy better than 1 % (the largest error bars of order 1 % are in the vicinity of the critical point where finite-size corrections are the largest; the errorbars for large $|X|$ shrink down to $0.3 \div 0.1$ %). Error bars are shown in all plots but typically they are much smaller than the point size. All the relevant data are mentioned in Table I. In Figures 1-5 the MC data are presented by dots and compared with the asymptotic analytic solutions shown by lines.

IV. SIMULATION RESULTS

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FIG. 2: The $\lambda(X)$ function compared to its asymptotic large $|X|$ expressions

FIG. 4: RG parameter $\kappa(X)$ obtained from finite-size scaling of the data according to Eqs. (34) and (35) and fitted using linear in X expansion with $d\kappa/dX = 0.61$

FIG. 3: The quasicondensate density dependence on X compared to the asymptotic large $|X|$ behavior predicted by Eq. (21)

FIG. 5: The superfluid density dependence on X compared to the asymptotic large $|X|$ behavior according to Eq. (28), and the Kosterlitz-Thouless theory for $X \leq 0.25$ according to Eq. (36)

In Figs. 1, 2, 3, and 5 we show our final results for the scaling functions. To compare them with analytic predictions we first determine the value of the θ_0 parameter from the $\theta(x)$ plot to verify that it agrees with the result predicted by Eq. (11). We find that

$$\theta_0 = 1.068 \pm 0.01 , \tag{39}$$

thus confirming the universality of parameter θ_0. Knowing ξ_μ is all we need to handle the asymptotic behavior at large $|X|$. The best fits correspond to $\xi_\mu = 13.4$, which, within the error bars, coincides with Eq. (6). The agreement between the data and asymptotic laws Eqs. (25) and (31) in Figs. 1 and 2 is remarkable.

The same is also true for the quasicondensate density (see Fig. 3). The data perfectly agree with the idea of quasicondensate which is not entirely obvious at a first glance for the system without long-range order. We thus confirm that n_0, rigorously defined through the correlation function Eq. (14), plays the same role as the genuine condensate density in the 3D theory.

As explained above, the analysis of the superfluid density data near the critical point is based on the Kosterlitz-Thouless RG equations. We found that for $X \in (-0.1, 0.25)$ the data scale according to the RG Eqs. (34) and (35) with negligible finite-U corrections, and $\kappa(X)$ dependence is well described by the linear in X expansion Eq. (37), see Fig. 4. Having determined the first derivative of κ at the critical point as

$$\kappa' = \left. \frac{d\kappa}{dX} \right|_{X=0} = 0.61 \pm 0.01 , \tag{40}$$

FIG. 6: Temperature dependence of n_s/n for small $mU = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-5}. Points are connected with lines to guide the eye. The dashed lines are the mean-field theory results with exact relation between T_c and particle density

FIG. 7: Temperature dependence of n_0/n for small $mU = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-5}. Points are connected with lines to guide the eye. The dashed lines are the mean-field theory results

we proceed with the solution of the thermodynamic value of $f(X)$ using Eq. (36). The results are plotted in Fig. 5 along with the asymptotic large X law $f(X) \to (\pi/2)\,\theta(X) - 1/4$. The two limits match almost perfectly around $X = 0.5$ and describe all the data points with exceptional accuracy.

V. CONCLUSIONS

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The most striking result of this study is that the Equation of state, $\theta(X)$, and the quasicondensate density, $g(X)$, are predicted by the MF theory with accuracy of

FIG. 8: Temperature dependence of n_0/n for $mU = 0.2$ derived from universal relations (filled circles) and simulated for the quantum lattice BG (open circles). Points are connected with lines to guide the eye

few percent all the way to the critical point. Even at the critical point the difference between the data and asymptotic relations is barely visible. This outcome is quite unexpected because the superfluid density does show deviations from the MF theory for $X < 0.5$.

Hence, from the side of the superfluid phase, the boundary of the fluctuation region corresponds to $X \sim 0.5$. To express this estimate in terms of temperature (density), we relate X and T at a constant density (X and n at a constant temperature). Equation (10) with n_c from Eq. (4) may be written as

$$\frac{mT}{n} = \frac{2\pi}{\ln(\xi/mU) + 2\pi\lambda(X)} \, ,$$

or

$$\frac{T_c(n)}{T} = \frac{n}{n_c(T)} = 1 + \frac{2\pi}{\ln(\xi/mU)} \, \lambda(X) \, , \qquad (41)$$

and maps the control parameter X onto $T/T_c(n)$ or $n/n_c(T)$. This relation is not universal because T_c, n_c, and the r.h.s. of Eq. (41) depend on the ultra-violet cutoff. For $X = 0.5$ and, say, $mU = 0.1$ we find that the fluctuation region is roughly $(T_c-T)/T_c \sim (n-n_c)/n_c \sim 0.3$.

It is instructive to compare results for the superfluid density of the weakly interacting BG in the more conventional n_s/n vs T/T_c plot at a constant particle density which may be immediately obtained from the universal relations. The superfluid density is given by

$$\frac{n_s}{n} = \frac{2mT}{\pi n} \, f(X) \equiv \frac{4(T/T_c)}{\ln(\xi/mU)} \, f(X) \, , \qquad (42)$$

which along with Eq. (41) defines a parametric dependence of n_s/n on T/T_c. One may construct then a modified MF theory result by substituting $f(X)$ in this relation with $\pi\theta(X)/2 - 1/4$, see Eq. (28), and $\theta(X)$ from

Eq. (25). An analogous expression for n_0 is obtained by replacing $f(X)$ with $g(X)$.

In Fig. 6 we show the comparison between the MF solution and the data for small effective interaction $mU = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-5}. We see that the fluctuation region is still of order T_c even for $mU = 0.01$ — clearly that small coupling parameter may not be obtained by going to the dilute limit for the gas of hard-core particles and thus any realistic discussion of the experimental data should involve the proper description of the fluctuation region. Even for $mU = 0.01$ the familiar MF formula $n_s/n \approx 1 - T/T_c$ does not work at all — the slope keeps changing with T and the left-most slope does not point to $n_s/n = 1$. However, if one uses an exact relation between T_c and n, then the MF approach described above is capable of reproducing the data for $X > 0.5$.

The other remark concerns the asymmetry of the fluctuation region at $T < T_c$ and $T > T_c$. The minimum in the Fig. 1 plot for $\theta(X)$ and the agreement with the MF laws already suggest that the fluctuation region is much broader on the normal side. When MF results for $n_s(T)$ in Fig. 6 are extrapolated to higher temperatures, the intersection with the temperature axis is still at $\approx 1.5\, T_c$ for mU as small as $\sim 10^{-5}$! We draw the same conclusion from the n_0/n plot shown in Fig. 7: The decay of the quasicondensate density $n_0 \equiv \sqrt{Q}$, which is a measure of local non-Gaussian correlations, is quite extended into the normal state.

Our universal relations are obtained in the limit $mU \ll 1$. From a practical point of view it is important to estimate a typical value of mU at which higher-order corrections to our results become unimportant. To this end we note that if we plot the quasicondensate density as a function of T/T_c for various values of mU, see Fig. 7 we will find that for $mU = 0.1$ the ratio n_0/n exceeds unity already at $T \approx 0.4\, T_c$. This unphysical result tells us that non-universal corrections for the quantum BG are *not* negligible even for $mU \sim 0.1$ (at least for the quasicondensate density; for numerical reasons they might be smaller for other quantities). We thus expect that universal expressions established in this study are likely to work without limitations only for mU significantly smaller than 0.1. To understand the situation with the quantum corrections quantitatively, we compare in Fig. 8 our results for n_0/n at $mU = 0.2$ with the previously reported [15] results for the quantum lattice model [18]. The comparison suggests that at $T \sim T_c$ the quantum correction to the quasicondensate density is $\sim mU$ (in relative units). The sign of the correction is negative, that is we are dealing with a quantum depletion of the quasicondensate.

For helium films on various substrates [19, 20], and spin-polarized atomic hydrogen on helium film [21], the value of mU is most probably of order unity [5].

In the recently created quasi-2D gas of sodium atoms [22], mV_0 is of order 10^{-2}, and this system is supposed to be described by our results very precisely. In experiments with *trapped* gases the quantity directly relevant

to the experimental setup is $\lambda(X)$ since it describes, according to Eq. (10), the density profile in the trapping potential which is smooth enough to guarantee the hydrostatic regime. In this regime the density variation over the mode-coupling radius $r_c \sim 1/k_c$ [the data of Ref. 8 suggest $r_c \approx 2/m(UT)^{1/2}$] is small, and the coordinate dependence of density reduces to $n \equiv n(T, \mu(\mathbf{r}))$, where $n(T, \mu)$ is the homogeneous equation of state, $\mu(\mathbf{r}) = V_{\text{ext}}(\mathbf{r}) + \text{const}$, and V_{ext} is the trapping potential. It follows from Fig. 7 that for $mU \sim 10^{-2}$ the size of the fluctuation region on the normal side is of order unity. Thus when the density at the trap center is tuned to the critical point, practically the whole density profile finds itself in the fluctuation region where MF equations do not work.

TABLE I: Final results (after taking care of finite-size and finite-U corrections) for the scaling functions $\theta(X)$, $g(X)$, and $f(X)$.

X	$\theta(X)$	$g(X)$	$f(X)$
-4.0056	2.8363(3)	0.2657(3)	
-3.0056	1.9603(5)	0.3094(4)	
-2.0056	1.1472(6)	0.3866(6)	
-1.5056	0.791(1)	0.4561(8)	
-1.0056	0.514(1)	0.581(1)	
-0.7556	0.434(3)	0.688(3)	
-0.5056	0.442(3)	0.869(2)	
-0.2556	0.630(4)	1.214(4)	
-0.1056	0.885(9)	1.560(7)	
-0.0556	0.973(9)	1.680(7)	
-0.0156	1.041(9)	1.774(8)	
-0.0056	1.061(4)	1.800(3)	
0.0044	1.075(5)	1.821(4)	1.077(5)
0.0444	1.137(5)	1.908(4)	1.274(4)
0.0944	1.208(5)	2.013(4)	1.433(5)
0.2444	1.415(6)	2.319(6)	1.823(6)
0.4944	1.734(6)	2.800(7)	2.37(1)
0.9944	2.334(9)	3.721(8)	3.348(6)
1.9944	3.469(9)	5.47(1)	5.135(10)
2.9944	4.554(9)	7.19(1)	6.87(1)
3.9944	5.631(8)	8.870(6)	8.58(1)

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