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Chiral Perturbation Theory: a Primer

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Abstract

Recently methods have been developed which exploit the chiral symmetry of QCD in order to make rigorous contact with low energy particle physics phenomenology. In these lectures we present a pedagogical introduction to these techniques.

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1 Introduction

For the past quarter century a primary goal of both particle and nuclear theorists has been the ability to make reliable predictions about experimental quantities directly from the Lagrangian of QCD. However, this quest has proven to be extraordinarily difficult because QCD, while formally similar to QED, also possesses important differences.

1.1 QED

In Quantum Electrodynamics the interaction between charged particles is mediated by the exchange of *neutral* gauge bosons—photons. Because of the neutrality of the photon there do *not* exist vertices where a photon interacts directly with another photon. Therefore in QED only a single vertex is required—*i.e.* the coupling of the photon to a fermion. The QED Lagrangian density is

$$\mathcal{L}_{\rm QED} = \bar{q}(i\not{D} - m)q - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{1}$$

where

$$iD_\mu = i\partial_\mu - Q_q e A_\mu \tag{2}$$

is the covariant derivative and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3}$$

is the electromagnetic field tensor. The coupling constant e in QED is related to the fine structure constant via $\alpha = e^2/4\pi \approx 1/137$, and, because of the smallness of α, the theory can be successfully treated perturbatively. Quantum Electrodynamics has thereby been confronted with numerous precise experimental tests and has proven remarkably successful in each case.

1.2 Quantum Chromodynamics

The remarkable success of QED leads quite naturally to a nonabelian generalization involving a triplet of color-charges interacting via the exchange of color gauge bosons called gluons. This is the theory of Quantum Chromodynamics with the Lagrange density

$$\mathcal{L}_{\rm QCD} = \bar{q}(i\not{D} - m)q - \frac{1}{2}{\rm Tr}\; G_{\mu\nu}G^{\mu\nu}\,. \tag{4}$$

Here the covariant derivative is

$$iD_\mu = i\partial_\mu - gA^a_\mu\frac{\lambda^a}{2}\,, \tag{5}$$

where λ^a (with $a = 1, \ldots, 8$) are the SU(3) Gell-Mann matrices, operating in color space. The color-field tensor is defined by

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]\,, \tag{6}$$

where the last term, *which has no QED analog*, arises from the non-abelian nature of the theory. Despite the *formal* similarity of the QCD and QED Lagrangians, more careful examination reveals crucial differences between the two theories:

- i) The coupling constant $g^2/4\pi \sim 1$ so that a perturbative treatment analogous to that used for QED is not possible.

- ii) In QCD gauge bosons themselves carry color-charge. Therefore we have, in addition to the fermion–gluon vertex, three- and four-gluon vertices, which makes the theory highly nonlinear. (A corresponding situation exists in general relativity, where gravitons themselves carry energy-momentum and therefore couple to one another).

These difficulties have heretofore prevented a precise confrontation of experiment with rigorous QCD predictions. Nevertheless there are at least two cases in which these problems can be ameliorated and reliable theoretical predictions can be generated from QCD:

- High energy limit: At very high energies, when the momentum transfer q^2 is large, QCD becomes "asymptotically free"—*i.e.* the running coupling constant $g(q^2)$ approaches zero. Hence, in this limit one can utilize perturbative methods. However, this procedure, "*perturbative QCD*," is not useful except for interactions at the very highest energies.[1]

- Symmetry: The second way to confront QCD with experimental test is to utilize the *symmetry* of $\mathcal{L}_{\rm QCD}$. In order to do so, we separate the quark components into two groups. That involving the heavy quarks– c,b,t—we shall not consider further in these lectures. Indeed the masses of such quarks are much larger than the QCD scale—$\Lambda_{QCD} \sim 300$ MeV—but can be treated using *heavy-quark symmetry* methods. On the other hand, the light quarks—u,d,s—have masses much smaller than the QCD scale and their interactions can be analyzed by exploiting the *chiral symmetry* of the QCD Lagrangian as will be developed further below.[3] As we shall see, this procedure is capable of rigor but is only useful for energies $E << 1$ GeV—it is a *low energy method.*

However, before going into detail about chiral techniques it is useful to review general ideas about symmetry and in particular about symmetry *breaking.*

2 Symmetry and Symmetry Breaking

2.1 Symmetry

The best definition of symmetry for our purposes is probably that due to the mathematician Herman Weyl who said that a system is symmetric when one can do something to it and, after making this change, the system looks the same as it did before.[4] The importance of symmetry in physics is due to an important result—Noether's theorem—which connects each symmetry of a system with a corresponding conserved current and conservation law.[5] Familiar examples include:

- $\mathcal{L}$ invariant under translation $\rightarrow$ momentum conservation

– $\mathcal{L}$ invariant under time translation → energy conservation

– $\mathcal{L}$ invariant under rotation → angular momentum conservation

However, because these symmetries are so familiar (and are exact) they are also not of interest here. Rather we shall be dealing in these lectures with examples of approximate symmetries which would obtain in some hypothetical universe which is not our own—in our world such symmetries will be seen to be broken in some fashion. In spite of this, such broken symmetries are of great importance and by their study we will be able to learn much about the underlying interactions.

2.2 Symmetry Breaking

In general there exist in physics three possible mechanisms for symmetry breaking

– explicit symmetry breaking

– spontaneous symmetry breaking

– quantum mechanical symmetry breaking

and in this section we study examples of each:

2.2.1 Explicit Symmetry Breaking

First consider a simple harmonic oscillator of frequency ω_0 described by the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 \,. \tag{7}$$

which is explicitly invariant under spatial inversion—$x \to -x$—since

$$V_0(x) = -\frac{1}{2}m\omega_0^2 x^2 = -\frac{1}{2}m\omega^2(-x)^2 = V_0(-x) \tag{8}$$

Thus it is clear from symmetry considerations that the equilibrium location x_E, which is determined by the condition $[\partial L/\partial x](x_E) = 0$, must occur at $x_E = 0$, since the equilibrium position should also manifest this symmetry.

Now, however, consider what happens if we add an term $V_1(x) = \lambda x$ *i.e.* a constant force, to the Lagrangian. The new Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 + \lambda x \tag{9}$$

which describes a displaced oscillator. This new Lagrangian is *not* invariant under spatial inversion, and consequently the new equilibrium location—$x_E = \lambda/m\omega^2 \neq 0$—is no longer required to be at the origin. This is an example of *explicit symmetry breaking* wherein the symmetry violation is manifested in the Lagrangian itself.

2.2.2 Spontaneous Symmetry Breaking

As our second example, consider a hoop rotating in the earth's gravitational field about a vertical axis.[6] Attached to the hoop is a bead which can slide along the circumference without friction. The lagrangian L for the system is then

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + \omega^2 R^2 \sin^2\theta) + mgR\cos\theta\,, \tag{10}$$

where θ measures the angular displacement of the bead from the nadir. L is clearly symmetric under the angular parity transformation $L(\theta) = L(-\theta)$, but the equilibrium condition for the bead is found to be

$$\frac{\partial L}{\partial\theta} = m\omega^2 R^2 \sin\theta(\cos\theta - \frac{g}{\omega^2 R}) = 0\,. \tag{11}$$

which is somewhat more complex than the displaced oscillator considered above. For slow rotation—*i.e* for $\omega^2 < \frac{g}{R}$, we have $\cos\theta - \frac{g}{\omega^2 R} \neq 0$, so that the ground (equilibrium) state configuration is given by $\theta_E = 0$ as expected from symmetry considerations. However, if we proceed to higher angular velocities such that $\omega^2 > \frac{g}{R}$ then the bead finds equilibrium at $\theta_E = \pm\cos^{-1}\frac{g}{\omega^2 R}$, where the choice of + vs. - is *not* determined by the physics but rather by the history of motion of the system as the critical angular velocity was reached. Note that neither of these equilibrium positions exhibits the symmetry of the underlying potential, which is invariant under the exchange of θ and $-\theta$. This is an example of *spontaneous symmetry breaking*, wherein the Lagrangian of a system possess a symmetry, but this symmetry is broken by the ground (equilibrium) state of the system.

2.2.3 Quantum Mechanical Symmetry Breaking

The third type of symmetry breaking is the least familiar to most physicists because it has no classical analog. It is called "quantum mechanical" or "anomalous" symmetry breaking and occurs when the classical Lagrangian of a system possesses a symmetry, but the symmetry broken in the process of quantization. As the simplest example and the only one (of which I am aware) that does not involve quantum field theory—just quantum mechanics!—consider a free particle, for which the stationary state Schrödinger equation is[7]

$$-\frac{1}{2m}\nabla^2\psi = E\psi \equiv \frac{k^2}{2m}\psi\,, \tag{12}$$

A partial wave solution in polar coordinates is

$$\psi(\vec{r}) = \frac{1}{r}\chi_k(r)P_l(\cos\theta)\,, \tag{13}$$

where $\chi_k(\vec{r})$ satisfies the radial Schrödinger equation

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + k^2\right)\chi_k(r) = 0\,. \tag{14}$$

Here the central piece in the above differential operator is the well-known "centrifugal potential." By inspection the radial Schrödinger equation is invariant under a "scale transformation"

$$r \to \lambda r \qquad k \to \frac{1}{\lambda} k\,. \tag{15}$$

This *scale invariance* has an important physical consequence, which can be seen if we expand a plane wave solution in terms of incoming and outgoing partial waves

$$e^{ikz} \stackrel{r\to\infty}{\longrightarrow} \frac{1}{2ikr}\sum_l (2l+1)P_l(\cos\theta)\left(e^{ikr} - e^{-i(kr-l\pi)}\right)\,, \tag{16}$$

We observe that in each partial wave the incoming and outgoing component of the wavefunction differ by the centrifugal phase shift $l\pi$. This phase shift must be *independent* of energy via scale invariance.

If we place the free particle in a potential $V(\vec{r})$ then the scale invariance is broken. The corresponding wave function expanded in partial waves then becomes

$$\psi^{(+)}(\vec{r}) \stackrel{r\to\infty}{\longrightarrow} \frac{1}{2ikr}\sum_l (2l+1)P_l(\cos\theta)(e^{i(kr+2\delta_l(k))} - e^{-i(kr-l\pi)}) \tag{17}$$

Usually this is written as

$$\psi^{(+)}(\vec{r}) = e^{ikx} + \frac{e^{ikr}}{r} f_k(\theta) \tag{18}$$

where the scattering amplitude is defined by

$$f_k(\theta) = \sum_l (2l+1)\frac{e^{2i\delta_l(k)} - 1}{2ik} P_l(\cos\theta)\,. \tag{19}$$

Of course, the phase shifts $\delta_l(k)$ of various angular momenta l now depend on energy, but this is to be expected since the scale invariance no longer obtains.

One can generalize the scattering formalism to two dimensions, in which case we obtain for the scattering wave function

$$\psi^{(+)}(\vec{r}) \stackrel{r\to\infty}{\longrightarrow} e^{ikz} + \frac{1}{\sqrt{r}} e^{i(kr+\frac{\pi}{4})} f_k(\theta) \tag{20}$$

and for the scattering amplitude

$$f_k(\theta) = -i \sum_{m=-\infty}^{\infty} \frac{e^{2i\delta_m(k)} - 1}{\sqrt{2\pi k}} e^{im\theta} \tag{21}$$

where we expand in terms of exponentials $e^{im\theta}$ rather than Legendre polynomials. What is *special* about two dimensions is that it is possible to introduce a *scale invariant* potential

$$V(\vec{r}) = g\delta^2(\vec{r}) \tag{22}$$

The associated differential scattering cross section is found to be[8]

$$\frac{d\sigma}{d\Omega} \propto \frac{\pi}{2k}\frac{1}{(\ln\frac{k}{\mu}^2)}\,. \tag{23}$$

which is somewhat of a surprise. Indeed since the cross section is isotropic, the scattering is pure $m = 0$, corresponding to a phase shift

$$\cot \delta_0(k) = \frac{1}{\pi} \ln \frac{k^2}{\mu^2} - \frac{2}{g}\,, \tag{24}$$

which depends on k—scale invariance has been broken as a result of quantization. Although this should not be completely unexpected (indeed while at the classical level non-zero impact parameter means no scattering, in quantum mechanics this is not the case because of the non-zero deBroglie wavelength), still the "physics" of this result is not completely clear.

3 Examples of Symmetries

In this section we study examples of symmetry and symmetry breaking found within the Lagrangian of QCD and discuss ways in which these features are manifested in the interactions of hadronic systems.

3.1 Explicitly Broken Symmetry

To begin, we assume a Lagrangian within only the u,d quark sectors

$$\begin{aligned} \mathcal{L} &= \bar{u}(i\not{D} - m_u)u + \bar{d}(i\not{D} - m_d)d \\ &\equiv \bar{q}(i\not{D} - m)q\,, \end{aligned} \tag{25}$$

where q and m are defined as

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \qquad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{26}$$

In the limit $m_u = m_d$ this Lagrangian is unchanged after arbitrary rotations

$$q \to \exp(i\frac{1}{2}\vec{\tau} \cdot \vec{\alpha})q\,, \tag{27}$$

where $\vec{\tau}$ represents the Pauli Matrices—*i.e.* the u,d quark Lagrangian is SU(2) flavor (isotopic-spin) invariant.

Now define the vector current density

$$\vec{V}_\mu = \bar{q}\gamma_\mu \frac{1}{2}\vec{\tau} q\,, \tag{28}$$

which is conserved for equal masses $m_u = m_d$—

$$\partial^\mu \vec{V}_\mu = 0\,. \tag{29}$$

Therefore the associated isospin charge, given by

$$\vec{I} \equiv \int d^3x \vec{V}_0(\vec{x}, t)\,. \tag{30}$$

is time-independent—

$$\frac{d}{dt}\vec{I} = \int d^3x \frac{\partial \vec{V}_0}{\partial t} = -\int d^3x \vec{\nabla}\cdot\vec{V} = -\int d\vec{S}\cdot\vec{V} = 0\,. \tag{31}$$

where we have used Gauss' theorem and locality in making the last step.

These isotopic charge operators form an SU(2) algebra with

$$[I_i, I_j] = i\varepsilon_{ijk} I_k\,, \tag{32}$$

Since these commutation relations are identical to those for ordinary spin we know that the eigenstates, eigenvalues must be identical to those for spin, *i.e.*

$$\begin{aligned} \hat{I}^2|I, I_z\rangle &= I(I+1)|I, I_z\rangle \\ \hat{I}_z|I, I_z\rangle &= I_z|I, I_z\rangle\,. \end{aligned}$$

Now since the Lagrangian $\mathcal{L}$ is unchanged under an isospin rotation, states which differ only by I_z must have identical spin-parity assignments and be degenerate, as seen in Nature.

Since a rotation in isospin space merely changes the orientation of the axes, $\mathcal{L}_{\text{eff}}$ is invariant, where $\mathcal{L}_{\text{eff}}$ represents an *effective* Lagrangian which describes the interaction in terms of *experimental* degrees of freedom (hadrons) instead of *fundamental* ones (quarks). An example of such an effective Lagrangian which describes the interactions of nucleons with pions is

$$\mathcal{L}_{\text{eff}}(\pi NN) = g\bar{N}\gamma_5\vec{\tau}N\cdot\vec{\phi}\,. \tag{33}$$

Of course, in the real world the masses of the light quarks are *unequal* and isospin invariance is *broken*.

$$\mathcal{L}_{\text{QCD}} \not\longrightarrow \mathcal{L}_{\text{QCD}} \quad \text{if} \quad m_u \neq m_d.$$

However, the concept of isospin remains a useful one provided that the breaking is not too large—*i.e.* provided that the u,d mass splitting is small compared to Λ_{QCD}. In this case we can write the mass matrix in the form

$$\begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \hat{m}\mathbf{1} + \frac{1}{2}(m_u - m_d)\tau_3\,, \tag{34}$$

with $\hat{m} = \frac{1}{2}(m_u + m_d)$ and can hope to treat the isospin breaking (τ_3) part perturbatively. For example, the nucleon mass is given to first order in perturbation theory by

$$\begin{aligned} m_N &= m_0\bar{N}N + m_1\bar{N}\tau_3 N \\ &= \bar{N}\begin{pmatrix} m_0 + m_1 & 0 \\ 0 & m_0 - m_1 \end{pmatrix} N\,, \end{aligned} \tag{35}$$

where m_0, m_1 are unknown constants, so that proton and neutron masses are no longer degenerate—

$$m_n - m_p = \frac{m_1}{m_0}(m_p + m_n). \tag{36}$$

The neutron and proton are distinguished in addition by their charge, so isotopic spin invariance is also broken by electromagnetism. Hence in order to completely understand the n,p mass difference we must also include electromagnetic effects. Now at the phenomenological level the nucleon is a simple three quark object and its mass contains a contribution from the Coulomb energy between quark pairs[9]

$$\begin{aligned} U_p &\sim \frac{\alpha}{<r>}\left((\frac{2}{3})^2 + 2\frac{2}{3}\cdot -\frac{1}{3}\right) = 0 \\ U_n &\sim \frac{\alpha}{<r>}\left((-\frac{1}{3})^2 + 2\frac{2}{3}\cdot -\frac{1}{3}\right) = -\frac{\alpha}{3<r>}\,, \end{aligned} \tag{37}$$

where $<r>$ represents some average radial quark separation distance within the nucleon. Then *e.g.* $m_p - m_n \sim \frac{\alpha}{3<r>} \sim 0.5$ MeV, suggesting $m_d - m_u \sim 2$ MeV, but this is only a rough estimate.

Similar considerations apply if we extend our discussion to SU(3) (*i.e.* Gell-Mann's *Eightfold Way*)[10] by including the mass of the strange quark. Since $m_s >> m_u, m_d$ the breaking effects would be expected to be somewhat larger, but SU(3) symmetry is still found to be a very useful concept. We begin by defining a free Lagrange density for the three quark system

$$\mathcal{L} = \bar{q}(i\not{D} - m)q\,, \tag{38}$$

where q and m are now defined as

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{39}$$

As in the SU(2) case, if $m_u = m_d = m_s$ this Lagrangian is manifestly invariant under rotations in this *three* dimensional space

$$q \longrightarrow \exp(i\frac{1}{2}\sum_j \lambda_j \alpha_j)q\,, \tag{40}$$

where λ_j are the Gell–Mann matrices and an arbitrary rotation is defined by the eight parameters $\alpha_j, j = 1, 2, \ldots 8$.

Analogous to the $2^2 - 1 = 3$ τ matrices, which satisfy

$$\begin{aligned} [\tau_i, \tau_j] &= 2i\epsilon_{ijk}\tau_k \\ \{\tau_i, \tau_j\} &= 2\delta_{ij}\,. \end{aligned} \tag{41}$$

the $3^2 - 1 = 8$ Gell-Mann matrices obey

$$\begin{aligned} [\lambda_i, \lambda_j] &= 2if_{ijk}\lambda_k \\ \{\lambda_i, \lambda_j\} &= 2id_{ijk}\lambda_k\,. \end{aligned} \tag{42}$$

Hence there exist *eight* vector currents

$$V_\mu^j = \bar{q}\gamma_\mu \frac{1}{2}\lambda_j q\,, \tag{43}$$

which are conserved—

$$\partial^\mu V^j_\mu = 0 \tag{44}$$

and

$$0 = \frac{d}{dt}\int d^3x V^j_0(\vec{x},t) \equiv \frac{d}{dt}F_j \tag{45}$$

These SU(3) charge operators F_j form an SU(3) algebra with commutation relations

$$[F_i, F_j] = if_{ijk}F_k\,. \tag{46}$$

Representations of the SU(2) and SU(3) Group

A unitary representation of the SU(2) [SU(3)] group is a mapping of the 2×2 matrices in Eq. 27 [the 3×3 matrices in Eq. 40] onto unitary matrices $D(U)$ which, in general, act in spaces with different dimensions. In SU(2) the representations are well-known:

SU(2)	$\{1\}$	$I_z = 0$	$\Lambda,\ \eta$
	$\{2\}$	$I_z = \frac{1}{2}, -\frac{1}{2}$	$(p,n)(K^+,K^-)$
	$\{3\}$	$I_z = 1,0,-1$	$(\pi^+,\pi^0,\pi^-)(\Sigma^+,\Sigma^0,\Sigma^-)$
	$\{4\}$	$I_z = \frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}$	$(\Delta^{++},\Delta^+,\Delta^0\Delta^-)$

Likewise the various representations of SU(3) can be worked out, with the most important representation being the well-known octet.

In order to represent SU(3) invariant effective interactions it is useful to define the 3×3 matrices

$$\frac{1}{\sqrt{2}}\sum_j \lambda_j\phi_j = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}}+\frac{\eta^0}{\sqrt{6}} & \pi^- & K^- \\ \pi^+ & -\frac{\pi^0}{\sqrt{2}}+\frac{\eta^0}{\sqrt{6}} & \bar{K}^0 \\ K^+ & K^0 & -\frac{2\eta^0}{\sqrt{6}} \end{pmatrix} \equiv \Phi$$

$$\frac{1}{\sqrt{2}}\sum_j \lambda_j B_j = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda^0}{\sqrt{6}} & \Sigma^- & \Xi^- \\ \Sigma^+ & -\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda^0}{\sqrt{6}} & \Xi^0 \\ p & n & -\frac{2\Lambda^0}{\sqrt{6}} \end{pmatrix} \equiv B \tag{47}$$

as the contraction of the SU(3) octet fields with the associated Gell-Mann matrices. Then the most general SU(3) invariant interaction describing the interactions of the octet baryons with the pseudoscalars can be written in terms of two arbitrary constants F, D as

$$\mathcal{L}_{\rm eff}(\Phi\bar{B}B) = D\mathrm{Tr}\bar{B}\gamma_5\{\Phi,B\} + F\mathrm{Tr}\bar{B}\gamma_5[\Phi,B] \tag{48}$$

which is the SU(3) analog of the SU(2) relation

$$\mathcal{L}_{\rm eff}(\pi\bar{N}N) = g\bar{N}\gamma_5\vec{\tau}N\cdot\vec{\phi} \tag{49}$$

The invariance of Eq. 49 is easily demonstrated since under an SU(3) rotation

$$U = \exp(i\frac{1}{2}\sum_j \lambda_j\alpha_j) \tag{50}$$

a matrix M must transform as

$$M \to UMU^{-1} \tag{51}$$

We see then that a structure such as

$$\mathrm{Tr}\bar{B}\Phi B \to \mathrm{Tr}U\bar{B}U^{-1}U\Phi U^{-1}UBU^{-1} = \mathrm{Tr}\bar{B}\Phi B \tag{52}$$

is clearly invariant, as required. According to Eq. 49 then arbitrary $\bar{B}BP$ vertices can be expressed in terms just two constants *e.g.*[11]

$$\begin{aligned} g(\pi^+\bar{p}n) &= F + D \\ g(K^+\bar{p}\Lambda) &= -\frac{1}{\sqrt{6}}(D + 3F) \end{aligned} \tag{53}$$

and experimentally this prediction is found to work extremely well. Having studied examples of explicit symmetry breaking via SU(2) and SU(3) methods, we now move on to the case of spontaneous symmetry breaking in QCD.

3.2 Spontaneous Symmetry Breaking

The classic example of spontaneous symmetry breaking is that of the ferromagnet. In this case one deals with a Hamiltonian of the form

$$H \sim \lambda \sum_{i,j} \vec{\sigma}_i \cdot \vec{\sigma}_j f_{ij} \tag{54}$$

which is clearly rotationally invariant. Yet a permanent magnet selects a definite direction in space along which it is magnetized—the ground state does not share the symmetry of the underlying interaction. Note that just as in the case of the rotating hoop the direction selected by the ground state is *not* a matter of physics but depends rather on the history of the system.

Chiral Symmetry

In order to understand how spontaneous symmetry breaking occurs in QCD we must introduce the idea of chirality, defined by the operators

$$\Gamma_{L,R} = \frac{1}{2}(1 \pm \gamma_5) = \frac{1}{2}\begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix} \tag{55}$$

which project left- and right-handed components of the Dirac wavefunction via

$$\psi_L = \Gamma_L \psi \qquad \psi_R = \Gamma_R \psi \quad \text{with} \quad \psi = \psi_L + \psi_R \tag{56}$$

In terms of these chirality states the quark component of the QCD Lagrangian can be written as

$$\bar{q}(i\not{D} - m)q = \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R - \bar{q}_L m q_R - \bar{q}_R m q_L \tag{57}$$

The reason that these chirality states are called left- and right-handed can be seen by examining helicity eigenstates of the free Dirac equation. In the high energy (or massless) limit we have

$$u(p)=\sqrt{\frac{E+m}{2E}}\left(\begin{array}{c}\chi\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi\end{array}\right)\stackrel{E\gg m}{\sim}\sqrt{\frac{1}{2}}\left(\begin{array}{c}\chi\\ \vec{\sigma}\cdot\hat{p}\chi\end{array}\right) \tag{58}$$

Left- and right-handed helicity eigenstates then can be identified as

$$u_L(p)\sim\sqrt{\frac{1}{2}}\left(\begin{array}{c}\chi\\ -\chi\end{array}\right),\qquad u_R(p)\sim\sqrt{\frac{1}{2}}\left(\begin{array}{c}\chi\\ \chi\end{array}\right) \tag{59}$$

But note that

$$\begin{array}{ll}\Gamma_L u_L=u_L & \Gamma_R u_L=0\\ \Gamma_R u_R=u_R & \Gamma_L u_R=0\end{array} \tag{60}$$

i.e. in this limit chirality is identical with helicity.

$$\Gamma_{L,R}\sim\text{helicity!}$$

With this background, we now return to QCD. We observe that if $m=0$ then

$$\mathcal{L}_{\rm QCD}=\bar{q}_L i\not{D}q_L+\bar{q}_R i\not{D}q_R \tag{61}$$

would be invariant under independent global left- and right-handed rotations

$$q_L\rightarrow\exp(i\sum_j\lambda_j\alpha_j)q_L,\qquad q_R\rightarrow\exp(i\sum_j\lambda_j\beta_j)q_R \tag{62}$$

(Of course, in this limit the heavy quark component is also invariant, but since $m_{c,b,t}>>\Lambda_{\rm QCD}$ it would be silly to consider this as even an approximate symmetry in the real world.) This invariance is called $SU(3)_L\bigotimes SU(3)_R$ or chiral $SU(3)\times SU(3)$. Continuing to neglect the light quark masses, we see that in a chiral symmetric world one would have sixteen—eight left-handed and eight right-handed—conserved currents

$$\bar{q}_L\gamma_\mu\frac{1}{2}\lambda_i q_L\,,\qquad \bar{q}_R\gamma_\mu\frac{1}{2}\lambda_i q_R \tag{63}$$

Equivalently, by taking the sum and difference we would have eight vector and eight axial vector conserved currents

$$V_\mu^i=\bar{q}\gamma_\mu\frac{1}{2}\lambda_i q,\qquad A_\mu^i=\bar{q}\gamma_\mu\gamma_5\frac{1}{2}\lambda_i q \tag{64}$$

In the vector case, we have already examined the consequences. There exist eight time-independent generators

$$F_i=\int d^3xV_0^i(\vec{x},t) \tag{65}$$

and there exist supermultiplets of particles in the configurations demanded by SU(3).

If chiral symmetry were realized in the conventional fashion one would expect there also to exist corresponding nearly degenerate opposite parity states generated by the action of the time-independent axial charges $F_i^\sigma = \int d^3x A_0^i(\vec{x}, t)$ on these states. Indeed since

$$\begin{aligned} H|P\rangle &= E_P|P\rangle \\ H(Q_5|P\rangle) &= Q_5(H|P\rangle) = E_P(Q_5|P\rangle) \end{aligned} \tag{66}$$

we see that $Q_5|P\rangle$ must also be an eigenstate of the Hamiltonian with the same eigenvalue as $|P>$, which would seem to require the existence of parity doublets. However, experimentally this does not appear to be the case.

3.3 Goldstone's Theorem

One can resolve this apparent problem by postulating that the theorem is avoided because the axial symmetry is *spontaneously broken.* Then according to a theorem due to Goldstone, when a continuous symmetry is broken in this fashion there must also be generated a massless boson having the quantum numbers of the broken generator, in this case a pseudoscalar, and when the axial charge acts on a single particle eigenstate one does not get a single particle eigenstate of opposite parity in return.[12] Rather one generates one or more of these massless pseudoscalar bosons

$$Q_5|P\rangle \sim |Pa\rangle + \cdots \tag{67}$$

This phenomenon is a well-known one in ferromagnetism, where, since it does not cost any energy to rotate the spin direction, one can find correlated groups of spins which develop in a wavelike fashion—a spin wave with

$$E \sim \frac{c}{\lambda} \sim \text{cp} \tag{68}$$

which is the dispersion formula associated with the existence of a massless excitation.

A simple example can be studied within the context of scalar field theory. The spin-zero Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{69}$$

which can be verified since by applying the Euler-Lagrange equation

$$0 = \partial^\mu \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)} - \frac{\delta\mathcal{L}}{\delta\phi} \tag{70}$$

we obtain the Klein-Gordon equation

$$\partial^\mu\partial_\mu\phi + m^2\phi = (\Box + m^2)\phi = 0 \tag{71}$$

Now, however, consider a *complex* field $\phi(x)$ which has the Lagrangian

$$\mathcal{L} = |\partial_\mu\phi|^2 - V(|\phi|) \tag{72}$$

If we write things in terms of the modulus and phase of the field $\phi(x)$

$$\phi = \frac{1}{\sqrt{2}}\rho e^{i\theta}, \qquad \partial_\mu \phi = \frac{1}{\sqrt{2}} e^{i\theta}(\partial_\mu \rho + i\rho\partial_\mu \theta) \tag{73}$$

then the potential depends only on the modulus $\rho(x)$ and therefore has the shape shown of a Mexican hat, for which each point along the minimum corresponds to a different value for the phase but has the same energy. The ground state of course selects one particular value for the phase and breaks the rotational symmetry. Let this value be $\rho(x) = \rho_0, \theta = 0$ and expand about this point $\rho = \rho^0 + \chi$. The Lagrangian then reduces to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}\rho_0^2(\partial_\mu \theta)^2 - V(\frac{\rho^0}{\sqrt{2}}) - \frac{1}{2f}\chi^2 V''(\frac{\rho^0}{\sqrt{2}}) + \cdots \tag{74}$$

i.e.

$$m_\chi^2 = \frac{1}{2}V''(\frac{\rho^0}{\sqrt{2}}) \quad m_\theta^2 = 0 \tag{75}$$

We see that there exists a massless excitation in the θ direction—this is the Goldstone mode.

According to this argument then one would expect there to exist eight massless pseudoscalar states, which are the Goldstone bosons of QCD. Examination of the particle data tables reveals that no such particles exist, however, and causes us to ask what has gone wrong. The answer is found in the fact that our discussion thus far has neglected the piece of the QCD Lagrangian which is associated with quark mass and can be written in the form

$$\mathcal{L}^m_{\rm QCD} = -(\bar{u}_L u_R + \bar{u}_R u_L)m_u - (\bar{d}_L d_R + \bar{d}_R d_L)m_d \tag{76}$$

Since clearly this term breaks the chiral symmetry—

$$\begin{aligned} \bar{q}_L q_R \quad &\rightarrow \quad \bar{q}_L \exp(-i\sum_j \lambda_j \alpha_j) \times \exp(i\sum_j \lambda_j \beta_j) q_R \\ &\neq \quad \bar{q}_L q_R \end{aligned} \tag{77}$$

—we have a violation of Goldstone's theorem. The associated pseudoscalar bosons are not required to be massless

$$m_G^2 \neq 0 \tag{78}$$

but since their mass arises only from the breaking of the symmetry the various would-be Goldstone boson masses are expected to be proportional to the breaking

$$m_G^2 \propto m_u, m_d, m_s$$

and therefore small to the extent that the quark masses are light. Indeed the pseudoscalar masses are considerably lighter than other hadronic masses in the spectrum, as expected in this scenario.

4 Effective Field Theory

Before proceeding further it is useful to discuss the concept of effective field theory, since it in this fashion that we will be able to make contact with QCD. An effective field theory is one which does not include all of the degrees of freedom of the underlying (true) field theory and for this reason the term *effective* is sometimes taken to mean *defective*. However, this is not at all necessarily the case, as the following example will show.

4.1 Superconductivity

An example which is somewhat close to the case of QCD is that of superconductivity. In this case the full degrees of freedom consist of free electrons and a lattice of ions. As is well known, the interaction of one of these electrons with the lattice deforms the latter, which in turn has an effect on a nearby electron, giving an effective binding between these electron pair states. In fact by integrating out the lattice completely one has an effective field theory expressed entirely in terms of electron pair states, which has the form[13]

$$L_{\rm eff} = \chi^* \left(-\frac{(\vec{\nabla} + ie^*\vec{A})^2}{2m^*} + e^*\phi \right) \chi a(T) + (\chi^*\chi)^2 b(T) + \chi^*\chi c(T) \tag{79}$$

with

$$e^* = 2e \qquad\qquad m^* = 2m \tag{80}$$

The important feature here is the coefficient

$$c(T) = K \ln \frac{T}{T_c} \tag{81}$$

which changes sign as a function of temperature. At temperatures $T > T_c$ the effective potential has the shape of a simple well *cf.* Figure 1a. The ground state occurs at $\chi^*\chi = 0$ which means that there is nothing remarkable going on. On the other hand for $T < T_c$ this linear term changes sign so that the effective potential now has the familiar double well behavior associated with spontaneous symmetry breaking *cf.* Figure 1b. The ground state now occurs at $\chi^*\chi \neq 0$ which means that there occurs a Bose condensation—the electron pairs condense into the same state and the superconducting phase occurs.

The connection with the QCD problem can now be made. Of course, instead of weakly bound electron pairs interacting weakly with a lattice (which is integrated out of the effective Lagrangian) we have quark-antiquark pairs interacting strongly with color gluons (which are integrated out of the effective Lagrangian). However, the idea is the same—in both cases we end up with a description of the physics in terms of an effective interaction which, even though not including all the relevant degrees of freedom, nevertheless simply encapsulates the relevant physics in terms of those degrees of freedom which are relevant experimentally.

weakly bound	$\rightarrow$	strongly bound
$L_{\rm eff}$ for (e^-e^-)	$\rightarrow$	$L_{\rm eff}^{\rm QCD}$ for $(q\bar{q})$
Lattice degrees of freedom gone	$\rightarrow$	Gluon degrees of freedom gone

Figure 1: Shape of the effective superconducting potential above and below the critical temperature.

4.2 Effective Chiral Lagrangian

Our goal then is to generate an effective field theory which describes the interactions of the pseudoscalar (Goldstone) bosons by exhibiting the chiral symmetry of QCD which we have previously discussed. This is done by defining a nonlinear function of the pseudoscalar fields $U = \exp(i\vec{\tau}\cdot\pi/v$ such that under the chiral transformations

$$\begin{aligned} \psi_L &\rightarrow L\psi_L \\ \psi_R &\rightarrow R\psi_R \end{aligned} \tag{82}$$

then

$$U \rightarrow LUR^\dagger \tag{83}$$

and a form such as

$$\mathrm{Tr}\partial^\mu U\partial_\mu U^\dagger \rightarrow \mathrm{Tr}L\partial^\mu UR^\dagger R\partial_\mu U^\dagger L^\dagger = \mathrm{Tr}\partial^\mu U\partial_\mu U^\dagger \tag{84}$$

is invariant under chiral rotations and can be used as part of the effective Lagrangian. However, this form is also not one which we can use in order to realistically describe Goldstone interactions in Nature since according to Goldstone's theorem a completely invariant Lagrangian must also have zero pion mass, in contradiction to experiment.

We infer then that the *lowest order* effective chiral Lagrangian is given by

$$\mathcal{L}_2 = \frac{v^2}{4}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) + \frac{m_\pi^2}{4}v^2\mathrm{Tr}(U + U^\dagger)\,. \tag{85}$$

where the subscript 2 indicates that we are working at two-derivative order or one power of chiral symmetry breaking—*i.e.* m_π^2. This Lagrangian is also *unique*—if we expand to lowest order in $\vec{\phi}$

$$\mathrm{Tr}\partial_\mu U\partial^\mu U^\dagger = \mathrm{Tr}\frac{i}{v}\vec{\tau}\cdot\partial_\mu\vec{\phi}\times\frac{-i}{v}\vec{\tau}\cdot\partial^\mu\vec{\phi} = \frac{2}{v^2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}\,, \tag{86}$$

we reproduce the free pion Lagrangian, as required,

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{2}m_\pi^2\vec{\phi}\cdot\vec{\phi} + \mathcal{O}(\phi^4)\,. \tag{87}$$

At the SU(3) level, including a generalized chiral symmetry breaking term, there is even predictive power—one has

$$\frac{v^2}{4}\mathrm{Tr}\partial_\mu U\partial^\mu U^\dagger = \frac{1}{2}\sum_{j=1}^{8}\partial_\mu\phi_j\partial^\mu\phi_j + \cdots \tag{88}$$

$$\begin{aligned}\frac{v^2}{4}\mathrm{Tr}2B_0 m(U+U^\dagger) &= \text{const.} - \frac{1}{2}(m_u+m_d)B_0\sum_{j=1}^{3}\phi_j^2 \\ &- \frac{1}{2}(\hat{m}+m_s)B_0\sum_{j=1}^{3}\phi_j^2 - \frac{1}{6}(m_u+m_d+4m_s)B_0\phi_8^2 + \cdots\end{aligned} \tag{89}$$

where B_0 is a constant and m is the quark mass matrix. We can then identify the meson masses as

$$\begin{aligned} m_\pi^2 &= (m_u+m_d)B_0 = 2\hat{m}B_0 \\ m_K^2 &= (\hat{m}+m_s)B_0 \\ m_\eta^2 &= \frac{1}{3}(m_u+m_d+4m_s)B_0 = \frac{2}{3}(\hat{m}+2m_s)B_0\,, \end{aligned} \tag{90}$$

This system of three equations is *overdetermined*, and we find by simple algebra

$$3m_\eta^2 + m_\pi^2 - 4m_K^2 = 0\ . \tag{91}$$

which is the Gell-Mann-Okubo mass relation and is well-satisfied experimentally.[15]

Currents

In order to proceed further in our analysis, we can identify the Noether currents via standard techniques. Suppose that the Lagrangian is invariant under the transformation $\phi \longrightarrow \phi + \varepsilon f(\phi)$—*i.e.*

$$\begin{aligned} 0 &= \mathcal{L}(\phi+\varepsilon f, \partial_\mu\phi + \varepsilon\partial_\mu f) - \mathcal{L}(\phi,\partial_\mu\phi) \\ &= \varepsilon f\frac{\delta\mathcal{L}}{\delta\phi} + \varepsilon\partial_\mu f\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = \varepsilon\partial_\mu\left(f\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\,. \end{aligned} \tag{92}$$

so that we can identify the associated conserved current as [1]

$$J^\mu = f\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,, \tag{96}$$

[1] Note: This is often written in an alternative fashion by introducing a *local* transformation $\varepsilon = \varepsilon(x)$, so

Since under a

$$\text{Vector, Axial transformation:}\alpha_L = \pm\alpha_R \tag{97}$$

we have

$$U \to \mathrm{LUR}^\dagger \overset{V}{\simeq} U + i\left[\sum_j \alpha_j\lambda_j, U\right] \overset{A}{\simeq} U + i\left\{\sum_j \alpha_j\lambda_j, U\right\}\,. \tag{98}$$

which leads to the vector and axial-vector currents

$$\{V, A\}_\mu^k = -i\frac{v^2}{4}\mathrm{Tr}\lambda^k(U^\dagger\partial_\mu U \pm U\partial_\mu U^\dagger) \tag{99}$$

At this point the constant v can be identified by using the axial current. In SU(2) we find

$$U^\dagger\partial_\mu U - U\partial_\mu U^\dagger \;=\; 2i\frac{1}{v}\vec{\tau}\cdot\partial_\mu\vec{\phi} + \cdots \tag{100}$$

so that

$$A_\mu^k = i\frac{v^2}{4}\mathrm{Tr}\tau^k 2i\frac{1}{v}\vec{\tau}\cdot\partial_\mu\vec{\phi} + \cdots = -v\partial_\mu\phi^k + \cdots\,. \tag{101}$$

If we set $k = 1 - i2$ then this represents the axial-vector component of the $\Delta S = 0$ charged weak current and

$$A_\mu^{1-i2} = -v\partial_\mu\phi^{1-i2} = -\sqrt{2}v\partial_\mu\phi^-\,. \tag{102}$$

Comparing with the conventional definition

$$\langle 0|A_\mu^{1-i2}(0)|\pi^+(p)\rangle = i\sqrt{2}F_\pi p_\mu\,, \tag{103}$$

we find that, to lowest order in chiral symmetry, $v = F_\pi$, where $F_\pi = 92.4$ MeV is the pion decay constant.[16]

Likewise in SU(2), we note that

$$U^\dagger\partial_\mu U + U\partial_\mu U^\dagger = \frac{2i}{v^2}\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi} + \cdots\,, \tag{104}$$

so that the *vector current* is

$$\begin{aligned} V_\mu^k &= -i\frac{v^2}{4}\mathrm{Tr}\tau^k\frac{2i}{v^2}\vec{\tau}\vec{\phi}\times\partial_\mu\vec{\phi} + \cdots \\ &= (\vec{\phi}\times\partial_\mu\vec{\phi})^k + \cdot\,. \end{aligned} \tag{105}$$

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that the Lagrangian transforms as

$$\mathcal{L}(\phi, \partial_\mu\phi) \to \mathcal{L}(\phi + \varepsilon f, \partial_\mu\phi + \varepsilon\partial_\mu f + f\partial_\mu\varepsilon)\,. \tag{93}$$

Then

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varepsilon)} = f\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,, \tag{94}$$

so that the Noether current can also be written as

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varepsilon)}\,. \tag{95}$$

We can identify V_μ^k as the electromagnetic current by setting $k=3$ so that

$$V_\mu^{\rm em} = \phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+ + \cdots \tag{106}$$

Comparison with the conventional definition

$$\langle\pi^+(p_2)|V_\mu^{\rm em}(0)|\pi^+(p_1)\rangle = G_\pi(q^2)(p_1+p_2)_\mu\,, \tag{107}$$

we identify the pion formfactor—$G(q^2)=1$. Thus to lowest order in chiral symmetry the pion has unit charge but is pointlike and structureless. We shall later see how to insert structure.

$\pi\pi$ **Scattering**

At two derivative level we can generate additional predictions by extending our analysis to the case of $\pi\pi$ scattering. Expanding $\mathcal{L}_2$ to order $\vec{\phi}^4$ we find

$$\mathcal{L}_2 : \phi^2 = \frac{1}{6v^2}\phi^2\vec{\phi}\cdot\Box\vec{\phi} + \frac{1}{2v^2}(\vec{\phi}\cdot\partial_\mu\vec{\phi})^2 + \frac{m_\pi^2}{24v^2}\phi^4 \tag{108}$$

which yields for the pi-pi T matrix

$$\begin{aligned} T(q_a,q_b,q_c,q_d) \;\; = \;\; & \frac{1}{F_\pi^2}\left[\delta^{ab}\delta^{cd}(s-m_\pi^2) + \delta^{ab}\delta^{bd}(t-m_\pi^2) + \delta^{ad}\delta^{bc}(u-m_\pi^2)\right] \\ & -\frac{1}{3F_\pi^2}(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})(q_a^2+q_b^2+q_c^2+q_d^2-4m_\pi^2)\,. \end{aligned} \tag{109}$$

Defining more generally

$$T_{\alpha\beta;\gamma\delta}(s,t,u) = A(s,t,u)\delta_{\alpha\beta}\delta_{\gamma\delta} + A(t,s,u)\delta_{\alpha\gamma}\delta_{\beta\delta} + A(u,t,s)\delta_{\alpha\delta}\delta_{\beta\gamma}\,, \tag{110}$$

we can write the chiral prediction in terms of the more conventional isospin language by taking appropriate linear combinations[17]

$$\begin{aligned} T^0(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s)\,, \\ T^1(s,t,u) &= A(t,s,u) - A(u,t,s)\,, \\ T^2(s,t,u) &= A(t,s,u) + A(u,t,s)\,. \end{aligned} \tag{111}$$

Partial wave amplitudes, projected out via

$$T_l^I(s) = \frac{1}{64\pi}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)T^I(s,t,u)\,, \tag{112}$$

can be used to identify the associated scattering phase shifts as

$$T_l^I(s) = \left(\frac{s}{s-4m_\pi^2}\right)^{\frac{1}{2}} e^{i\delta_l^I}\sin\delta_l^I\,. \tag{113}$$

	Experimental	Lowest Order[3]	First Two Orders[3]
a_0^0	0.26 ± 0.05	0.16	0.20
b_0^0	0.25 ± 0.03	0.18	0.26
a_0^2	-0.028 ± 0.012	-0.045	-0.041
b_2^2	-0.082 ± 0.008	-0.089	-0.070
a_1^1	0.038 ± 0.002	0.030	0.036
b_1^1 −	0	0.043	
a_2^0	$(17 \pm 3) \times 10^{-4}$	0	20×10^{-4}
a_2^2	$(1.3 \pm 3) \times 10^{-4}$	0	3.5×10^{-4}

Table 1: The pion scattering lengths and slopes compared with predictions of chiral symmetry.

From the lowest order chiral form

$$A(s,t,u) = \frac{s - m_\pi^2}{F_\pi^2} \tag{114}$$

we find the predicted values for the pion scattering lengths and effective ranges

$$\begin{aligned} a_0^0 &= \frac{7m_\pi^2}{32\pi F_\pi^2}\,, \quad a_0^2 = -\frac{m_\pi^2}{16\pi F_\pi^2}\,, \quad a_1^1 = -\frac{m_\pi^2}{24\pi F_\pi^2}\,, \\ b_0^0 &= \frac{m_\pi^2}{4\pi F_\pi^2}\,, \quad b_0^2 = \frac{m_\pi^2}{8\pi F_\pi^2}\,, \end{aligned} \tag{115}$$

comparison of which with experimentally measured values is shown in Table 1.

Difficulties

As seen in this Table, these experimental data agree fairly well with the lowest order theoretical predictions. However, there exist also obvious problems, which show up at higher energy.

i) Consider first the S-wave I=0 channel for which

$$T_0^0 = \frac{1}{32\pi F_\pi^2}(2s - m_\pi^2)\,, \tag{116}$$

Obviously this form cannot be extended too far in energy since the unitary condition

$$\sqrt{\frac{s - 4m_\pi^2}{s}}|T_l^I|^2\,, \tag{117}$$

is violated for $\sqrt{s} > 700$ MeV.

ii) A second indication of problems can be seen in that unitarity requires the existence of an imaginary component for each partial wave amplitude

$$\mathrm{Im}T_l^I = \left(\frac{s-4m_\pi^2}{s}\right)^{1/2} |T_l^I|^2\,, \tag{118}$$

while our simple tree-level analysis yields only real values.

iii) A third indication of shortcomings is that simple analytic forms such as result from our analysis cannot possibly reproduce the resonant behavior seen in $\pi\pi$ scattering such as, for example, the ρ resonance at 767.0 MeV seen in the P-wave, I=1 channel.

iv) Our final example of limitations of the simple lowest order chiral analysis is provided by the pion electromagnetic form factor. The unitarity relation reads in general

$$I = S^\dagger S = (I - iT^\dagger)(I + iT) \tag{119}$$

i.e.

$$i(T - T^\dagger) = -T^\dagger T\,, \tag{120}$$

If we apply this stricture to the $\gamma\pi^+\pi^-$ matrix element

$$i\langle\gamma|T - T^\dagger|\pi^+\pi^-\rangle = -\sum_n \langle\gamma|T^\dagger|\pi\pi\rangle\langle\pi\pi|T|\pi^+\pi^-\rangle \tag{121}$$

we find

$$\begin{aligned} 2\mathrm{Im}G_\pi(q^2)(p_1-p_2)_\mu &= \sum \frac{d^3q_1 d^3q_2}{(2\pi)^6 2q_1^0 2q_2^0} \\ \times \quad (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)&(q_1-q_2)_\mu\langle\pi_{q_1}^+\pi_{q_2}^-|T|\pi_{p_1}^+\pi_{p_2}^-\rangle\,, \end{aligned} \tag{122}$$

which shows that the unitarity stricture requires an imaginary component to the pion form factor, in contradiction to our simple result given in equation

The solution of these problems with unitarity are well known—the inclusion of loop corrections to these simple tree level calculations. Insertion of such loop terms removes the unitarity violations but comes with a high price—numerous divergences are introduced and this difficulty prevented progress in this field for nearly a decade until a paper by Weinberg suggested the solution.[18] One can deal with such divergences, just as in QED, by introducing counterterms into the Lagrangian in order to absorb the infinities.

5 Renormalization

5.1 Effective Chiral Lagrangian

We can now apply these lessons to the effective chiral Lagrangian, Eqs. 88,89. In this case also when loop corrections are made to lowest order amplitudes in order to enforce unitarity, divergences inevitably arise. However, there is an important difference from the familiar case of QED in that the form of the divergences is *different* from their lower order counterparts. The reason for this can be seen from a simple example. Thus consider pi-pi scattering. In lowest order there exists a tree level contribution from $\mathcal{L}_2$ which is $\mathcal{O}(p^2/F_\pi^2)$ where p represents some generic external energy-momentum. The fact that p appears to the second power is due to the feature that its origin is the two-derivative Lagrangian $\mathcal{L}_2$. Now suppose that pi-pi scattering is examined at one loop order. Since the scattering amplitude must still be dimensionless but now the amplitude involves a factor $1/F_\pi^4$ the numerator must involve *four* powers of energy-momentum. Thus any counterterm which is included in order to absorb this divergence must be four derivative in character. Gasser and Leutwyler have studied this problem and have written the most general form of such an order four counterterm as[19]

$$\begin{aligned}
\mathcal{L}_4 \;=\;& \sum_{i=1}^{10} L_i \mathcal{O}_i = L_1 \Big[\mathrm{Tr}(D_\mu U D^\mu U^\dagger)\Big] + L_2 \mathrm{Tr}(D_\mu U D_\nu U^\dagger)\cdot \mathrm{Tr}(D^\mu U D^\nu U^\dagger) \\
+\;& L_3 \mathrm{Tr}(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger) + L_4 \mathrm{Tr}(D_\mu U D^\mu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) \\
+\;& L_5 \mathrm{Tr}\left(D_\mu U D^\mu U^\dagger \left(\chi U^\dagger + U\chi^\dagger\right)\right) + L_6 \Big[\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)\Big]^2 \\
+\;& L_7 \Big[\mathrm{Tr}\left(\chi^\dagger U - U\chi^\dagger\right)\Big]^2 + L_8 \mathrm{Tr}\left(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger\right) \\
+\;& iL_9 \mathrm{Tr}\left(F^L_{\mu\nu} D^\mu U D^\nu U^\dagger + F^R_{\mu\nu} D^\mu U^\dagger D^\nu U\right) + L_{10} \mathrm{Tr}\left(F^L_{\mu\nu} U F^{R\mu\nu} U^\dagger\right)
\end{aligned} \tag{123}$$

where the constants $\alpha_i, i = 1, 2, \ldots 10$ are arbitrary and $F^L_{\mu\nu}, F^R_{\mu\nu}$ are external field strength tensors defined via

$$F^{L,R}_{\mu\nu} = \partial_\mu F^{L,R}_\nu - \partial_\nu F^{L,R}_\mu - i[F^{L,R}_\mu, F^{L,R}_\nu], \qquad F^{L,R}_\mu = V_\mu \pm A_\mu \tag{124}$$

Now just as in the case of QED the bare parameters L_i which appear in this Lagrangian are not physical. Rather the "physical" (renormalized) values of these parameters are obtained by appending to these bare values the divergent one-loop contributions having the appropriate form

$$L_i^r = L_i - \frac{\gamma_i}{32\pi^2}\left[\frac{-2}{\epsilon} - \ln(4\pi) + \gamma - 1\right] \tag{125}$$

By comparing with experiment, Gasser and Leutwyler have determined experimental values for each of these ten parameters. While ten sounds like a rather large number, we shall see below that this picture is actually predictive. Typical values for the parameters are shown in Table 2.

The question which one should ask at this point is why stop at order four? Clearly if two loop corrections from $\mathcal{L}_2$ or one-loop corrections from $\mathcal{L}_4$ are calculated, divergences will arise which are of six derivative character. Why not include these? The answer is that the chiral procedure represents an expansion in energy-momentum. Corrections to the tree level predictions from one loop corrections from $\mathcal{L}_2$ or tree level contributions from $\mathcal{L}_4$ are $\mathcal{O}(E^2/\Lambda_\chi^2)$ where $\Lambda_\chi \sim 4\pi F_\pi \sim 1$ GeV is the chiral scale.[20] Thus chiral perturbation

Coefficient	Value	Origin
L_1^r	0.65 ± 0.28	$\pi\pi$ scattering
L_2^r	1.89 ± 0.26	and
L_3^r	-3.06 ± 0.92	$K_{\ell 4}$ decay
L_5^r	2.3 ± 0.2	F_K/F_π
L_9^r	7.1 ± 0.3	π charge radius
L_{10}^r	-5.6 ± 0.3	$\pi \to e\nu\gamma$

Table 2: Gasser-Leutwyler counterterms and the means by which they are determined.

theory is a *low energy* procedure. It is only to the extent that the energy is small compared to the chiral scale that it makes sense to truncate the expansion at the four-derivative level. Realistically this means that we deal with processes involving $E < 500$ MeV, and, as we shall describe below, for such reactions the procedure is found to work very well.

Now let's give an example of a chiral perturbation theory calculation in order to see how it is performed and in order to see how the experimental counterterm values are actually determined. Consider the pion electromagnetic form factor, which by Lorentz- and gauge-invariance has the structure

$$\langle \pi^+(p_2)|J^\mu_{\rm em}|\pi^+(p_1)\rangle = G_\pi(q^2)(p_1+p_2)^\mu \tag{126}$$

We begin by identifying the electromagnetic current as

$$\begin{aligned} J^\mu_{\rm em} &= -\frac{\partial \mathcal{L}}{\partial(eA_\mu)} = (\varphi \times \partial^\mu \varphi)_3 \left[1 - \frac{1}{3F^2}\varphi\cdot\varphi + \mathcal{O}(\varphi^4)\right] \\ &+ (\varphi \times \partial^\mu \varphi)_3 \left[16L_4 + 8L_5\right]\frac{m_\pi^2}{F^2} + \frac{4L_9}{F^2}\partial^\nu(\partial^\mu\varphi \times \partial_\nu\varphi)_3 + \cdots \end{aligned} \tag{127}$$

where we have expanded to *fourth* order in the pseudoscalar fields. Defining

$$\begin{aligned} \delta_{jk} I(m^2) &= i\Delta_{Fjk}(0) = \langle 0|\varphi_j(x)\varphi_k(x)|0\rangle, \\ I(m^2) &= \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = \frac{\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)(m^2)^{\frac{d}{2}-1}, \\ \delta_{jk} I_{\mu\nu}(m^2) &= -\partial_\mu\partial_\nu i\Delta_{Fjk}(0) = \langle 0|\partial_\mu\varphi_j(x)\partial_\nu\varphi_k(x)|0\rangle, \\ I_{\mu\nu}(m^2) &= \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}k_\mu k_\nu \frac{i}{k^2-m^2} = g_{\mu\nu}\frac{m^2}{d}I(m^2) \end{aligned} \tag{128}$$

we calculate the one loop correction shown in Figure 2a to be

$$J^\mu_{\rm em}|_{(2a)} = -\frac{5}{3F_\pi^2}(\varphi \times \partial^\mu \phi)_3 I(m_\pi^2) \tag{129}$$

We also need the one loop correction shown in Figure 2b. For this piece we require the form of the pi-pi scattering amplitude which arise from $\mathcal{L}_2$

Figure 2: Loop corrections to the pion form factor.

$$\langle \pi^+(k_1)\pi^-(k_2)|\pi^+(p_1)\pi^-(p_2)\rangle = \frac{i}{3F_0^2}\left(2m_0^2 + p_1^2 + p_2^2 + k_1^2 + k_2^2 - 3(p_1 - k_1)^2\right) \tag{130}$$

and we use the results of dimensional integration

$$\begin{aligned}
&\int \frac{d^dp}{(2\pi)^d}\frac{1}{[(p-q)^2 - m_1^2 + i\varepsilon]^{n_1}[p^2 - m_2^2 + i\varepsilon]^{n_2}} \\
= \quad &(-1)^{n_1+n_2}\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(n_1+n_2-d/2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1 dx\frac{x^{n_1-1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}}
\end{aligned}$$

$$\begin{aligned}
&\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu}{[(p-q)^2 - m_1^2 + i\varepsilon]^{n_1}[p^2 - m_2^2 + i\varepsilon]^{n_2}} \\
= \quad &(-1)^{n_1+n_2}q^\mu\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(n_1+n_2-d/2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1 dx\frac{x^{n_1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}}
\end{aligned}$$

$$\begin{aligned}
&\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{[(p-q)^2 - m_1^2 + i\varepsilon]^{n_1}[p^2 - m_2^2 + i\varepsilon]^{n_2}} \\
= \quad &\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)}\Bigg[q^\mu q^\nu\Gamma(n_1+n_2-d/2)\int_0^1 dx\frac{x^{n_1+1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}} \\
&-\frac{g^{\mu\nu}}{2}\Gamma(n_1+n_2-1-d/2)\int_0^1 dx\frac{x^{n_1-1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-1-d/2}}\Bigg]
\end{aligned} \tag{131}$$

where

$$\mathcal{D} \equiv m_1^2 x + m_2^2(1-x) - q^2 x(1-x) - i\varepsilon \tag{132}$$

In the limit $d \to 4$ we find that

$$\Gamma\left(2 - \frac{d}{2}\right) = \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{133}$$

The integration in Figure 2b may now be done, yielding

$$
\begin{aligned}
\langle J^{\mu}_{\rm em}\rangle_{(2b)} &= \frac{1}{(4\pi F_\pi)^2}(p_1+p_2)^\mu \int_0^1 dx(m_\pi^2 - q^2x(1-x)) \\
&\times \left[\left(-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi\right)+\ln\frac{m_\pi^2-q^2x(1-x)}{\mu^2}\right]
\end{aligned}
\tag{134}
$$

Performing the x-integration we find, finally

$$
\begin{aligned}
\langle J^{\mu}_{\rm em}\rangle_{(2b)} &= \frac{1}{(4\pi F_\pi)^2}(p_1+p_2)^\mu \Big\{ \left(m_\pi^2-\frac{1}{6}q^2\right)\left[-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\right] \\
&+\frac{1}{6}(q^2-4m_\pi^2)H\left(\frac{q^2}{m_\pi^2}\right)-\frac{1}{18}q^2\Big\}
\end{aligned}
\tag{135}
$$

Here the function H(a) is given by

$$
\begin{aligned}
H(a) &\equiv \int_0^1 dx\ln(1-ax(1-x)) \\
&= \begin{cases} 2-2\sqrt{\frac{4}{a}-1}\cot^{-1}\sqrt{\frac{4}{a}-1} & (0<a<4) \\ 2+\sqrt{1-\frac{4}{a}}\left[\ln\frac{\sqrt{1-\frac{4}{a}}-1}{\sqrt{1-\frac{4}{a}}+1}+i\pi\theta(a-4)\right] & (\text{otherwise}) \end{cases}
\end{aligned}
\tag{136}
$$

and contains the imaginary component required by unitarity.

We are not done yet, however, since we must also include mass and wavefunction effects. In order to do so, we expand $\mathcal{L}_2$ to fourth order in $\varphi(x)$, and $\mathcal{L}_4$ to second order:

$$
\begin{aligned}
\mathcal{L}_2 &= \frac{1}{2}[\partial^\mu\varphi\cdot\partial_\mu\varphi-m_0^2\varphi\cdot\varphi]+\frac{m_0^2}{24F_0^2}(\varphi\cdot\varphi)^2 \\
&+\frac{1}{6F_0^2}[(\varphi\cdot\partial^\mu\varphi)(\varphi\cdot\partial_\mu\varphi)-(\varphi\cdot\varphi)(\partial^\mu\varphi\cdot\partial_\mu\varphi)]+\mathcal{O}(\varphi^6), \\
\mathcal{L}_4 &= \frac{m_0^2}{F_0^2}[16L_4+8L_5]\frac{1}{2}\partial_\mu\varphi\cdot\partial^\mu\varphi \\
&-\frac{m_0^2}{F_0^2}[32L_6+16L_8]\frac{1}{2}m_0^2\varphi\cdot\varphi+\mathcal{O}(\varphi^4).
\end{aligned}
\tag{137}
$$

Performing the loop integrations on the $\phi^4(x)$ component of the above yields

$$
\begin{aligned}
\mathcal{L}_{\rm eff} &= \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi-\frac{1}{2}m_0^2\varphi\cdot\varphi+\frac{5m_\pi^2}{12F_\pi^2}I(m_\pi^2)\varphi\cdot\varphi \\
&+\frac{1}{6F_\pi^2}(\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl})I(m_\pi^2)(\delta_{ij}\partial^\mu\varphi_k\partial_\mu\varphi_l+\delta_{kl}m_\pi^2\varphi_i\varphi_j) \\
&+\frac{1}{2}\partial_\mu\varphi\cdot\partial^\mu\varphi\frac{m_\pi^2}{F_\pi^2}[16L_4+8L_5)]-\frac{1}{2}m_\pi^2\varphi\cdot\varphi\frac{m_\pi^2}{F_\pi^2}[32L_6+16L_8] \\
&= \frac{1}{2}\partial^\mu\varphi\cdot\partial_\mu\varphi\left[1+(16L_4+8L_5)\frac{m_\pi^2}{F_\pi^2}-\frac{2}{3F_\pi^2}I(m_\pi^2)\right] \\
&-\frac{1}{2}m_0^2\varphi\cdot\varphi\left[1+(32L_6+16L_8)\frac{m_\pi^2}{F_\pi^2}-\frac{1}{6F_\pi^2}I(m_\pi^2)\right]
\end{aligned}
\tag{138}
$$

from which we can now read off the wavefunction renormalization term Z_π.

When this is done we find

$$\begin{aligned}
Z_\pi G_\pi^{(\text{tree})}(q^2) &= \Big[1-\frac{8m_\pi^2}{F_\pi^2}(2L_4+L_5 \\
&\quad +\frac{m_\pi^2}{24\pi^2F_\pi^2}\Big\{-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\Big\}\Big] \\
&\quad \times\Big[1+\frac{8m_\pi^2}{F_\pi^2}(2L_4+L_5)+2q^2\frac{L_9}{F_\pi^2}\Big] \\
&= \Big[1+\frac{m_\pi^2}{24\pi^2F_\pi^2}\Big(-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\pi^2}\Big)+\frac{2L_9}{F_\pi^2}q^2\Big]
\end{aligned} \tag{139}$$

while from the loop diagrams given earlier

$$\begin{aligned}
G_\pi(q^2)\Big|_{(2a)} &= -\frac{5m_\pi^2}{48\pi^2F_\pi^2}\Big\{-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\Big\} \\
G_\pi(q^2)\Big|_{(2b)} &= \frac{1}{16\pi^2F_\pi^2}\Big\{\Big(m_\pi^2-\frac{1}{6}q^2\Big)\Big[-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\Big] \\
&\quad +\frac{1}{6}(q^2-4m_\pi^2)H\left(\frac{q^2}{m_\pi^2}\right)-\frac{1}{18}q^2\Big\}
\end{aligned} \tag{140}$$

Adding everything together we have the final result, which when written in terms of the renormalized value $L_9^{(r)}$ is *finite*!

$$G_\pi(q^2) = 1+\frac{2L_9^{(r)}}{F_\pi^2}q^2+\frac{1}{96\pi^2F_\pi^2}\left[(q^2-4m_\pi^2)H\left(\frac{q^2}{m_\pi^2}\right)-q^2\ln\frac{m_\pi^2}{\mu^2}-\frac{q^2}{3}\right] \tag{141}$$

If we expand to lowest order in q^2 we find

$$G_\pi(q^2) = 1+q^2\left[\frac{2L_9^{(r)}}{F_\pi^2}-\frac{1}{96\pi^2F_\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right)\right]+\cdots \tag{142}$$

which can be compared with the phenomenological description in terms of the pion charge radius

$$G_\pi(q^2)=1+\frac{1}{6}\langle r_\pi^2\rangle q^2+\cdots \tag{143}$$

By equating these two expressions and using the experimental value of the pion charge radius—$\langle r_\pi^2\rangle_{\text{exp}} = (0.44\pm 0.01)\text{fm}^2$[21]—we determine the value of the counterterm $L_9^{(r)}$ shown in Table 2.

Although due to lack of space, we have in these lectures limited our attention to only a very limited number of reactions, chiral perturbative techniques have been applied to many other processes involving Goldstone interaction, some examples of which are indicated in Table 2.

5.2 Anomalous Symmetry Breaking

Thus far we have given examples of *explicit* and of *spontaneous* symmetry breaking within QCD. Remarkably QCD also involves *anomalous* symmetry breaking, which can be characterized in terms of an effective Lagrangian which has *no* free parameters. The form of this interaction has been given by Witten as[22]

$$\mathcal{L}_{\rm A} = -\frac{N_c}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}[eA_\mu \mathrm{Tr}(QL_\nu L_\alpha L_\beta - QR_\nu R_\alpha R_\beta) + ie^2 F_{\mu\nu}A_\alpha T_\beta] \tag{144}$$

where

$$\begin{aligned} L_\mu &\equiv \partial_\mu U U^\dagger, \qquad R_\mu \equiv \partial_\mu U^\dagger U, \\ T_\beta &= \mathrm{Tr}\left(Q^2 L_\beta - Q^2 R_\beta + \frac{1}{2}QUQU^\dagger L_\beta - \frac{1}{2}QU^\dagger QUR_\beta\right) \end{aligned} \tag{145}$$

and N_c is the number of colors. The best known manifestation of the anomaly is its prediction for the process $\pi^0 \to \gamma\gamma$. The component of $\mathcal{L}_{\rm A}$ which is responsible for this process can be identified as

$$\mathcal{L}_A = \frac{e^2 N_c}{48\pi^2 F_\pi} 3\mathrm{Tr}(Q^2\tau_3)\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}A_\alpha \partial_\beta \pi^0 = \frac{\alpha N_c}{24\pi F_\pi}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\pi^0 \tag{146}$$

Defining the decay amplitude as

$$\mathcal{M}_{\pi^0\to\gamma\gamma} = -iA_{\gamma\gamma}\varepsilon^{\mu\nu\alpha\beta}\varepsilon^*_\mu k_\nu \varepsilon'^*_\alpha k'_\beta \tag{147}$$

we find the decay rate

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{m_\pi^3}{64\pi}|A_{\gamma\gamma}|^2 \tag{148}$$

The decay amplitude predicted by the anomaly is found to be

$$A_{\gamma\gamma} = \frac{\alpha N_c}{3\pi F_\pi} \stackrel{N_c=3}{\Longrightarrow} 0.025\,\mathrm{GeV}^{-1} \tag{149}$$

which is in excellent agreement with the value determined by experiment[23]

$$A_{\gamma\gamma} = 0.0025 \pm 0.001\,\mathrm{GeV}^{-1} \tag{150}$$

and gives eloquent proof that the number of colors is precisely three. Although this is the best known example, there are numerous additional manifestations of the anomaly, *e.g.* $\gamma \longrightarrow 3\pi, K \longrightarrow \pi\pi e\nu_e, \eta \longrightarrow \pi\pi\gamma$, *etc.*

5.3 Comparison with Experiment

Of course, having gone to such effort to set up this formalism, the real question is "Does it make successful predictions?" We do not have the space here to give a detailed answer to this question, so a simple example will have to suffice. A particularly interesting indication

	Data	Lowest Order	Order (E^4)
$f_1(0)$	1.47 ± 0.04	1.00	1.45
$f_2(0)$	1.25 ± 0.07	1.00	1.24
λ_1	0.08 ± 0.02	0.00	0.08
λ_2	0.08 ± 0.02	0.00	0.06
$g(0)$	0.96 ± 0.24	0.00	1.00

Table 3: Experimental values of $K_{\ell 4}$ parameters and their chiral predictions.

of the predictive power of chiral perturbation theory is found in the semileptonic weak process $K \to \pi\pi l\nu_l$ for which one defines the matrix element[24]

$$\begin{aligned}
\langle\pi^+(p_+)\pi^-(p_-)|\bar{s}\gamma_\mu\gamma_5 u|K^+(k)\rangle &= -\frac{i}{\sqrt{2}F_\pi} \\
\times\left[(p_+ + p_-)_\mu f_1 + (p_+ - p_-)_\mu\, f_2\right. &+ \left.(k - p_+ - p_-)_\mu f_3\right] \\
\langle\pi^+(p_+)\pi^-(p_-)|\bar{s}\gamma_\mu u|K^+(k)\rangle &= \frac{2g}{\pi F_\pi^3}\varepsilon_{\mu\nu\alpha\beta}k^\nu p_+^\alpha p_-^\beta
\end{aligned} \tag{151}$$

Parameterizing the form factors as

$$f_i(k^2) = f_i(0)\left[1 + \lambda_i\frac{k^2}{m_\pi^2}\right] \quad \text{with} \quad k^2 = \frac{1}{4}\left((p_+ + p_-)^2 - 4m_\pi^2\right) \tag{152}$$

one finds predicted values which are in excellent agreement with those determined experimentally as shown in Table 3. Notice also that higher order corrections are essential here.

A second example of predictive power can be seen by examining the connection between radiative pion decay and pion Compton scattering. For the former we can define[40]

$$\begin{aligned}
\mathcal{M}_{\pi^+\to e^+\nu_e\gamma} &= -\frac{eG_F}{\sqrt{2}}\cos\theta_1 M_{\mu\nu}(p,q)\varepsilon^{\mu *}(q)\bar{u}(p_\nu)\gamma^\nu(1+\gamma_5)v(p_e) \\
\mathcal{M}_{\pi^+\to e^+\nu_e e^+e^-} &= -\frac{e^2G_F}{\sqrt{2}}\cos\theta_1 M_{\mu\nu}(p,q)\frac{1}{q^2} \\
&\quad\times\bar{u}(p_2)\gamma^\mu v(p_1)\bar{u}(p_\nu)\gamma^\nu(1+\gamma_5)v(p_e)
\end{aligned} \tag{153}$$

where the hadronic component of $M_{\mu\nu}$ has the structure

$$\begin{aligned}
M_{\mu\nu}(p,q) &= \int d^4x e^{iq\cdot x} < 0|T(J_\mu^{\rm em}(x)J_\nu^{1-i2}(0)|\pi(\vec{p}) >= \text{Born terms} \\
&- h_A((p-q)_\mu q_\nu - g_{\mu\nu}q\cdot(p-q)) - r_A(q_\mu q_\nu - g_{\mu\nu}q^2) \\
&+ ih_V\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta
\end{aligned} \tag{154}$$

where h_A, r_A, h_V are unknown structure functions. (Note that r_A can only be measured via the rare Dalitz decay $\pi^+ \to e^+\nu_e e^+e^-$.) Likewise we can define the amplitude for Compton scattering as

$$\begin{aligned}
-iT_{\mu\nu}(p,p',q) &= -i\int d^4x e^{iq_1\cdot x} < \pi^+(\vec{p}')|T(J_\mu^{\rm em}(x)J_\nu^{\rm em}(0)|\pi^+(\vec{p}) > \\
&= \text{Born terms} + \sigma(q_{2\mu}q_{1\nu} - g_{\mu\nu}q_1\cdot q_2) + \cdots
\end{aligned} \tag{155}$$

Chiral symmetry makes *four* predictions among these parameters, and three of the four are found to be in good agreement with experiment. The possible exception involves a relation between the charged pion polarizability and the axial structure constant h_A measured in radiative pion decay. In this case there exist two conflicting experimental results, one of which agrees and one of which does not agree with the theoretical prediction. It is important to resolve this potential discrepancy, since these chiral predictions are firm ones. There is no way (other than introducing perversely large higher order effects) to bring things into agreement were some large violation of a chiral prediction to be verified, since the only ingredient which goes into such predictions is the chiral symmetry of QCD itself!

6 Baryon Chiral Perturbation Theory

Our discussion of chiral methods given above was limited to the study of the interactions of the pseudoscalar mesons with leptons and with each other. In the real world, of course, interactions with baryons also take place and it is an important problem to develop a useful predictive scheme based on chiral invariance for such processes. Again much work has been done in this regard, but there remain important problems. Writing down the lowest order chiral Lagrangian at the SU(2) level is straightforward—

$$\mathcal{L}_{\pi N} = \bar{N}(i\not{D} - m_N + \frac{g_A}{2}\not{u}\gamma_5)N \tag{156}$$

where g_A is the usual nucleon axial coupling in the chiral limit, the covariant derivative $D_\mu = \partial_\mu + \Gamma_\mu$ is given by

$$\Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u] - \frac{i}{2}u^\dagger(V_\mu + A_\mu)u - \frac{i}{2}u(V_\mu - A_\mu)u^\dagger \tag{157}$$

and u_μ represents the axial structure

$$u_\mu = iu^\dagger \nabla_\mu U u^\dagger \tag{158}$$

Expanding to lowest order we find

$$\begin{aligned} \mathcal{L}_{\pi N} &= \bar{N}(i\not{\partial} - m_N)N + \frac{g_A}{F_\pi}\bar{N}\gamma^\mu\gamma_5\frac{1}{2}\vec{\tau}N\cdot\vec{\pi} \\ &- \frac{1}{4F_\pi^2}\bar{N}\gamma^\mu\vec{\tau}N\cdot\vec{\pi}\times\partial\vec{\pi} + \ldots \end{aligned} \tag{159}$$

which yields the Goldberger-Treiman relation, connecting strong and axial couplings of the nucleon system[25]

$$F_\pi g_{\pi NN} = m_N g_A \tag{160}$$

Using the present best values for these quantities, we find

$$92.4\text{MeV} \times 13.0 = 1201\text{MeV} \quad \text{vs.} \quad 1183\text{MeV} = 939\text{MeV} \times 1.26 \tag{161}$$

The agreement to better than two percent strongly confirms the validity of chiral symmetry in the nucleon sector.

6.1 Heavy Baryon Methods

Extension to SU(3) gives additional successful predictions—the linear Gell-Mann-Okubo relation as well as the generalized Goldbeger-Treiman relation. However, difficulties arise when one attempts to include higher order corrections to this formalizm. The difference from the Goldstone case is that there now exist *two* dimensionful parameters—m_N and F_π—in the problem rather than *one*—F_π. Thus loop effects can be of order $(m_N/4\pi F_\pi)^2 \sim 1$ and we no longer have a reliable perturbative scheme. A consistent power counting mechanism can be constructed provided that we eliminate the nucleon mass from the Lagrangian. This is done by considering the nucleon to be very heavy. Then we can write its four-momentum as[26]

$$p_\mu = Mv_\mu + k_\mu \tag{162}$$

where v_μ is the four-velocity and satisfies $v^2 = 1$, while k_μ is a small off-shell momentum, with $v\cdot k << M$. One can then construct eigenstates of the projection operators $P_\pm = \frac{1}{2}(1\pm \not{v})$, which in the rest frame project out upper, lower components of the Dirac wavefunction, so that[27]

$$\psi = e^{-iMv\cdot x}(H_v + h_v) \tag{163}$$

where

$$H_v = P_+\psi, \qquad h_v = P_-\psi \tag{164}$$

The effective Lagrangian can then be written in terms of N, h as

$$\mathcal{L}_{\pi N} = \bar{H}_v\mathcal{A}H_v + \bar{h}_v\mathcal{B}H_v + \bar{H}_v\gamma_0\mathcal{B}^\dagger\gamma_0 h_v - \bar{h}_v\mathcal{C}h_v \tag{165}$$

where the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have the low energy expansions

$$\begin{aligned}
\mathcal{A} &= iv\cdot D + g_A u\cdot S + \ldots \\
\mathcal{B} &= i\not{D}^\perp - \frac{1}{2}g_A v\cdot u\gamma_5 + \ldots \\
\mathcal{C} &= 2M + iv\cdot D + g_A u\cdot S + \ldots
\end{aligned} \tag{166}$$

Here $D^\perp_\mu = (g_{\mu\nu} - v_\mu v_\nu)D^\nu$ is the transverse component of the covariant derivative and $S_\mu = \frac{i}{2}\gamma_5\sigma_{\mu\nu}v^\nu$ is the Pauli-Lubanski spin vector and satisfies the relations

$$S\cdot v = 0, \quad S^2 = -\frac{3}{4}, \quad \{S_\mu, S_\nu\} = \frac{1}{2}(v_\mu v_\nu - g_{\mu\nu}), \quad [S_\mu, S_\nu] = i\epsilon_{\mu\nu\alpha\beta}v^\alpha S^\beta \tag{167}$$

We see that the two components H,h are coupled in this expression for the effective action. However, this may be undone by the field transformation

$$h' = h - \mathcal{C}^{-1}\mathcal{B}H \tag{168}$$

in which case the Langrangian becomes

$$\mathcal{L}_{\pi N} = \bar{H}_v(\mathcal{A} + (\gamma_0\mathcal{B}^\dagger\gamma_0)\mathcal{C}^{-1}\mathcal{B})H_v - \bar{h}'_v\mathcal{C}h'_v \tag{169}$$

The piece of the Lagrangian involving H no longer contains the mass as a parameter and is the effective Lagrangian that we desire. The remaining piece involving h'_v can be thrown away, as it does not couple to the H_v physics. (In path integral language we simply integrate out this component yielding an uninteresting overall constant.) Of course, when loops are calculated a set of counterterms will be required and these are given at leading (two-derivative) order by

$$\begin{aligned}
\mathcal{A}^{(2)} &= \frac{M}{F_\pi^2}(c_1\mathrm{Tr}\chi_+ + c_2(v\cdot u)^2 + c_3 u\cdot u + c_4[S^\mu, s^\nu]u_\mu u_\nu \\
&+ c_5(\chi_+ - \mathrm{Tr}\chi_+) - \frac{i}{4M}[S^\mu, S^\nu]((1+c_6)F^+_{\mu\nu} + c_7\mathrm{Tr}f^+_{\mu\nu})) \\
\mathcal{B}^{(2)} &= \frac{M}{F_\pi^2}((-\frac{c_2}{4}i[u^\mu, u^\nu] + c_6 f_+^{\mu\nu} + c_7 Trf_+^{\mu\nu})\sigma_{\mu\nu} \\
&- \frac{c_4}{2}v_\mu\gamma_\nu Tru^\mu u^\nu) \\
\mathcal{C}^{(2)} &= -\frac{M}{F_\pi^2}(c_1 Tr\chi_+ + (-\frac{c_2}{4}i[u^\mu, u^\nu] + c_6 f_+^{\mu\nu} + c_7 trF_+^{\mu\nu})\sigma_{\mu\nu} \\
&- \frac{c_3}{4}Tru^\mu u_\nu - (\frac{c_4}{2} + Mc_5)v_\mu v_\nu Tru^\mu u^\nu)
\end{aligned} \tag{170}$$

Expanding $\mathcal{C}^{-1}$ and the other terms in terms of a power series in $1/M$ then leads to an effective heavy nucleon Lagrangian of the form (to $\mathcal{O}(q^3)$)

$$\begin{aligned}
\mathcal{L}_{\pi N} &= \bar{H}_v\{\mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + (\gamma_0\mathcal{B}^{(1)\dagger}\gamma_0)\frac{1}{2M}\mathcal{B}^{(1)} \\
&+ \frac{(\gamma_0\mathcal{B}^{(1)\dagger}\gamma_0)\mathcal{B}^{(2)} + (\gamma_0\mathcal{B}^{(2)\dagger}\gamma_0)\mathcal{B}^{(1)}}{2M} \\
&- (\gamma_0\mathcal{B}^{(1)\dagger}\gamma_0)\frac{i(v\cdot D) + g_A(u\cdot S)}{(2M)^2}\mathcal{B}^{(1)}\}H_v + \mathcal{O}(q^4)
\end{aligned} \tag{171}$$

A set of Feynman rules can now be written down and a consistent power counting scheme developed, as shown by Meissner and his collaborators.[28]

6.2 Applications

As our first example consider the nucleon-photon interaction. To lowest (one derivative) order we have from $\mathcal{A}^{(1)}$

$$\mathcal{L}^{(1)}_{\gamma NN} = ie\bar{N}\frac{1}{2}(1+\tau_3)\epsilon\cdot vN \tag{172}$$

while at two-derivative level we find

$$\mathcal{L}^{(2)}_{\gamma NN} = \bar{N}\left\{\frac{e}{4M}(1+\tau_3)\epsilon\cdot(p_1+p_2) + \frac{ie}{2M}[S\cdot\epsilon, S\cdot k](1+\kappa_S+\tau_3(1+\kappa_V)\right\}N \tag{173}$$

whre we have made the identifications $c_6 = \kappa_V, \quad c_7 = \frac{1}{2}(\kappa_S - \kappa_V)$. We can now reproduce the low energy theorems for Compton scattering. Consider the case of the proton. At the two derivative level, we have the tree level prediction from

$$(\gamma_0\mathcal{B}^{(1)\dagger}\gamma_0)\frac{1}{2M}\mathcal{B}^{(1)}|_{\gamma pp} = \frac{e^2}{2M}\vec{A}_\perp^2 \tag{174}$$

which yields the familiar Thomson amplitude

$$\mathrm{Amp}_{\gamma pp} = -\frac{e^2}{M}\hat{\epsilon}'\cdot\hat{\epsilon} \tag{175}$$

On the other hand at order q^3 we find a contribution from the pole diagrams shown in Figure 3 with two-derivative terms at each vertex. This yields

$$\begin{aligned}\mathrm{Amp}_{\gamma pp} &= (\frac{e}{M}^2)\frac{1}{\omega}\bar{p}[(\hat{\epsilon}'\cdot\vec{k}\vec{S}\cdot\hat{\epsilon}\times\vec{k}-\hat{\epsilon}\cdot\vec{k}'\vec{S}\cdot\hat{\epsilon}'\times\vec{k}')(1+\kappa_p)\\ &+ i\vec{S}\cdot(\hat{\epsilon}\times\vec{k})\times(\hat{\epsilon}'\times\vec{k}')(1+\kappa_p)^2]\end{aligned} \tag{176}$$

The full result must also include contact terms at order q^3 from the last piece of Eq. 171

$$-eP_+\not{A}^\perp\frac{iv\cdot D}{(2M)^2}e\not{A}^\perp P_+ = -\frac{e^2}{2M^2}\vec{S}\cdot\vec{A}\times\dot{\vec{A}} \tag{177}$$

and from the third

$$\frac{1}{2M}P_+\left\{e\not{A}^\perp,\kappa_p\sigma_{\mu\nu}F^{\mu\nu}\right\}P_+ = \kappa_p\frac{e^2}{M^2}\vec{S}\cdot\vec{A}\times\dot{\vec{A}} \tag{178}$$

When added to the pole contributions the result can be expressed in the general form[28]

$$\begin{aligned}\mathrm{Amp} &= \hat{\epsilon}\cdot\hat{\epsilon}'A_1+\hat{\epsilon}'\cdot\vec{k}\hat{\epsilon}\cdot\vec{k}'A_2+i\vec{\sigma}\cdot(\hat{\epsilon}'\times\hat{\epsilon})A_3\\ &+ i\vec{\sigma}\cdot(\vec{k}'\times\vec{k})\hat{\epsilon}'\cdot\hat{\epsilon}A_4+i\vec{\sigma}\cdot[(\hat{\epsilon}'\times\vec{k})\hat{\epsilon}\cdot\vec{k}'-(\hat{\epsilon}\times\vec{k}')\hat{\epsilon}'\cdot\vec{k}]A_5\\ &+ i\vec{\sigma}\cdot[(\hat{\epsilon}'\times\vec{k}')\hat{\epsilon}\cdot\vec{k}'-(\hat{\epsilon}\times\vec{k})\hat{\epsilon}'\cdot\vec{k}]A_6\end{aligned} \tag{179}$$

with[2]

$$\begin{aligned}A_1 &= -\frac{e^2}{M},\quad A_2=\frac{1}{M^2\omega},\quad A_3=\frac{e^2\omega}{2M^2}(1+2\kappa-(1+\kappa)^2\hat{k}\cdot\hat{k}')\\ A_4 &= -A_5=-\frac{e^2(1+\kappa)^2}{2M^2\omega},\quad A_6=-\frac{e^2(1+\kappa)}{2M^2\omega}\end{aligned} \tag{180}$$

which agrees with the usual result derived in this order via Low's theorem.[29]

A full calculation at order q^3 must also, of course, include loop contributions. Using the lowest order (one-derivative) pion-nucleon interactions

$$\begin{aligned}\mathcal{L}_{\pi NN} &= \frac{g_A}{F_\pi}\bar{N}\tau^a S\cdot qN\\ \mathcal{L}_{\pi\pi NN} &= \frac{1}{4F_\pi^2}v\cdot(q_1+q_2)\epsilon^{abc}\bar{N}\tau_c N\\ \mathcal{L}_{\gamma\pi NN} &= \frac{ieg_A}{F_\pi}\epsilon^{a3b}\bar{N}\epsilon\cdot S\tau_b N\end{aligned} \tag{181}$$

[2]Here we have used the identity

$$\vec{\sigma}\cdot(\hat{\epsilon}'\times\vec{k}')\times(\hat{\epsilon}\times\vec{k}) = \vec{\sigma}\cdot(\vec{k}'\times\vec{k})\hat{\epsilon}\cdot\hat{\epsilon}'+\vec{\sigma}\cdot(\hat{\epsilon}'\times\hat{\epsilon})\vec{k}'\cdot\vec{k}+\vec{\sigma}\cdot(\hat{\epsilon}\times\vec{k}')\hat{\epsilon}'\cdot\vec{k}-\vec{\sigma}\cdot(\hat{\epsilon}'\times\vec{k})\hat{\epsilon}\cdot\vec{k}'$$

Figure 3: Pole diagrams for Compton scattering.

these can be calculated using the diagrams shown in Figure 4. Of course, from Eq. 171 the propagator for the nucleon must have the form $1/iv \cdot k$ where k is the off-shell momentum. Thus, for example, the seagull diagram, Figure 4a, is of the form

$$\mathrm{Amp} = 4e^2(\frac{g_A}{F_\pi})^2\hat{\epsilon}\cdot\hat{\epsilon}'\int\frac{d^4k}{(2\pi)^4}\frac{S\cdot kS\cdot k}{v\cdot k(k^2-m_\pi^2)((k+q_1-q_2)^2-m_\pi^2)} \tag{182}$$

Since there are no additional counterterms at this order q^3, the sum of loop diagrams must be finite and yields, to lowest order in energy and after considerable calculation

$$\begin{aligned} A_1^{loop} &= \xi(\frac{11\omega^2}{24m_\pi}+\frac{t}{48m_\pi}), \quad A_2^{loop}=\xi(\frac{1}{24m_\pi}), \quad A_3^{loop}=\xi(\frac{\omega t}{\pi m_\pi^2}+\frac{\omega^3}{3\pi m_\pi^2}) \\ A_4^{loop} &= \xi(\frac{\omega}{6\pi m_\pi^2}), \quad A_5^{loop}=-A_6^{loop}=-\xi(\frac{13\omega}{12\pi m_\pi^2}), \end{aligned} \tag{183}$$

and $\xi = g_A^2/8\pi F_\pi^2$.

The experimental implications of these results may be seen by first considering the case of an unpolarized proton target. Then writing

$$\mathrm{Amp}_{unpol} = \{\hat{\epsilon}\cdot\hat{\epsilon}'(-\frac{e^2}{M}+4\pi\alpha_E\omega^2)+(\hat{\epsilon}\times\vec{k})\cdot(\hat{\epsilon}'\times\vec{k}')4\pi\beta_M\} \tag{184}$$

where α_E, β_M are the proton electric and magnetic polarizabilities, we identify the one loop chiral predictions[30]

$$\alpha_E^{theo} = 10\beta_M^{theo} = \frac{5e^2g_A^2}{384\pi^2F_\pi^2m_\pi} = 13.6\times10^{-4}\mathrm{fm}^3 \tag{185}$$

which are in reasonable agreement with the recently measured values[31]

$$\alpha_E^{exp} = (10.4\pm0.6)\times10^{-4}\mathrm{fm}^3, \qquad \beta_M^{exp} = (3.8\mp0.6)\times10^{-4}\mathrm{fm}^3 \tag{186}$$

Figure 4: Loop diagrams for Compton scattering. Diagrams b,c,d must also include cross terms.

For the case of spin-dependent forward scattering, we find

$$\frac{1}{4\pi}\mathrm{Amp} = f_1(\omega^2)\hat{\epsilon}\cdot\hat{\epsilon}' + i\omega f_2(\omega^2)\vec{\sigma}\cdot\hat{\epsilon}'\times\hat{\epsilon} \tag{187}$$

Then we find

$$\begin{aligned} f_1(\omega^2) &= -\frac{e^2}{4\pi M} + (\alpha_E + \beta_M)\omega^2 + \mathcal{O}(\omega^4) \\ f_2(\omega^2) &= -\frac{e^2\kappa_p^2}{8\pi^2 M^2} + \gamma_S\omega^2 + \mathcal{O}(\omega^4) \end{aligned} \tag{188}$$

where γ_S is a sort of spin-polarizability. Assuming that the amplitudes f_1, f_2 obey once-subtracted and unsubtracted dispersion relations respectively we find the sum rules

$$\begin{aligned} \alpha_E + \beta_M &= \frac{1}{4\pi^2}\int_{\omega_0^\infty}\frac{d\omega}{\omega^2}(\sigma_+(\omega) + \sigma_-(\omega)) \\ \frac{\pi e^2\kappa_p^2}{2M^2} &= \int_{\omega_0}^{\infty}\frac{d\omega}{\omega}[\sigma_+(\omega) - \sigma_-(\omega)] \\ \gamma_S &= \frac{1}{4\pi^2}\int_{\omega_0}^{\infty}\frac{d\omega}{\omega^3}[\sigma_+(\omega) - \sigma_-(\omega)] \end{aligned} \tag{189}$$

where here $\sigma_\pm(\omega)$ denote the photoabsorption cross sections for scattering cirvularly polarized photons on polarized nucleons for total γN helicity 3/2 and 1/2 respectively. Here the first is the well-known unitarity sum rule for the sum of the electric and magnetic polarizabilities, while the second is the equally familiar Drell-Hearn-Gerasimov sum rule.[32] The third is less well known, but follows from that of DHG and offeres a new check of the chiral predictions.

A second venue wherein chiral methods offer new predictive power is that of threshhold photoproduction. Here what is measured is the s-wave or E_{0+} multipole, defined via

$$\mathrm{Amp} = 4\pi(1+\mu)E_{0+}\vec{\sigma}\cdot\hat{\epsilon} + \ldots \tag{190}$$

where $\mu = m_\pi/M$. In the case of charged photoproduction, things are relatively simple, as the dominant contribution of the amplitude, occurring at one derivative level is the so-called Kroll-Ruderman term given in Eq. 166.[33] In addition, at the two derivative level there exists a second contact term which arises from

$$\mathcal{L}^{(2)}_{\pi\gamma NN} = \frac{eg_A}{8MF_\pi} v\cdot q P_+[(1+\tau_3)\not{A}^\perp, \gamma_5\tau^a]P_+ = \frac{eg_A}{2MF_\pi} S\cdot\epsilon v\cdot q(\tau^a + \delta^{a3}) \tag{191}$$

Adding these two contributions yields the result[34]

$$\begin{aligned} E_{0+} &= \pm\frac{1}{4\pi(1+\mu)}\frac{eg_A}{\sqrt{2}F_\pi}(1\mp\frac{\mu}{2}) = \frac{eg_A}{4\sqrt{2}F_\pi}\left(\begin{array}{cc} 1-\frac{3}{2}\mu & \pi^+ \\ -1+\frac{1}{2}\mu & \pi^- \end{array}\right) \\ &= \left\{\begin{array}{ll} +26.3\times 10^{-3}/m_\pi & \pi^+ n \\ -31.3\times 10^{-3}/m_\pi & \pi^- p \end{array}\right. \end{aligned} \tag{192}$$

The numerical predictions are found to be in excellent agreement with the present experimental results,

$$E^{exp}_{0+} = \left\{\begin{array}{ll} (+27.9\pm 0.5)\times 10^{-3}/m_\pi[35] & \pi^+ n \\ (+28.8\pm 0.7)\times 10^{-3}/m_\pi[36] & \\ (-31.4\pm 1.3)\times 10^{-3}/m_\pi[35] & \pi^- p \\ (-32.2\pm 1.2)\times 10^{-3}/m_\pi[37] & \end{array}\right. \tag{193}$$

However, these results are old emulsion measurements involving significant extrapolation to threshold. A new experiment is being run this summer at Saskatoon which will explore the region only 1 MeV above threshold.

More challenging is the case of neutral photoproduction, for which the one-derivative contribution vanishes. In this case the leading contribution arises from the two derivative term given in Eq. 171, augmented by the three derivative contribution from the pole terms shown in Figure 5a. The net result is

$$\mathrm{Amp}^{(2)} = \frac{eg_A}{2F_\pi}\mu\hat{\epsilon}\cdot\vec{\sigma}\times\left\{\begin{array}{ll} 1 & \pi^0 p \\ 0 & \pi^0 n \end{array}\right. \tag{194}$$

for the contact term and

$$\mathrm{Amp}^{(3)} = -\frac{e}{2M}[S\cdot\epsilon, S\cdot k](1+\kappa_p)\frac{1}{v\cdot q}\frac{g_A}{2MF_\pi}S\cdot(2p-q)m_\pi = -\frac{eg_A}{4F_\pi}\mu^2(1+\kappa_p)\vec{\sigma}\cdot\vec{\epsilon} \tag{195}$$

for the pole terms (Note: only the cross term is nonvanishing at threshhold.) Finally we must append the contribution of the loop contributions which arise from the graphs shown in Figure 5b,c

$$\mathrm{Amp}^{loop} = -\frac{eg_A M}{64\pi F_\pi^2}\mu^2\vec{\sigma}\cdot\vec{\epsilon} \tag{196}$$

The result is the prediction[38]

$$E_{0+} = \frac{eg_A}{8\pi M}\mu\{1 - [\frac{1}{2}(3+\kappa_p) + (\frac{M}{4F_\pi})^2]\mu + \mathcal{O}(\mu^2)\} \tag{197}$$

Figure 5: Diagrams for neutral pion photoproduction. Each should be accompanied by an appropriate cross term.

However, comparison with experiment is tricky because of the existence of isotopic spin breaking in the pion and nucleon masses and the feature that there exist two thresholds—one for $\pi^0 p$ and the second for $\pi^+ n$—only 7 MeV apart. When the physical masses of the pions are used the soon to be published data from both Mainz and from Saskatoon are rumored to agree with the chiral prediction.[39]

There also exists a chiral symmetry prediction for the reaction $\gamma n \rightarrow \pi^0 n$

$$E_{0+} = -\frac{eg_A}{8\pi M}\mu^2\{\frac{1}{2}\kappa_n + (\frac{M}{4F_\pi})^2\} \tag{198}$$

However, the experimental measurement of such an amplitude involves considerable challenge, and must be accomplished either by use of a deuterium target with the difficult subtraction of the proton contribution and of meson exchange contributions or by use of a polarized ^{3}He target, wherein one must account for the $\sim 10\%$ component of the wavefunction which is not a simple polarized neutron. Neither of these will be easy.

Other areas wherein chiral predictions can be confronted with experiment include the electric dipole amplitude in electroproduction as well as the P-wave multipoles in the ordinary photoproduction case. One the challenges which remains in this regard is the inclusion of the effects of the delta resonance. If one does this using a relativistic formalism, then the power counting is no longer valid. However, the problem of including the delta in a heavy baryon formalism is not yet solved.

7 Back to the Future

We have spent a good deal of time now discussing the formalism of chiral perturbation theory and I hope that I haven't given the impression that this field is basically cut and dried and that there are few remaining challenges, for this certainly is *not* the case. I shall close these lectures then by outlining where I believe that there exists room for future work.

7.1 Electroweak Goldstone Sector

For the electroweak interactions of the pseudoscalar mesons the implications of chiral symmetry have been well-developed by the work of Gasser, Leutwyler and others. In my view the remaining challenges are primarily experimental. We have seen good agreement obtains in nearly all cases where chiral predictions have been confronted with experimental tests, Nevertheless there remain two possible problems. One involves the discrepant values for the charged pion polarizability discussed above, one of which disagrees substantially from the value required by chiral invariance—

$$\bar{\alpha}_E^{\rm exp} = (6.8 \pm 1.4) \times 10^{-4}{\rm fm}^3 \qquad {\rm vs.} \qquad \bar{\alpha}_E^{\rm theo} = (2.8 \pm 0.3) \times 10^{-4}{\rm fm}^3 \tag{199}$$

Experiments to resolve this problem are proposed at DAΦNE, at Fermilab, and at MAMI so we should have an answer before too long. The other possible difficulty of which I am aware involves a probe of the anomaly via $\gamma \rightarrow \pi^+\pi^-\pi^0$, where a disagreement exists at the 3σ level[41]

$$\begin{aligned} {\rm Amp}(\gamma \rightarrow 3\pi)^{\rm exp} &= 12.9 \pm 0.9 \pm 0.5{\rm GeV}^{-3} \\ &{\rm vs.} \\ {\rm Amp}(\gamma \rightarrow 3\pi)^{\rm theo} &= 9.7{\rm GeV}^{-3} \end{aligned} \tag{200}$$

In this case there exists an approved experiment for the CLAS detector at CEBAF.

7.2 Nonleptonic Goldstone Sector

Above we have explored the utility of chiral symmetry methods applied to the semileptonic weak interactions of the Goldstone particles. An additional realm of Goldstone interactions opens if one considers the arena of nonleptonic kaon decay, for which possible reactions are[42]

i) $K_S \rightarrow \pi\pi$

ii) $K_L \rightarrow \pi\pi\pi$

iii) $K_S \rightarrow \gamma\gamma$

iv) $K_L \rightarrow \pi^0\gamma\gamma$

vi) *etc.*

Although a great deal of work has been done on such processes, there remain unanswered problems such as the origin of the $\Delta I = \frac{1}{2}$ rule, the discrepency between predicted and measured rates for the reaction $K_L \rightarrow \pi^0\gamma\gamma$, quadratic and higher effects in the $K_L \rightarrow \pi\pi\pi$ spectra, *etc.*.

7.3 High Energy Extension

As we have emphasized above, the strength of chiral perturbation theory is that it gives predictions which are model-independent—deriving solely from the symmetry properties of the underlying QCD Lagrangian. However, this carries with it a corresponding weakness in that such predictions can only rigorously be applied at energies small compared to the chiral scale of 1 GeV, while experiments are not subject to such limitations. Thus an important challenge is to find ways by which to extend the validity of chiral methods to higher energies. This attempt has been made in a number of recent works, often exploiting analyticity properties in the form of dispersion relations in order to make this extension. An example of such a program can be seen in the analysis of the reaction $\gamma\gamma \to \pi^0\pi^0$ for which there exist no tree-level contributions at either the two- or four-derivative level but for which a finite one-loop prediction obtains. Although this one-loop prediction is *not* in agreement with recent experimental results from SLAC which extend up to about 1 GeV, the use of dispersion relations in order to include the effects of loops to all orders has been shown to give very good agreement over this entire energy range.[43] However, this marriage between chiral perturbative and dispersive techniques is still new and it remains to be seen whether it will be a lasting one.

7.4 Calculating L_i from Theory

The original development of chiral perturbative techniques was phenomenological is that values of the Gasser-Leutwyler counterterms were obtained purely by empirical means. A successful theory of everything (TOE) would be able to predict the size of such coefficients directly from the underlying QCD Lagrangian, and this program remains an open challenge to future theoretical work. It is important to point out that important progress has been made in this regard, and various techniques have been employed, including

i) Nambu-Jona-Lasinio models

ii) Lippman-Schwinger approach

iii) vector dominance

iv) lattice techniques

However, no approach is completely successful and much remains to be done.

7.5 Extension to the Baryon Sector

The success of application to the processes $\gamma N \to \pi N$ and $\gamma N \to \gamma N$ is by no means clear—the convergence of the series may be too slow to be of utility. A marriage of heavy quark and dispersive methods may be helpful here, but the verdict is still out. Another important issue is inclusion of the Δ degrees of freedom. It is clear that this must be done, as the Δ couples strongly and its influence occurs even at near threshold energies. The problem is in order to do this in a consistent power counting scheme, a heavy baryon expansion must be carried out for a spin 3/2 system, which is challenging inasmuch as a Rarita-Schwinger spinor, used to describe such a system, contains both spin 3/2 and two independent spin

1/2 degrees of freedom. Nevertheless, progress on this front has recently been reported and such calculations should be forthcoming.[44]

8 Conclusions

We have spent a great deal of time studying the consequences of symmetry breaking in QCD. We have learned that by exploiting this breaking one can make rigorous contact between experimental processes and the QCD Lagrangian which presumably underlies them. It is interesting that of the three symmetry breaking mechanisms which are possible in physics:

- explicit: where the Lagrangian itself breaks the symmetry
- spontaneous: where the Lagrangian is symmetric but the ground state is not
- anomalous: where the symmetry is broken as a result of quantization

all three are associated with $\mathcal{L}_{\rm QCD}$ and by study of such symmetries and their breaking one can learn more about both the symmetries and about QCD itself.

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