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SUPERSYMMETRY AND STATIONARY SOLUTIONS

IN DILATON-AXION GRAVITY

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ABSTRACT

New stationary solutions of 4-dimensional dilaton-axion gravity are presented, which correspond to the charged Taub-NUT and Israel-Wilson-Perjés (IWP) solutions of Einstein-Maxwell theory. The charged axion-dilaton Taub-NUT solutions are shown to have a number of interesting properties: i) manifest $SL(2, R)$ symmetry, ii) an infinite throat in an extremal limit, iii) the throat limit coincides with an exact CFT construction.

The IWP solutions are shown to admit supersymmetric Killing spinors, when embedded in $d = 4$, $N = 4$ supergravity. This poses a problem for the interpretation of supersymmetric rotating solutions as physical ground states. In the context of 10-dimensional geometry, we show that dimensionally lifted versions of the IWP solutions are dual to certain gravitational waves in string theory.

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1 Introduction

A good deal is now known about static black hole solutions in the dilaton-axion theory of gravity, which arises in the low energy limit of string theory. Solutions have been found [1], [2], [3] which correspond to the Reissner-Nordström black holes of ordinary Einstein-Maxwell theory and also to the Majumdar-Papapetrou (MP) multi-black hole solutions. Analogues of the charged C-metrics, which describe pairs of black holes accelerating away from one another, have been found as well [4]. It is natural to expect that this correspondence between solutions should continue to hold in the more general stationary case. Here, the only solutions known thus far are the basic rotating black hole solutions with either pure electric or pure magnetic charge [5]. In this paper we present two additional classes of stationary solutions to dilaton-axion gravity, which may be considered to be of black hole *type*⁵. These new solutions correspond to the charged Taub-NUT solutions [6] and the Israel-Wilson-Perjés (IWP) solutions [7] of ordinary Einstein-Maxwell theory.

We find a compact form for the general charged Taub-NUT solution, which reduces to the form of the charged black hole solution given in [3], when the NUT parameter is set to zero. The charged Taub-NUT solution in Einstein-Maxwell theory has no curvature singularities. In contrast, the dilaton-axion solution is singular, if the charge exceeds a certain critical value. In the limit of zero NUT parameter, this critical charge vanishes and the singularity corresponds to the singular inner horizon of the charged black holes. There is thus a competition between the NUT parameter and the charge in determining the global structure of these solutions. The charged Taub-NUT solution in Einstein-Maxwell theory has an extremal limit in which an infinite throat arises. We find that a similar infinite throat exists for the rescaled string metric of certain extremal Taub-NUT solutions in dilaton-axion gravity, which have predominantly magnetic charge. Again, when the NUT parameter is set to zero, this reproduces a known property of extremal magnetically charged black holes in this theory [1]. This throat solution, with nonzero NUT parameter, was found recently in an exact conformal field theory construction by Johnson [8].

The original IWP solutions describe collections of charged objects in a state of equipoise, the charges being such that each object satisfies a ‘no force’ condition with respect to all of the others. Amongst the IWP solutions are the static MP solutions, which describe collections of extremally charged black holes. The simplest example of a nonstatic IWP solution is the Kerr-Newman solution with arbitrary angular momentum per unit mass a , charge Q , and mass M satisfying

$$Q^2 = M^2. \tag{1}$$

⁵We will use the annotation black hole *type* to encompass natural families of solutions, which include, in addition to black holes, such things as naked singularities and spacetimes carrying NUT charge.

For $a = 0$, this is the same as the extremality condition, the maximum charge that a static black hole can have. For $a \neq 0$, however, the extremality condition is

$$Q^2 = M^2 - a^2, \quad (2)$$

so the Kerr-Newman solution in the IWP limit (*i.e.* satisfying (1)), with $a \neq 0$, actually describes a naked singularity, rather than a black hole. Another simple example of a non-static IWP solution is the extremal charged Taub-NUT solution mentioned above. In this case, as with all Taub-NUT metrics, the inverse metric is singular unless the time direction is taken to be periodic. These then also fail to be black holes. More generally, Hartle and Hawking [9] have shown that amongst the IWP solutions, only the MP solutions describe black holes. All of the more general stationary solutions have naked singularities or other pathological features.

This result is troubling because it has been shown that the IWP solutions admit Killing spinors of $N = 2$ supergravity [10] and, hence, should be considered supersymmetric ground states of the theory. The general IWP solutions, however, do not seem like ground states. For rotating black holes, the extremal ones satisfying (2) and having vanishing Hawking temperature, would seem altogether better suited for the title of physical ground state. It was recently shown [11] that the nonstatic IWP solutions fail to be supersymmetric at the quantum level, due to the trace anomaly of $N = 2$ supergravity. Thus, they are, in this sense, ‘less supersymmetric’ than the static solutions, failing to be ground states once quantum corrections are taken into account. This provides a possible resolution to the paradox of having unphysical supersymmetric ground states in the Einstein-Maxwell system.

The IWP solutions of dilaton-axion gravity, which we present below, share many of the properties of the ordinary IWP solutions, including their various pathologies. Dilaton-axion gravity, however, may be embedded in $N = 4$ supergravity and has vanishing trace anomaly. We show that the new IWP solutions are again supersymmetric ground states, admitting Killing spinors of $N = 4$ supergravity. Hence, the possible resolution of ref. [11] fails in this case, and, if one still would like to associate the unbroken supersymmetry of asymptotically flat spacetimes with the absence of naked singularities, one is faced with a new challenge.

A completely different interpretation of the new IWP solutions arises in the context of embedding into the 10-dimensional geometry of critical superstring theory. It was recently shown [12] that 4-dimensional extremal black holes embedded in a certain way in 10-dimensional geometry are dual to a set of supersymmetric string wave (SSW) solutions in $d = 10$, $N = 1$ supergravity [13]. Here, we show that a similar construction can be used to relate the stringy IWP solutions to a class of SSW’s. In addition to showing that these solutions are identical from the point of view of string propagation, this gives an independent demonstration of the supersymmetry of the IWP solutions.

2 Taub-Nut Solutions in Dilaton-Axion Gravity

We follow the conventions of references [14], [3]. The signature of the metric is $(+ - - -)$. The action is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -\mathcal{R} + 2(\partial\phi)^2 + \frac{1}{2}e^{4\phi}(\partial a)^2 - e^{-2\phi}F^2 + iaF^*F \right\}, \quad (3)$$

where a is the axion, ϕ is the dilaton, \mathcal{R} is the scalar curvature and F is the field strength⁶ of a $U(1)$ gauge field A_μ . It is useful to define the complex scalar field $\lambda = a + ie^{-2\phi}$ and the $SL(2, R)$ -dual to the field strength

$$\tilde{F} = e^{-2\phi} {}^*F - iaF. \quad (4)$$

In terms of these fields, the action reads

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -\mathcal{R} + \frac{1}{2} \frac{\partial_\mu \lambda \partial^\mu \bar{\lambda}}{(\text{Im } \lambda)^2} - F_{\mu\nu} {}^* \tilde{F}^{\mu\nu} \right\}. \quad (5)$$

The advantage of using \tilde{F} is that the equations of motion imply the local existence of a vector potential \tilde{A} , satisfying

$$\tilde{F} = i d\tilde{A}. \quad (6)$$

If A_t plays the role of the electrostatic potential, then \tilde{A}_t plays the role of magnetostatic potential.

The action (3) describes the bosonic part of the 10-dimensional effective action of string theory, dimensionally reduced to four dimensions. The 4-dimensional vector field in this theory may be either an Abelian part of a Yang-Mills multiplet or it may come from the non-diagonal component of the metric in extra dimensions, as in Kaluza-Klein theory, and/or from components of the 2-form gauge field. Only when the supersymmetric embedding of the action (3) has been specified, can one attribute a definite origin of the vector field. In particular, in sec. 4 we will consider the embedding of the action (3) into $N = 4$ supergravity without additional matter multiplets. In terms of the 10-dimensional theory, this means that the 4-dimensional vector field arises from non-diagonal components of the metric, $g_{\mu 4}$, and also from the corresponding components of the 2-form gauge field $B_{4\mu} = -g_{4\mu}$, where $\mu = 0, 1, 2, 3$ and 4 stands for one of the six compactified directions.

⁶The spacetime duals are ${}^*F^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, with flat $\epsilon^{0123} = -\epsilon_{0123} = +i$ and curved $\epsilon^{\hat{0}\hat{1}\hat{2}\hat{3}} = \frac{1}{g} \epsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = +i$. We will sometimes use a flat 3-dimensional ϵ_{ijk} such that $\epsilon_{123} = +1$.

The metric of the Taub-NUT solution with charge Q in Einstein-Maxwell theory [6] is given by

$$\begin{aligned} ds^2 &= f(r)(dt + 2l \cos \theta d\varphi)^2 - f^{-1}(r)dr^2 - (r^2 + l^2)d\Omega^2, \\ f(r) &= \frac{r^2 - 2mr - l^2 + Q^2}{r^2 + l^2}. \end{aligned} \quad (7)$$

The Taub-NUT spacetimes have no curvature singularities. The metric, however, does have so-called ‘wire’ singularities along the axes $\theta = 0$, $\theta = \pi$ on which the metric fails to be invertible. Misner [15] has shown that the wire singularities may be removed by making the time coordinate periodic. This may be seen in the following way. To make the metric regular at the north ($\theta = 0$) and south ($\theta = \pi$) poles, we take separate coordinate patches. Define new time coordinates t_N , t_S on the patches by

$$t = t_N - 2l\varphi = t_S + 2l\varphi. \quad (8)$$

If φ is to be an angular coordinate with period 2π , then for consistency on the overlaps we must take the time coordinates to be periodic, with period $8\pi l$. The surfaces of constant r then turn out to have S^3 topology.

The metric function $f(r)$ in (7) has roots at $r_{\pm} = m \pm \sqrt{m^2 + l^2 - Q^2}$. For $r > r_+$ and $r < r_-$ the metric has closed time-like curves. Thus, although the form of the metric is similar to Schwarzschild, no black hole interpretation is possible. For r in the range $r_- < r < r_+$, the coordinate t is spacelike and r is timelike. This region describes a nonsingular, anisotropic, closed cosmological model. It can be thought of as a closed universe containing electromagnetic and gravitational radiation having the longest possible wavelength [6]. There is a limit of (7), with $Q^2 = m^2 + l^2$, in which the two roots r_{\pm} coincide at $r = m$. We will call this the extremal limit. If we let $x = r - m$ in this limit, then near $x = 0$ the metric becomes

$$ds^2 \approx \frac{x^2}{m^2 + l^2} (dt + 2l \cos \theta d\varphi)^2 - \frac{m^2 + l^2}{x^2} dx^2 - (m^2 + l^2)d\Omega^2, \quad (9)$$

which has the form of an infinite spatial throat of constant cross-sectional area, though it does not have the simple direct product form of the $l = 0$ throats.

The uncharged Taub-NUT solution, eqn. (7) with $Q = 0$, continues to be a solution in dilaton-axion gravity. The charged solution could in principle be found from the uncharged one using solution generating techniques (see *e.g.* refs. [16]), but this would most likely lead to a relatively unwieldy form of the solution. Rather, by using less direct methods, we have found the general charged Taub-NUT solutions in dilaton-axion gravity in a form

which naturally extends that given for the general charged black hole solutions in ref. [3]. This new solution is given by

$$\begin{aligned}
ds^2 &= f (dt + 2l \cos \theta d\varphi)^2 - f^{-1} dr^2 - R^2 d\Omega^2, \\
f &= \frac{(r - r_+)(r - r_-)}{R^2}, \quad R^2 = r^2 + l^2 - |\Upsilon|^2, \\
r_{\pm} &= m \pm r_0, \quad r_0^2 = m^2 + l^2 + |\Upsilon|^2 - 4|\Gamma|^2, \\
\lambda &= \frac{\lambda_0(r + il) + \bar{\lambda}_0 \Upsilon}{(r + il) + \Upsilon}, \quad \Upsilon = -2 \frac{\bar{\Gamma}^2}{M}, \quad M = m + il, \\
A_t &= \pm \frac{e^{\phi_0}}{R^2} \{ \Gamma(r + il + \Upsilon) + c.c. \}, \\
\tilde{A}_t &= \mp \frac{e^{\phi_0}}{R^2} \{ i\Gamma[\lambda_0(r + il) + \bar{\lambda}_0 \Upsilon] + c.c. \}.
\end{aligned} \tag{10}$$

The solution depends through the two complex parameters $\Gamma = (Q + iP)/2$ and $M = m + il$ on four real parameters; electric charge Q , magnetic charge P , mass m and NUT charge l .

The equations of motion of dilaton-axion gravity have an $SL(2, R)$ -symmetry, which was used in [2] to generate black holes with combined electric and magnetic charge, and hence nontrivial axion field, from those with either pure electric or magnetic charge. In ref. [3] the generally charged black hole solution was presented in a compact form which had the property of manifest $SL(2, R)$ symmetry. The dilaton-axion charged Taub-Nut solution (10) is already in such a manifestly symmetric form. In the present context, this means the following. The $SL(2, R)$ symmetry of the equations of motion of dilaton-axion gravity allows one, in principle, to generate new solutions from a known one by applying an $SL(2, R)$ -rotation. However, our solutions already describe the whole $SL(2, R)$ -family of solutions. It suffices to perform an $SL(2, R)$ rotation only on the parameters describing the solution. In particular, one has to substitute the $SL(2, R)$ -rotated values of the axion-dilaton field at infinity as well as the $SL(2, R)$ -rotated values of the charges

$$\lambda'_0 = \frac{\alpha \lambda_0 + \beta}{\gamma \lambda_0 + \delta}, \quad \Upsilon' = e^{-2i \text{Arg}(\gamma \lambda_0 + \delta)} \Upsilon, \tag{11}$$

$$\Gamma' = e^{+i \text{Arg}(\gamma \lambda_0 + \delta)} \Gamma. \tag{12}$$

where α, β, γ and δ are the elements of a $SL(2, R)$ matrix

$$R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{13}$$

This gives the $SL(2, R)$ -rotated solution.

The solutions (10) reduce to known solutions in a number of limits. If the complex electromagnetic charge Γ is set to zero, then the dilaton-axion charge Υ vanishes as well, and (10) reduces to the uncharged Taub-NUT solution. When the NUT charge l is taken to zero, (10) reduces to the general static charged black hole solution of ref. [3].

There is an extremal limit of (10) in which $r_+ = r_- = m$. This corresponds to fixing the parameters such that $2|\Gamma|^2 = |M|^2$, which in turn implies that $|\Upsilon|^2 = |M|^2$. If we write the metric in terms of a shifted radial coordinate $\bar{r} = r - m$, then in the extremal limit it becomes

$$ds^2 = \left(1 + \frac{2m}{\bar{r}}\right)^{-1} (dt + 2l \cos \theta d\varphi)^2 - \left(1 + \frac{2m}{\bar{r}}\right) (d\bar{r}^2 + \bar{r}^2 d\Omega^2). \quad (14)$$

We will see below that this is an example of a dilaton-axion IWP metric.

The metric (14) does not have an infinite throat near $\bar{r} = 0$. However, an infinite throat does arise in the rescaled string metric $d\bar{s}^2 = e^{2\phi} ds^2$ for certain cases. To see this, fix the asymptotic value of the dilaton-axion field to $\lambda_0 = i$, for simplicity. The dilaton field in (10) then becomes

$$e^{-2\phi} = \frac{R^2}{|r + il + \Upsilon|^2}. \quad (15)$$

The string metric corresponding to (10) is then given by

$$d\bar{s}^2 = |r + il + \Upsilon|^2 \left\{ \frac{(r - r_+)(r - r_-)}{R^4} (dt + 2l \cos \theta d\varphi)^2 - \frac{1}{(r - r_+)(r - r_-)} dr^2 - d\Omega^2 \right\}. \quad (16)$$

Now take the extremal limit with the particular choice of parameters $\Gamma = \pm i\bar{M}/\sqrt{2}$, $\Upsilon = +M$. Note that in terms of the real charges this is $P = \pm\sqrt{2}m$, $Q = \pm\sqrt{2}l$, so that this case reduces to an extremal magnetically charged black hole in the $l = 0$ limit. With this choice for the charges eqn. (16) becomes, in terms of the radial coordinate \bar{r} defined above,

$$d\bar{s}^2 = |\bar{r} + 2m + 2il|^2 \left\{ \frac{1}{(\bar{r} + 2m)^2} (dt + 2l \cos \theta d\varphi)^2 - \frac{1}{\bar{r}^2} d\bar{r}^2 - d\Omega^2 \right\}, \quad (17)$$

which does have the form of an infinite throat as $\bar{r} \rightarrow 0$,

$$d\bar{s}^2 \approx 4(m^2 + l^2) \left\{ \frac{1}{4m^2} (dt + 2l \cos \theta d\varphi)^2 - \frac{1}{\bar{r}^2} d\bar{r}^2 - d\Omega^2 \right\}. \quad (18)$$

There is a second interesting way to take the throat limit, analogous to the ‘black hole plus throat’ limit of the extremal magnetic black hole [17]. To reach this limit, start again

at (16) and make the coordinate transformation $r = m + r_0(1 + 2 \sinh^2(\sigma/2))$. Taking the extremal limit, $r_0 \rightarrow 0$, with the charges fixed as in the last paragraph, gives

$$d\bar{s}^2 = 4(m^2 + l^2) \left\{ \frac{\sinh^2 \sigma}{(2m \cosh \sigma + 2\sqrt{m^2 + l^2})^2} (dt + 2l \cos \theta d\varphi)^2 - d\sigma^2 - d\Omega^2 \right\}. \quad (19)$$

This limit of the extremal Taub-NUT solutions was recently found in an exact conformal field theory construction by Johnson [8].

The scalar curvature of the charged Taub-NUT solutions is given by

$$\mathcal{R} = \frac{2|\Upsilon|^2(r - r_+)(r - r_-)}{R^6}, \quad (20)$$

from which we see that the curvature is singular wherever R^2 vanishes. From (10) we can see that this can happen, if $|\Upsilon|^2 \geq l^2$, which corresponds to $(Q^2 + P^2)^2 \geq 4l^2(m^2 + l^2)$. We see that there is a sort of competition between the electromagnetic charge and the NUT charge in determining whether or not the geometry is singular. These singularities will occur at $r_{sing} = \pm \sqrt{|\Upsilon|^2 - l^2}$. In the limit $l = 0$, r_{sing} coincides with r_- , giving the singular inner black hole horizon. In the extremal limit we have $r_+ = r_- = r_{sing} = m$. There may also in certain cases be curvature singularities, such as the Schwarzschild one, which do not contribute to the scalar curvature and which we have not examined.

3 IWP Solutions

The IWP solutions in dilaton-axion gravity, like the IWP solutions in Einstein-Maxwell theory, can be compactly expressed in terms of a single complex function on the 3-dimensional plane. In the present case, this function gives the value of the complex dilaton-axion field λ , which is independent of the time coordinate. The metric and gauge field are then given in terms of the real and imaginary parts of this function by

$$ds^2 = e^{2\phi} (dt + \omega_i dx^i)^2 - e^{-2\phi} d\vec{x}^2, \quad A_\mu = \pm \frac{1}{\sqrt{2}} e^{2\phi} (1, \vec{\omega}), \quad (21)$$

where $d\vec{x}^2$ is the Euclidean 3-dimensional metric and the components ω_i of the 1-form $\omega \equiv \omega_i dx^i = \vec{\omega} \cdot d\vec{x}$ are solutions to the equation

$$\epsilon_{ijk} \partial_j \omega_{\hat{k}} = -\partial_i a. \quad (22)$$

Given these relations between the fields, the entire set of equations of motion reduce to the single equation

$$\partial_i \partial_i \lambda = 0, \quad (23)$$

i.e. the 3-d Euclidean Laplacian of λ is zero. Note that (23) includes the integrability condition for (22). Given this, it is natural to define a second 1-form $\eta = \eta_i dx^i$ by

$$\epsilon_{ijk} \partial_j \eta_k = -\partial_i e^{-2\phi}. \quad (24)$$

The $SL(2, R)$ dual gauge potential is then given by

$$\tilde{A}_\mu = \mp \frac{1}{\sqrt{2}} (ae^{2\phi}, ae^{2\phi} \vec{\omega} + \vec{\eta}). \quad (25)$$

The scalar curvature of the IWP spacetimes is given, again as an equation in the flat background metric, by

$$R = -2(\partial\phi)^2 - \frac{1}{2}e^{4\phi}(\partial a)^2 = -\frac{1}{2} \frac{\partial_i \lambda \partial_i \bar{\lambda}}{(\text{Im } \lambda)^2}. \quad (26)$$

We see that there will be a scalar curvature singularity if $\text{Im } \lambda = e^{-2\phi} = 0$ and $\partial_i \lambda \partial_i \bar{\lambda}$ does not vanish sufficiently rapidly.

There are many solutions to eqn. (23) corresponding to different IWP solutions. We will be primarily interested in solutions which are asymptotically locally flat, and for which λ is singular only at isolated points in the 3-dimensional plane. We can then write $\lambda(\vec{x})$ in the following form,

$$\lambda(\vec{x}) = i \left(1 + \sum_{i=1}^N \frac{2(m_i + in_i)}{r_i} \right), \quad r_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2, \quad (27)$$

where the parameters m_i, n_i are real and correspond to the masses and NUT charges of the objects. The parameters \vec{x}_i may be complex.

If we take all the $n_i = 0$ and \vec{x}_i real, then the axion a , and hence also ω , vanishes, and the IWP solution (21) reduces to the static MP type solutions of ref. [1] with pure electric charge. These correspond to collections of extremal charged black holes. As with the ordinary IWP solutions there are two basic ways of moving away from the static limit; either by allowing the positions \vec{x}_i to have imaginary parts, or by taking some of the $n_i \neq 0$. The first option again introduces angular momentum and the second NUT charge. By exercising these two options in turn for a single source, we will see that we recover special limits of known solutions.

For example, take $\lambda = i(1 + 2m/R)$ with $R^2 = x^2 + y^2 + (z - i\alpha)^2$ α and m are real parameters. If we change to spheroidal coordinates (r, θ, φ) given by

$$x \pm iy = \sqrt{r^2 + \alpha^2} \sin \theta \exp(\pm i\varphi), \quad z = r \cos \theta. \quad (28)$$

then λ has the form

$$\lambda = i \left(1 + \frac{2M}{r + i\alpha \cos \theta} \right) = \frac{2M\alpha \cos \theta}{r^2 + \alpha^2 \cos^2 \theta} + i \left(1 + \frac{2Mr}{r^2 + \alpha^2 \cos^2 \theta} \right), \quad (29)$$

and a solution of (22) is given by

$$\omega_\varphi = \frac{2M\alpha r \sin^2 \theta}{r^2 + \alpha^2 \cos^2 \theta}. \quad (30)$$

The metric and gauge field are then determined to be

$$\begin{aligned} ds^2 &= \left(1 + \frac{2Mr}{r^2 + \alpha^2 \cos^2 \theta} \right) \left(dt + \frac{2M\alpha r \sin^2 \theta}{r^2 + \alpha^2 \cos^2 \theta} d\varphi \right)^2 - \left(1 + \frac{2Mr}{r^2 + \alpha^2 \cos^2 \theta} \right) d\vec{x}^2, \\ d\vec{x}^2 &= \frac{r^2 + \alpha^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\varphi^2, \\ A_t &= \pm \frac{r^2 + \alpha^2 \cos^2 \theta}{\sqrt{2}(r^2 + \alpha^2 \cos^2 \theta + 2Mr)}, \\ A_\varphi &= \pm \frac{2M\alpha r \sin^2 \theta}{r^2 + \alpha^2 \cos^2 \theta + 2Mr}. \end{aligned} \quad (31)$$

This coincides, after correcting for changes in convention and a shift by m in the radial coordinate, with Sen's rotating solution [5] in the limit in which the electric charge q and mass m are related by $q = \sqrt{2}m$. Hence, the situation is the same as with the original IWP solutions. The object has the same charge to mass ratio as an extremal static black hole, but for $\alpha \neq 0$ corresponds to a naked singularity.

Next consider the second option, taking $\lambda = i[1 + 2(m + il)/r]$, where m, l are real and r is the usual spherical radial coordinate. The metric and gauge field are then given by

$$\begin{aligned} ds^2 &= \left(1 + \frac{2m}{r} \right)^{-1} (dt + 2l \cos \theta d\varphi)^2 - \left(1 + \frac{2m}{r} \right) (dr^2 + r^2 d\Omega^2), \\ A_t &= \pm \frac{1}{\sqrt{2}} \left(1 - \frac{2m}{r} \right), \\ A_\varphi &= \pm \frac{1}{\sqrt{2}} \left(1 - \frac{2m}{r} \right) 2l \cos \theta. \end{aligned} \quad (32)$$

This corresponds to the extremal charged Taub-NUT solution given in the last section, with a particular choice made for the charges.

Finally, it is clear that the IWP solutions presented in this section are not the most general ones. They do not have the property of manifest $SL(2, R)$ duality. As stated above, the present solutions reduce, in the static limit, to the MP solutions with pure electric

charge. Given the $SL(2, R)$ invariance of the equations of motion, one can certainly find a more general class of IWP solutions, which would reduce in the static limit to the generally charged multi-black hole solution given in [3]. Amongst these more general solutions, there will be examples, including the extremal, magnetic Taub-NUT solution already given above, for which the corresponding string metrics will have infinite throats.

4 Supersymmetry of dilaton-axion IWP Solutions

So far, in dilaton-axion gravity, the only solutions known to have unbroken supersymmetries were the extreme static dilaton-axion black holes [14, 18, 3, 19]. It is natural to ask now if the dilaton-axion IWP metrics we have presented in the previous section are supersymmetric. The analogy with Einstein-Maxwell IWP solutions [10] would suggest that the answer to this question is affirmative.

Before proceeding to check this statement let us explain its meaning. The action eqn. (3) can be considered as a consistent truncation of $N = 4$, $d = 4$ ungauged supergravity where all the fermionic fields (and some bosonic fields as well) are set to zero. The solutions we have presented can therefore be considered as solutions of $N = 4$, $d = 4$ ungauged supergravity with all the fermionic fields set to zero. It is reasonable to ask now whether these solutions are invariant under local supersymmetry transformations. Since the bosonic supersymmetry transformation rules are proportional to the (vanishing) fermionic fields, the bosonic fields are always invariant. However, the possibility remains that after a local supersymmetry transformation non-vanishing fermion fields appear. In fact this is what generally happens. For very special configurations there are a finite number of local supersymmetry transformations that leave the fermionic part of the solution vanishing. In this case one says that the configurations are supersymmetric (i.e. have unbroken supersymmetries) and the corresponding supersymmetry parameters are called Killing spinors.

The $N = 4$, $d = 4$ fermionic supersymmetry rules are

$$\frac{1}{2}\delta_\epsilon \Psi_{\mu I} = \nabla_\mu \epsilon_I - \frac{i}{4}e^{2\phi}\partial_\mu a \epsilon_I - \frac{1}{2\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^+\gamma_\mu \alpha_{IJ}\epsilon^J, \quad (33)$$

$$\frac{1}{2}\delta_\epsilon \Lambda_I = -\frac{i}{2}(e^{2\phi}\not{\partial}\lambda)\epsilon_I + \frac{1}{\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^-\alpha_{IJ}\epsilon^J. \quad (34)$$

The statement that the IWP solutions are supersymmetric then means that the equations $\delta_\epsilon \Psi_{\mu I} = 0$, $\delta_\epsilon \Lambda_I = 0$ have at least one set of solutions $\epsilon_I \neq 0$ when we substitute the IWP fields into eqns. (33), (34).

To show that this is indeed the case, we first calculate the self-dual spin connections and

self-dual vector field strengths. A basis of vierbeins for the metric (21) is provided by the 1-forms and vectors

$$\begin{aligned}
e^0 &= e^\phi(dt + \omega), \\
e^i &= e^{-\phi}dx^i, \\
e_0 &= e^{-\phi}\partial_{\hat{0}}, \\
e_i &= e^\phi[-\omega_i\partial_{\hat{0}} + \partial_i].
\end{aligned} \tag{35}$$

The self-dual (in the upper Lorentz indices) part of the spin connection 1-form is given by

$$\begin{aligned}
\omega^{+0i} &= \frac{i}{4}e^{3\phi}[\partial_i\bar{\lambda}e^0 + i\epsilon_{ijk}\partial_j\bar{\lambda}e^k], \\
\omega^{+ij} &= \frac{1}{4}e^{3\phi}[\epsilon_{ijk}\partial_k\bar{\lambda}e^0 + 2i\partial_{[i}\bar{\lambda}\delta_{j]k}e^k].
\end{aligned} \tag{36}$$

The different components of the self-dual part of the electromagnetic tensor F are

$$\begin{aligned}
F_{0i}^+ &= \pm\frac{i}{2\sqrt{2}}e^{4\phi}\partial_i\bar{\lambda}, \\
F_{ij}^+ &= \mp\frac{1}{2\sqrt{2}}e^{4\phi}\epsilon_{ijk}\partial_k\bar{\lambda}.
\end{aligned} \tag{37}$$

Now, let us examine the dilatino supersymmetry rule eqn. (34). First observe that

$$\begin{aligned}
-\frac{i}{2}e^{2\phi}\not{\partial}\lambda &= \gamma^i(-\frac{i}{2}e^{3\phi}\partial_i\lambda), \\
\frac{1}{\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^-\alpha_{IJ}\epsilon^J &= \gamma^i(-\sqrt{2}F_{0i}^-\alpha_{IJ}\gamma^0\epsilon^J) = \\
&= \gamma^i(\pm\frac{i}{2}e^{3\phi}\partial_i\lambda\alpha_{IJ}\gamma^0\epsilon^J).
\end{aligned} \tag{38}$$

All this implies

$$\frac{1}{2}\delta_\epsilon\Lambda_I = -\frac{i}{2}e^{2\phi}\not{\partial}\lambda[\epsilon_I \mp \alpha_{IJ}\gamma^0\epsilon^J] = 0, \tag{39}$$

which gives the constraint on the Killing spinors

$$\epsilon_I \mp \alpha_{IJ}\gamma^0\epsilon^J = 0. \tag{40}$$

This is a first clear indication that supersymmetry may be working. In the dilatino part the spin connection does not enter, it was the relation between the dilaton and vector field which was relevant.

Now we take the $\hat{0}$ -component of the gravitino supersymmetry rule Eq. (33) imposing time-independence of the Killing spinor set ϵ_I , i.e.

$$\partial_{\hat{0}}\epsilon_I = 0. \tag{41}$$

The equation is

$$-\frac{1}{2}\omega_{\hat{0}}^{+ab}\sigma_{ab}\epsilon_I - \frac{1}{2\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^+\alpha_{IJ}\gamma_{\hat{0}}\epsilon_J = 0. \quad (42)$$

First, using equation (36), we get

$$-\frac{1}{2}\omega_{\hat{0}}^{+ab}\sigma_{ab}\epsilon_I = \gamma^i(-\frac{i}{4}e^{3\phi}\partial_i\bar{\lambda}\gamma^0\epsilon_I). \quad (43)$$

Using eq. (37), we have

$$-\frac{1}{2\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^+\alpha_{IJ}\gamma_{\hat{0}}\epsilon_J = \gamma^i(\mp\frac{i}{4}e^{3\phi}\partial_i\bar{\lambda}\alpha_{IJ}\epsilon_J). \quad (44)$$

Putting both pieces together we get

$$\frac{1}{2}\delta_\epsilon\Psi_{\hat{0}I} = -\frac{i}{4}e^{2\phi}\not{\partial}\bar{\lambda}\gamma^0[\epsilon_I \mp \alpha_{IJ}\gamma^0\epsilon^J] = 0, \quad (45)$$

which is obviously satisfied on account of the constraint given in eq. (40).

The last part is to verify the space component of the gravitino supersymmetry transformation. This is a differential equation on the Killing spinor ϵ_I which has the form

$$\frac{1}{2}\delta_\epsilon\Psi_{iI} = \partial_i\epsilon_I - \frac{1}{2}\omega_i^{+ab}\sigma_{ab}\epsilon_I - \frac{i}{4}e^{2\phi}\partial_ia\epsilon_I - \frac{1}{2\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^+\gamma_i\alpha_{IJ}\epsilon^J = 0. \quad (46)$$

First we have

$$-\frac{1}{2}\omega_i^{+ab}\sigma_{ab}\epsilon_I = \gamma^j[-e^\phi\omega_i\omega_0^{+0j} - e^{-\phi}\omega_i^{+0j}]\gamma^0\epsilon_I, \quad (47)$$

and secondly

$$-\frac{1}{2\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^+\alpha_{IJ}\gamma_i\epsilon_J = \gamma^j[-\frac{1}{\sqrt{2}}\omega_iF_{0j}^+ + \frac{1}{\sqrt{2}}e^{-2\phi}F_{0j}^+\gamma^i\gamma^0]\alpha_{IJ}\epsilon_J. \quad (48)$$

Now we sum the last two equations and use our last result (the $\hat{0}$ component of the gravitino supersymmetry transformation rule) and the constraint Eq. (40) to simplify the sum. We get

$$-e^{-\phi}\gamma^j[\omega_i^{+0j}\gamma^0 - \frac{1}{\sqrt{2}}e^{-\phi}F_{0j}^+\gamma^i]\epsilon_I. \quad (49)$$

To further simplify this equation we use the identity

$$\gamma^j\gamma^iF_{0j}^+ = F_{0i}^+ + \gamma^jF_{ij}^+\gamma^0\gamma_5. \quad (50)$$

Substituting it into Eq. (49), noting that ϵ_I has negative chirality and using the explicit form of the spin connection and the electromagnetic field of the IWP solutions eqns. (36) and (37) we get

$$-\frac{1}{2}\omega_i^{+ab}\sigma_{ab}\epsilon_I - \frac{1}{2\sqrt{2}}e^{-\phi}\sigma^{ab}F_{ab}^+\gamma_i\alpha_{IJ}\epsilon^J = \pm\frac{i}{4}e^{2\phi}\partial_i\bar{\lambda}\epsilon_I. \quad (51)$$

This can be substituted back into Eq. (46) getting, at last, a much simpler differential equation for the Killing spinors

$$\partial_i \epsilon_I + \frac{1}{4} e^{2\phi} (\partial_i e^{-2\phi}) \epsilon_I = 0, \quad (52)$$

which can be rewritten as

$$\partial_i (e^{e^{-\phi/2}} \epsilon_I) = 0. \quad (53)$$

Thus, the Killing spinor set ϵ_I exists and is given in terms of a set of constant spinors $\epsilon_{I(0)}$ by

$$\epsilon_I = e^{\phi/2} \epsilon_{I(0)}, \quad (54)$$

where the constant spinors satisfy the same constraint Eq. (40) as the Killing spinors themselves:

$$\epsilon_{I(0)} \mp \alpha_{IJ} \gamma^0 \epsilon_{(0)}^J = 0. \quad (55)$$

This constraint limits the number of independent components of the Killing spinor set ϵ_I to half of the total. Thus, the dilaton-axion IWP solutions have two unbroken supersymmetries when embedded into $N = 4$, $d = 4$ ungauged supergravity.

Let us now analyze this result. It is known that supersymmetry enforces a Bogomolnyi-Gibbons-Hull (BGH) bound [20] on the charges, which is saturated by those configurations having unbroken supersymmetries. These charges are defined at asymptotic infinity and usually consist of the mass and central charges of the supersymmetry algebra (in the case of extended supersymmetry theories). For static, asymptotically flat black holes, there is a fascinating coincidence between the BGH bound, which implies supersymmetry, and the extremal bound, which implies the absence of naked singularities [14]. If one tries to extend this to configurations which are not asymptotically flat or are not static, one finds that the rule no longer holds. For instance, static asymptotically anti-deSitter solutions with both unbroken supersymmetries are known to have naked singularities [21]. Likewise, there are Einstein-Maxwell IWP metrics which are both supersymmetric and have naked singularities. In this latter case (take for instance the Kerr-Newman metric with $Q^2 = M^2$) the difficulty can be traced to the fact that the angular momentum of the hole is present in the condition for the absence of naked singularities, but not in the BGH bound.

For the IWP solutions, NUT charge plays an interesting role in the analysis of BGH bounds as well. We discuss this briefly here for our dilaton-axion solutions, but a parallel discussion would hold for the Einstein-Maxwell IWP solutions and, to our knowledge, has not previously been given. We hope to return to a more complete discussion of this point in future work. The IWP solutions being supersymmetric, must saturate some BGH bound. One might naively think that, the supersymmetry algebra being the same, the BGH bound should be the same as that derived in [18] for the extreme static dilaton-axion black holes, namely

$$m^2 + |\Upsilon|^2 - 4|\Gamma|^2 \geq 0. \quad (56)$$

However, from the discussion in the previous sections, it is clear that the correct BGH bound saturated by the IWP solutions is

$$m^2 + l^2 + |\Upsilon|^2 - 4|\Gamma|^2 \geq 0. \quad (57)$$

The NUT charge l appears here as a sort of dual to the usual ADM mass. At first, this is a bit surprising. However, the NUT charge is well known to play such a role; Taub-NUT space being interpreted as a gravitational dyon (see e.g. [23] and references therein). If asymptotic conditions are relaxed to allow for nonzero NUT charge, then the NUT charge must arise as a boundary term in a positive energy construction on a spatial slice.

Finally, note that once again the angular momentum does not appear in the bound (57) and that, in the cases in which the NUT charge vanishes and we have asymptotically flat geometries in the usual sense, we will have the same problems discussed above.

From the dilaton-axion IWP solutions, which we have presented in the previous section, one can generate more general solutions using $SL(2, R)$ -duality rotations [2, 22]. The new solutions will have the same Einstein-frame metric but different dilaton, axion and vector fields, and, therefore, different string-frame metric. In addition they will have the same number of unbroken supersymmetries [18, 19]. The same problems with supersymmetry and naked singularities will be present in the $SL(2, R)$ -rotated solutions.

The relation between extreme black holes of ordinary Einstein-Maxwell theory and IWP metrics, which are both supersymmetric at the classical level when embedded into $N = 2$ supergravity, was analysed in [10], [11]. In particular the Killing spinors of the IWP solution depend on space coordinates through some complex harmonic function V .

$$\epsilon_I(x) = V^{1/2}(x)\epsilon_{I(0)}, \quad I = 1, 2. \quad (58)$$

where $\epsilon_{I(0)}$ are some constant spinors. The same function V is also used in the ansatz for the metric and for the vector field. If we choose the imaginary part of the function V to become zero, we reduce the IWP solution to the extreme electrically charged Reissner-Nordström solution. Simultaneously the Killing spinor dependence on the space-time is reduced to the dependence on the real function V in eq. (58).

In $N = 4$ supergravity the Killing spinors of $SL(2, R)$ -form invariant dilaton-axion black holes also have a dependence on space coordinates through a complex function [18, 19]. The real part of this function is defined by the g_{tt} component of the metric and the imaginary part of it is related to the $Arg(\gamma\lambda + \delta)$, where λ is the dilaton-axion field and γ, δ are parameters of $SL(2, R)$ transformations defined in equation (13).

We have found that the Killing spinors for dilaton-axion IWP solutions, described in the previous sections of this paper, depend on space coordinates through a real function $e^{\frac{1}{2}\phi}$.

Such dependence is known to exist for pure electric dilaton black holes [14]. If we would consider the more general class of dilaton-axion IWP solutions, related to those presented above by a generic $SL(2, R)$ transformations, we would get a complex function, defining the dependence of a Killing spinor on space coordinates [18, 19]. However here we have studied the special form of IWP solutions, whose Killing spinors depend on x only through a dilaton field. In stringy frame electric black holes as well as our IWP solutions have Killing spinors with dependence on space coordinates in the form e^ϕ . This gives us a nice bridge to the supersymmetric gravitational waves and their dual partners. We have studied such dependence before and know that the relevant gravitational waves have constant Killing spinors [13]. The Killing spinors of the dual waves have the following dependence on space coordinates [26]

$$\epsilon_{str}(x) = e^{\phi(x)} \epsilon_0 . \tag{59}$$

5 IWP Solutions From Stringy Waves

The previous section shows that to find the unbroken supersymmetry of the 4-dimensional IWP dilaton-axion solutions one has to solve a rather involved set of equations. In the present situation, where the result poses a conceptual puzzle, it is useful to have an independent means of checking supersymmetry.

Interesting enough, there does exist a completely independent method of deriving the general IWP dilaton-axion solutions, which automatically insures that they are supersymmetric. There exist gravitational wave solutions in 10-dimensional $N = 1$ supergravity, which have been shown to be supersymmetric [13] and which are known as supersymmetric string waves (SSW's). Further, it was recently shown [12] that certain SSW's are dual, under a sigma model duality transformation [27], to extremal 4-dimensional black holes, embedded in 10-dimensional geometry. Here, we show that certain other SSW's transform under sigma model duality into 4-dimensional IWP solutions embedded into 10-dimensional geometry. One can preserve the unbroken supersymmetry of higher-dimensional solutions by using dimensional reduction of supergravity [25]. The fact that sigma-model duality preserves supersymmetry is proved in [24]. Also a direct proof of supersymmetry of dual partners of the wave solution in 10-dimensional theory is available [26].

Here we recount the construction briefly. More details can be found in [12]. Our conventions in this section are those in [26], [12] .

We consider the zero slope limit of the effective string action. This limit corresponds to 10-dimensional $N = 1$ supergravity. The Yang-Mills multiplet will appear in first order α'

string corrections. The bosonic part of the action is

$$S = \frac{1}{2} \int d^{10}x e^{-2\phi} \sqrt{-g} [-\mathcal{R} + (\partial\phi)^2 - \frac{3}{4}H^2], \quad (60)$$

where the 10-dimensional fields are the metric g_{MN} , the 3-form field strength $H_{MNL} = \partial_{[M}B_{NL]}$ and the dilaton ϕ . The zero slope limit of the SSW solutions [13] in $d = 10$ are given by the Brinkmann metric and 2-form gauge potential

$$\begin{aligned} ds^2 &= 2d\tilde{u}d\tilde{v} + 2A_M d\tilde{x}^M d\tilde{u} - \sum_{i=1}^{i=8} d\tilde{x}^i d\tilde{x}^i, \\ B &= 2A_M d\tilde{x}^M \wedge d\tilde{u}, \quad A_v = 0, \end{aligned} \quad (61)$$

where the indices run over the values $i = 1, \dots, 8$, $M = 0, 1, \dots, 8, 9$ and we are using the notation $x^M = \{\tilde{u}, \tilde{v}, \tilde{x}^i\}$ for the 10-dimensional coordinates. We have put the tilde over the 10-dimensional coordinates for this solution, because we will have to compare this original 10-dimensional solution after dual rotation with the 4-dimensional one, embedded into the 10-dimensional space. A rather non-trivial identification of coordinates, describing these solutions will be required.

The equations of motion reduce to equations for $A_u(\tilde{x}^i)$ and $A_i(\tilde{x}^j)$, which are

$$\Delta A_u = 0, \quad \Delta \partial^i A^j = 0, \quad (62)$$

where the Laplacian Δ is taken over the transverse directions only.

Application of a sigma-model duality transformation [27] to the SSW solutions given in eq. (61) leads to the following new supersymmetric solutions of the equations of motion in the zero slope limit [26]:

$$\begin{aligned} ds^2 &= 2e^{2\phi} \{d\tilde{u}d\tilde{v} + A_i d\tilde{u}d\tilde{x}^i\} - \sum_{i=1}^{i=8} d\tilde{x}^i d\tilde{x}^i, \\ B &= -2e^{2\phi} \{A_u d\tilde{u} \wedge d\tilde{v} + A_i d\tilde{u} \wedge d\tilde{x}^i\}, \\ e^{-2\phi} &= 1 - A_u, \end{aligned} \quad (63)$$

where, as before, the functions $A_M = \{A_u = A_u(\tilde{x}^j), A_v = 0, A_i = A_i(\tilde{x}^j)\}$ satisfy eqns. (62). We call this new solution the dual partner of the SSW, or ‘dual wave’ for simplicity.

The next step after the dual rotation of the wave is to dimensionally reduce from $d = 10$ to $d = 4$. Both steps can be performed in a way which keeps the unbroken supersymmetry of the original SSW configuration intact. The rules for this procedure have been worked out in refs. [12] and [24], and more details of the procedure may be found there.

The 4-dimensional action, related to (60) by our dimensional reduction is⁷

$$S = \frac{1}{2} \int d^4x e^{-2\phi} \sqrt{-g} [-R + 4(\partial\phi)^2 - \frac{3}{4}H^2 + \frac{1}{2}F^2(V)], \quad (64)$$

where

$$\begin{aligned} F_{\mu\nu}(V) &= 2\partial_{[\mu}V_{\nu]}, \\ H_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} + V_{[\mu}F_{\nu\rho]}(V). \end{aligned} \quad (65)$$

The embedding of the 4-dimensional fields in this action in $d = 10$ are as follows:

$$\begin{aligned} g_{\mu\nu}^{(10)} &= g_{\mu\nu} - V_\mu V_\nu, \\ g_{4\nu}^{(10)} &= -V_\nu, \\ g_{44}^{(10)} &= -1, \\ g_{IJ}^{(10)} &= \eta_{IJ} = -\delta_{IJ}, \\ B_{\mu\nu}^{(10)} &= B_{\mu\nu}, \\ B_{4\nu}^{(10)} &= V_\nu, \\ \phi^{(10)} &= \phi, \end{aligned} \quad (66)$$

where the indices μ, ν run over the range $0, 1, 2, 3$ and I, J run over the range $5, 6, 7, 8, 9$. This formula can be used to uplift a $U(1)$ 4-dimensional field configuration, including the dilaton and axion fields, to a 10-dimensional field configuration, in a way which is consistent with supersymmetry. Or vice versa, one may bring down to 4 dimensions a 10-dimensional configuration which has the property $g_{4\nu}^{(10)} = -B_{4\nu}^{(10)}$.

To generate a dual-wave, which upon dimensional reduction gives the 4-dimensional IWP solutions, we make the following choices for the vector A_M in the 10-dimensional SSW (61). The components A_M are taken to depend only on three of the transverse coordinates, $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3$, which will ultimately correspond to our 3-dimensional space. We choose one component, *e.g.*, A_4 to be related to according to $A_4 = \xi A_u$, where $\xi^2 = \pm 1$ depending on the signature of spacetime⁸. Recall that from (63) A_u is related to the dilaton by $A_u = 1 - e^{-2\phi}$. For the remaining components, we take only A_1, A_2, A_3 to be non-vanishing, and we relabel these as $\omega_1, \omega_2, \omega_3$ for obvious reasons. To summarize, we then have

$$A_4 = \xi A_u, \quad A_1 \equiv \omega_1, \quad A_2 \equiv \omega_2, \quad A_3 \equiv \omega_3, \quad A_5 = \dots = A_8 = 0. \quad (67)$$

⁷Note that the 4-vector field V_μ in this action is related to the 4-dimensional 4-vector field A_μ in the action (3) as $V_\mu = \sqrt{2}A_\mu$. The difference in notation between the 4-dimensional action (3) and equation below is explained in [24].

⁸See [12] for the details.

The equations of motion (60), in terms of these degrees of freedom, become

$$\Delta e^{-2\phi} = 0, \quad \Delta \partial^{[i} \omega^{j]} = 0, \quad (68)$$

where again the Laplacian is taken over the transverse directions only.

A few more steps are required to dimensionally reduce the dual wave and recover the IWP solutions of the previous sections. These include a coordinate change,

$$\hat{x} = \tilde{x}^4 + \xi \tilde{u}, \quad \hat{v} = \tilde{v} + \xi \tilde{x}^4, \quad (69)$$

shifting B by a constant value and a particular identification of the coordinates in the dual wave solution with those in the uplifted IWP solution,

$$\begin{aligned} t &= \hat{v} = \tilde{v} + \xi \tilde{x}^4, \\ x^4 &= \tilde{u}, \\ x^9 &= \hat{x} = \tilde{x}^4 + \xi \tilde{u}, \\ x^{1,2,3,5,\dots,8} &= \tilde{x}^{1,2,3,5,\dots,8}, \end{aligned} \quad (70)$$

After all these steps our 10-dimensional dual wave becomes

$$\begin{aligned} ds^2 &= 2e^{2\phi} dx^4 (dt + \vec{\omega} \cdot d\vec{x}) - \sum_4^9 dx^i dx^i - d\vec{x}^2, \\ B &= -2e^{2\phi} dx^4 \wedge (dt + \vec{\omega} \cdot d\vec{x}). \end{aligned} \quad (71)$$

To recognize this as the lifted IWP solution, add and subtract from the metric the term $e^{4\phi} (dt + \vec{\omega} \cdot d\vec{x})^2$. We can then rewrite the dual wave metric (70) as

$$ds^2 = e^{4\phi} (dt + \vec{\omega} \cdot d\vec{x})^2 - d\vec{x}^2 - \left(dx^4 - e^{2\phi} (dt + \vec{\omega} \cdot d\vec{x}) \right)^2 - \sum_5^9 dx^i dx^i. \quad (72)$$

The first two terms now give the string metric for the 4-dimensional IWP solutions. The non-diagonal components $g_{\mu 4}$, in the third term, are interpreted as the 4-dimensional gauge field components, showing the Kaluza-Klein origin of the gauge field in this construction. Note that the 4-dimensional vector field components are also equal to the off-diagonal components of the 2-form gauge field, giving the overall identifications

$$g_{t4} = B_{t4} = e^{2\phi} = -V_t, \quad g_{i4} = B_{i4} = e^{2\phi} \omega_i = -V_i. \quad (73)$$

The dilaton of the IWP solution, is identified with the fundamental dilaton of string theory, rather than with one of the modulus fields. The axion is identified with the 4-dimensional part of the 3-form field strength H given in eq. (65). Note that these components of H come

totally from the second term in (65), since the 4-dimensional $B_{\mu\nu}$ vanishes. The equations of motion (68) coincide with the equation (23) which we had earlier for the IWP solutions.

This provides an independent proof of unbroken supersymmetry of dilaton-axion IWP solutions. Here we have demonstrated only the relation between the SSW's and IWP solutions. The proof that the unbroken supersymmetry survives the duality transformation and special compactification is the subject of other publications [12], [24].

6 Discussion

We have presented new, stationary, black hole type solutions to dilaton-axion gravity. These new solutions share many basic properties with their counterparts in Einstein-Maxwell theory, while differing from these counterparts in ways which are familiar from previous studies of dilaton-axion black holes. For the Taub-NUT solutions, two particularly interesting features are the singularities which arise for charge sufficiently large and the existence of an infinite throat for the string metric in the extremal limit with predominantly magnetic charge. This latter feature makes contact with recently derived exact CFT results [8]. It should be possible to find a more general stationary solution in dilaton-axion gravity, analogous to the general type D metric of Einstein-Maxwell theory [28], including mass, NUT charge, electromagnetic charge, rotation and an acceleration parameter. This grand solution would encompass the present Taub-NUT solutions, as well as the rotating solutions [5] and C-metric type solutions [4], as subclasses.

As explained in the introduction, the discovery of the new, supersymmetric, dilaton-axion IWP solutions brings back an apparent paradox, to which a resolution had recently been posed in the case of Einstein-Maxwell theory. The paradox arises if one interprets unbroken supersymmetry of a bosonic solution as indicating that the solution is a physical ground state of the theory. In Einstein-Maxwell theory the fact that the IWP solutions are supersymmetric implies a large degeneracy of the ground state in a given charge sector; single object solutions with vanishing NUT parameter, for example, satisfy $Q^2 + P^2 = M^2$ with arbitrary angular momenta. This degeneracy, however, is removed when quantum corrections are taken into account. The trace anomaly of $N = 2$ supergravity is found to contradict the integrability condition for supersymmetry for nonzero angular momentum [11]. This provides a simple possible resolution of the paradox, since the trace anomaly, related to the Gauss-Bonnet Lagrangian, is well known in supergravity, being gauge independent and finite. Any other quantum corrections in gravitational theories are significantly more difficult to control. The gauge independent trace anomaly, however, vanishes in the present case of $N = 4$ supergravity, making it much more difficult to understand any effect of quantum corrections

in removing the degeneracy of the supersymmetric state. This will be the subject of future investigations.

The fact that the new stationary solutions of $N = 4$ supersymmetric theory have naked singularities in the canonical geometry does not contradict the conjecture [14] that supersymmetry may act as a cosmic censor for static, asymptotically flat configurations. In $N = 2$ theory, it was possible to establish that the more general stationary solutions are not actually supersymmetric, when quantum corrections are taken into account. At present, we do not know whether any analogous situation may hold in $N = 4$ theory.

The presence of a fundamental dilaton field in our theory may be considered as the source of a completely different interpretation of the geometry. In Einstein-Maxwell theory there is no natural source of a conformal transformation to another metric. However, in string theory the natural geometry of the target space is the so-called stringy frame, where the string metric is related to the canonical one by the Weyl transformation $e^{2\phi}$. Moreover, in string theory the 4-dimensional spacetime is not the only relevant space for understanding of configurations which solve the equations of motion of the full ten-dimensional theory. A possible radical point of view in seeking a resolution to our paradox would be to ignore the properties of the canonical four-dimensional configuration and to study our solutions instead in the stringy frame by “lifting” them up to ten dimensions. We have done this in section 6 and shown that the IWP solutions are dual partners of the supersymmetric string waves, which were found in [13]. We hope to return in future work to the implications of this dual relationship between black-hole-type solutions and gravitational waves.

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