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$K(S)\text{-}]\text{GAMMA-GAMMA, } K(L)\text{-}]\text{PI}(0)\text{GAMMA-GAMMA AND UNITARITY}$

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$K_S \rightarrow \gamma\gamma, K_L \rightarrow \pi^0\gamma\gamma$ and Unitarity *

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Abstract

Agreement between the experimental value $\Gamma(K_S \rightarrow \gamma\gamma)$ and the number predicted via a one-loop chiral perturbation theory calculation has been cited as a success for the latter. On the other hand the one-loop prediction for the closely related process $K_L \rightarrow \pi^0\gamma\gamma$ has been found to be a factor three below the experimental value. Using the inputs of unitarity and dispersion relations, we demonstrate the importance of higher order loop effects to both of these processes.

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1 Introduction

During the past decade, we have learned (at last) how to make rigorous contact between experimental low energy hadronic physics and QCD which is presumed to underlie such processes. This contact is provided by chiral perturbation theory (ChPT) [1], which exploits the (broken) chiral invariance of the light quark component of the QCD Lagrangian and provides a representation for interaction amplitudes as an expansion in energy-momentum divided by the chiral scale parameter $4\pi F_\pi \sim 1$ GeV [2]. A review of ChPT techniques will not be attempted here, but has been presented in a number of sources, wherein it is shown that, at least in the Goldstone boson sector, such a chiral approach provides a remarkably successful and predictive representation of a wide variety of experimental processes [3].

The chiral technology begins by writing down an effective chiral Lagrangian, the simplest (two-derivative) form of which is, in the Goldstone sector, [4]

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{\bar{F}^2}{4} \text{Tr} D_\mu U D^\mu U^\dagger + \frac{\bar{F}^2}{4} \text{Tr} 2B_0 m (U + U^\dagger) + \dots \quad (1)$$

where

$$U = \exp \left(\frac{i}{\bar{F}} \sum_{j=1}^8 \lambda_j \phi_j \right) \quad (2)$$

is a nonlinear function of the pseudoscalar fields, $m = (m_u, m_d, m_s)_{\text{diag}}$ is the quark mass matrix,

$$2B_0 = \frac{2m_K^2}{m_u + m_s} = \frac{2m_\pi^2}{m_u + m_d} = \frac{6m_\eta^2}{m_u + m_d + 4m_s} \quad (3)$$

is a phenomenological constant, D_μ is the covariant derivative, and \bar{F} is the pion decay constant in the limit of chiral symmetry. Although these are only two of an infinite number of terms, already at this level there exists predictive power—e.g., tree level evaluation of $\mathcal{L}^{(2)}$ yields the familiar Weinberg predictions (at $\mathcal{O}(p^2, m^2)$) for S-wave $\pi - \pi$ scattering lengths [5] which are approximately borne out experimentally. Loop diagrams required by unitarity produce terms of $\mathcal{O}(p^4, p^2 m^2, m^4)$ and contain divergences. However, just as in QED such infinities can be absorbed into renormalizing phenomenological chiral couplings, and the most general "four-derivative" Lagrangian has

been given by Gasser and Leutwyler [1]

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{(4)} = & L_1(\text{Tr}D_\mu U D^\mu U^\dagger)^2 + L_2(\text{Tr}D_\mu U D_\nu U^\dagger)^2 \\ & + L_3\text{Tr}(D_\mu U D^\mu U^\dagger)^2 + \dots \end{aligned} \quad (4)$$

Here the bare L_i coefficients are themselves unphysical and are related to empirical quantities $L_i^r(\mu)$ measured at scale μ via

$$L_i^r(\mu) = L_i + \frac{\Gamma_i}{32\pi^2} \left(\frac{1}{\epsilon} + \ln \frac{4\pi}{\mu^2} + 1 - \gamma \right), \quad (5)$$

where Γ_i are constants defined in ref. [1] and $\epsilon = 4 - d$ is the usual parameter arising in dimensional regularization, with d being the number of dimensions. Gasser and Leutwyler have obtained empirical values for the phenomenological constants L_1^r, \dots, L_{10}^r .

A wide range of electroweak and strong interactions of these Goldstone bosons have been successfully treated within this formalism, but there is at least one recurring problem—whenever the S-wave I=0 $\pi - \pi$ interaction is involved the simple one loop predictions have in general been found to be wanting [6]. This is perhaps not surprising, as the associated phase shift $\delta_0^0(s)$ passes through 90° somewhere in the vicinity of $\sqrt{s} \sim 700 - 900$ MeV, which has sometimes been associated with the existence of a broad resonance in this region. Such resonant behavior can certainly not be treated in any perturbative fashion and thus in this channel the chiral expansion must break down well before this energy is reached.

There are a number of ways which have been used in order to avoid this difficulty. One is simply to confine predictions to a low enough energy that one-loop corrections should be sufficient. However, it is also possible to treat such effects nonperturbatively either by inclusion of S-wave I=0 (σ or ϵ) pole effects or by use of dispersion relations which effectively treat the $\pi - \pi$ interaction to all loop orders—but at the price of introduction of model dependence [7]. This cost is usually considered worth paying, however, as by inclusion of such well-motivated model-dependence one can often reliably extend the usual region of validity of chiral predictions ($E < \sim 400 - 500$ MeV) up to much higher values ($E < \sim 1$ GeV).

A particularly useful example of this procedure has recently been provided for the process $\gamma\gamma \rightarrow \pi^0\pi^0$. Because only neutral particles are involved

there exists no tree level two-derivative or four-derivative contribution to this reaction, which guarantees that the one loop chiral perturbative prediction must itself be finite. This has been calculated as [8]

$$\text{Amp}(\gamma\gamma \rightarrow \pi^0\pi^0) = 2e^2\epsilon_1^\mu\epsilon_2^\nu \frac{s - m_\pi^2}{16\pi^2 F_\pi^2} \left(\frac{g_{\mu\nu}k_1 \cdot k_2 - k_{2\mu}k_{1\nu}}{k_1 \cdot k_2} \right) F(s, m_\pi^2). \quad (6)$$

Here

$$F(x, y) = 1 + \frac{y}{x} \left[\ln \left(\frac{1 + \beta(x)}{1 - \beta(x)} \right) - i\pi \right]^2 = 1 + \frac{y}{x} \ln^2 \left(\frac{\beta(s) + 1}{\beta(s) - 1} \right) \\ \text{with } \beta(x) = \sqrt{\frac{x - 4y}{x}}. \quad (7)$$

However, the associated $\gamma\gamma \rightarrow \pi\pi$ cross section is given by

$$\sigma(\gamma\gamma \rightarrow \pi^0\pi^0) = \frac{\alpha^2}{256\pi^3 F_\pi^4} \frac{(s - m_\pi^2)^2}{s} \sqrt{1 - \frac{4m_\pi^2}{s}} |F(s, m_\pi^2)|^2 \quad (8)$$

and is found to bear little resemblance to its recently measured experimental analog, as shown in Figure 1.

The solution to this problem has recently been explored by a number of authors and has been found to be related to the inadequacy of the one-loop approach in the $I=0$ S-wave $\pi - \pi$ channel [9]. The solution is most clearly presented in terms of a dispersion relation approach. We assume, consistent with the chiral expansion, that when we are in the near-threshold region the only relevant higher order effects are in the helicity conserving S-wave amplitude, which we write as

$$\begin{aligned} \gamma\gamma \rightarrow \pi^+\pi^- : \quad f^C(s) &= \frac{1}{3} [2f_0(s) + f_2(s)] \\ \gamma\gamma \rightarrow \pi^0\pi^0 : \quad f^N(s) &= \frac{2}{3} [f_0(s) - f_2(s)], \end{aligned} \quad (9)$$

where $I = 0, 2$ refers to the isospin of the final $\pi\pi$ state. For neutral pion production and working in the gauge wherein $\epsilon_2 \cdot k_2 = \epsilon_2 \cdot k_1 = \epsilon_1 \cdot k_2 = \epsilon_1 \cdot k_1 = 0$ the transition amplitude is

$$\gamma\gamma \rightarrow \pi^0\pi^0 : \quad \text{Amp} = 2ie^2\epsilon_1 \cdot \epsilon_2 f^N(s). \quad (10)$$

In the charged pion case the Born and seagull contributions to this multipole must also be included, so that the full amplitude becomes

$$\gamma\gamma \rightarrow \pi^+\pi^- : \text{Amp} = 2ie^2 \left[\epsilon_1 \cdot \epsilon_2 a(s) - \frac{\epsilon_1 \cdot p_+ \epsilon_2 \cdot p_-}{p_+ \cdot k_1} - \frac{\epsilon_1 \cdot p_- \epsilon_2 \cdot p_+}{p_+ \cdot k_2} \right]. \quad (11)$$

Here

$$a(s) = 1 + f^C(s) - f_{\text{Born}}^C(s), \quad (12)$$

where

$$f_{\text{Born}}^C(s) = \frac{1 - \beta^2(s)}{2\beta(s)} \ln \left(\frac{1 + \beta(s)}{1 - \beta(s)} \right) = f_0^{\text{Born}}(s) = f_2^{\text{Born}}(s) \quad (13)$$

is the Born approximation value for the helicity conserving S-wave multipole. In the threshold region the phase of $f_I(s)$ is required by unitarity to be equal to the corresponding $\pi\pi$ phase shift $\delta_I(s)$. When $s > 16m_\pi^2$, inelastic reactions involving four pions are allowed. However, the inelasticity is small, being of order E^8 in the chiral expansion and will be neglected.

The functions $f_I(s)$ are analytic functions of s except for cuts along the positive and negative real axis. For positive s , the right hand cut extends from $4m_\pi^2 < s < \infty$ and is due to the s channel $\pi\pi$ state. For negative s , the left hand cut is due to t, u -channel intermediate states such as $\gamma\pi \rightarrow \pi \rightarrow \gamma\pi$ or $\gamma\pi \rightarrow \rho \rightarrow \gamma\pi$, and extends from $-\infty < s < 0$. The single-channel final state unitarization problem has a simple solution in terms of the Omnès function [10]

$$D_I^{-1}(s) = \exp \left(\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\delta_0^I(s')}{s' - s - i\epsilon} \right) \quad (14)$$

—the result must have the form

$$f_I(s) = g_I(s) D_I^{-1}(s) \quad (15)$$

where $g_I(s)$ is an analytic function with no cuts along the positive real axis. Morgan and Pennington consider a function $p_I(s)$ which has the same left hand singularity structure as $f_I(s)$, but which is real for $s > 0$. They then write a twice subtracted dispersion relation for the difference $(f_I(s) - p_I(s))D_I(s)$, with the result [11]

$$f_I(s) = D_I^{-1}(s) \left[p_I(s) D_I(s) + (c_I + s d_I) - \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{p_I(s') \text{Im} D_I(s')}{s' - s - i\epsilon} \right], \quad (16)$$

where c_I, d_I are subtraction constants. Picking $p_I(s)$ to be given by its Born value and matching onto the known form of the low energy amplitude required by chiral symmetry determines these constants unambiguously to be

$$c_I = 0, \quad d_I = \frac{2}{F_\pi^2}(L_9^r + L_{10}^r) + \frac{1}{384\pi^2} \times \begin{cases} -1 & I = 0 \\ +2 & I = 2 \end{cases} \quad (17)$$

One can now address the origin of the large corrections found in the $\gamma\gamma \rightarrow \pi^0\pi^0$ amplitude. Do they arise simply from the unitarization of the amplitude (*i.e.* $D_I(s) \neq 1$) or are new inputs needed in the amplitude? It turns out that the rescattering physics in $D_I^{-1}(s)$ is most important, and that the main corrections are due to well-known ingredients. In our subsequent discussion, we will use a full phenomenological treatment but it is useful here to explore the case with a simple analytic form for $D_I^{-1}(s)$. The condition $ImD_I(s) = -\beta(s)t_I^{CA}(s)$ defines the $[0,1]$ Padé approximation for the Omnès function [12], *i.e.*

$$D_I^{-1}(s) = \frac{1}{1 - k_I s + t_I^{CA}(s)(h(s) - h(0))}$$

with $h(s) = \frac{\beta(s)}{\pi} \ln \left(\frac{\beta(s) + 1}{\beta(s) - 1} \right), \quad h(0) = \frac{2}{\pi} \quad (18)$

and allows one an approximate but simple analytic representation for the $\gamma\gamma \rightarrow \pi\pi$ amplitude. The constant $k_0 \cong \frac{1}{25m_\pi^2}$ is chosen to match the small s behavior of the experimental $D_0^{-1}(s)$ function, and $k_2 \cong -\frac{1}{30m_\pi^2}$ is chosen from a fit to $I = 2$ $\pi\pi$ scattering. The resulting form for the $\gamma\gamma \rightarrow \pi^0\pi^0$ amplitude is

$$\begin{aligned} f^N(s) &= -\frac{1}{48\pi^2 F_\pi^2} F(s, m_\pi^2) \\ &\times \left[(2s - m_\pi^2) D_0^{-1}(s) + (s - 2m_\pi^2) D_2^{-1}(s) \right] \\ &+ \frac{4}{3F_\pi^2} (L_9^r + L_{10}^r) s (D_0^{-1}(s) - D_2^{-1}(s)) \end{aligned} \quad (19)$$

which, when the Padé forms of $D_I^{-1}(s)$ are used, provides a consistent analytic solution to the dispersion relation while also displaying the correct chiral properties to $\mathcal{O}(s)$. In Figure 1, we plot the resulting cross section, in

comparison with the data and the lowest order result. It can be seen that the Omnès functions produce a substantial modification even near threshold. Of these, the most important is $D_0^{-1}(s)$ which reflects the strong attractive $\pi\pi$ scattering in the $I = 0, J = 0$ channel [6].

A much more satisfactory fit is found by use of an Omnès function $D_0^{-1}(s)$ determined via the use of *experimental* values of the pion-pion phase shifts [13] as well as including contributions to the left hand cut due to $A1, \rho, \omega$ exchange diagrams, which leads to the very good fit given in Figure 1. Details of this calculation can be found in ref. [9].

The lesson to be learned from this example is the importance of inclusion of $I=0$ S-wave $\pi - \pi$ rescattering corrections especially in processes which have no counterterm contributions and are generated from a simple one-loop chiral calculation. We shall see in the next sections two additional examples of this type, when we extend our formalism to cover the nonleptonic weak-radiative decays $K_S \rightarrow \gamma\gamma$ and $K_L \rightarrow \pi^0\gamma\gamma$.

2 $K_S \rightarrow \gamma\gamma$

A good deal of work has been done extending the chiral formalism into the regime of nonleptonic weak processes. To two-derivative order the form of the effective SU(3) octet chiral Lagrangian is unique

$$\mathcal{L}_{\Delta S=1} = F_\pi^4 G_8 \text{Tr} \left(\lambda_6 D_\mu U D^\mu U^\dagger \right) \quad (20)$$

where G_8 is a constant whose value can be determined empirically. The corresponding four-derivative weak effective Lagrangian has also been written down and involves thirty-seven additional terms [14], whose explicit form will not be needed here, since as in the case of $\gamma\gamma \rightarrow \pi^0\pi^0$ there is no tree level contributions to the weak-radiative decay process $K_S \rightarrow \gamma\gamma$. Nevertheless there does exist a finite one-loop piece. Defining [15]

$$\text{Amp}(K_S \rightarrow \gamma\gamma) = \epsilon_1^\mu \epsilon_2^\nu \left(\frac{-q_1 \cdot q_2 g_{\mu\nu} + q_{1\nu} q_{2\mu}}{q_1 \cdot q_2} \right) \alpha F_\pi B(m_K^2) \quad (21)$$

the one-loop chiral prediction is found to be

$$B(s) = G_8^{CA} \left(\frac{1}{\pi} (m_\pi^2 - s) F(s, m_\pi^2) - [m_\pi^2 \rightarrow m_K^2] \right). \quad (22)$$

With the value

$$G_8^{CA} \approx 9.1 \times 10^{-6} GeV^{-2} \quad (23)$$

determined from the tree level prediction for $K_S \rightarrow \pi\pi$

$$\text{Amp}^8(K_S \rightarrow \pi^+\pi^-) \equiv 2F_\pi A_0^{CA}(m_K^2) = 2F_\pi G_8^{CA}(m_K^2 - m_\pi^2) \quad (24)$$

we find the one-loop chiral prediction to be in good agreement with the recently determined experimental number

$$\begin{aligned} \text{B.R.}(K_S \rightarrow \gamma\gamma)_{\text{ChPT}} &= 2.0 \times 10^{-6} && \text{vs.} \\ \text{B.R.}(K_S \rightarrow \gamma\gamma)_{\text{exp}} &= (2.4 \pm 1.2) \times 10^{-6}. \end{aligned} \quad (25)$$

An alternative derivation of this result is provided by the use of a dispersion relation. We note that the function $B(s)$ has a cut along the line element $4m_\pi^2 < s < \infty$ and by unitarity has the imaginary part [16]

$$\text{Im}B(s) = \theta(s - 4m_\pi^2)\beta(s)A_0(s)f_0^*(s), \quad (26)$$

where $A_0(s)$ is the amplitude for the weak decay $K_S \rightarrow \pi\pi(I=0)$ and $f_0(s)$ is the amplitude for S-wave radiative pion annihilation $\pi\pi(I=0) \rightarrow \gamma\gamma$ discussed in the previous section. Note that since we are using an SU(3) octet assumption for the weak transition, both di-pions are required to be in an isoscalar configuration. To lowest order in chiral perturbation theory we have

$$\begin{aligned} A(s) &\approx A_0^{CA}(s) = G_8^{CA}(s - m_\pi^2) \\ f_0(s) &\approx f_{\text{Born}}(s) = \frac{1 - \beta^2(s)}{2\beta(s)} \ln \left(\frac{1 + \beta(s)}{1 - \beta(s)} \right). \end{aligned} \quad (27)$$

We now write a doubly subtracted dispersion relation for the function $B(s)$ using as subtraction constants the requirements that the amplitude vanish at both $s = 0$ and $s = m_\pi^2$, as given in the lowest order chiral analysis, yielding

$$B(s) = \frac{s(s - m_\pi^2)}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im}B(s')ds'}{s'(s' - m_\pi^2)(s' - s - i\epsilon)}. \quad (28)$$

If the lowest order chiral values—Eq. 27—are used to determine $\text{Im} B(s')$ then the integration can be performed analytically to yield

$$B(s) = \frac{1}{\pi}(m_\pi^2 - s)F(s, m_\pi^2)G_8^{CA}, \quad (29)$$

which is precisely the (pion contribution to the) one-loop chiral result Eq. 22.

At one level then the dispersive technique represents merely an alternative (and simpler!) way by which to perform the one-loop calculation. However, at a deeper level the dispersion relation provides a means to undertake a much more complete calculation of the radiative decay process by using for $\text{Im } B(s')$ not just the lowest order chiral forms for these amplitudes but instead values which have more experimental validity. Of course, the subtraction constants must be fixed by some other means — we assume that chiral perturbation theory to one loop is accurate enough to describe the amplitude at very low energies.

The $K_S \rightarrow \pi\pi$ decay amplitude $A_0(s)$ itself is an analytic function with a discontinuity along a cut $4m_\pi^2 < s < \infty$ given in terms of the S-wave I=0 $\pi - \pi$ scattering phase shift

$$\text{Im}A_0(s) = \theta(s - 4m_\pi^2)e^{-i\delta_0^0(s)} \sin \delta_0^0(s)A_0(s). \quad (30)$$

The general solution of such an equation is given

$$A_0(s) = P(s)D_0^{-1}(s), \quad (31)$$

where $P(s)$ is an arbitrary polynomial and $D_0^{-1}(s)$ is the Omnès function discussed in section 1. In order to determine the polynomial $P(s)$ we demand that the full $K_S \rightarrow \pi\pi$ amplitude given in Eq. 31 match the simple chiral form Eq. 27 in the absence of rescattering (*i.e.* when $D_0^{-1}(s) = 1$). We have then

$$A_0(s) = G_8(s - m_\pi^2)D_0^{-1}(s) \quad (32)$$

while the corresponding solution for $f_0(s)$ was derived in ref. [9] and has been outlined in section 1 of this paper.¹ Substitution into Eq. 26 and numerical evaluation of the dispersive integral then provides a prediction for the $K_S \rightarrow \gamma\gamma$ amplitude which is much more complete and founded in experiment than is its simple one-loop chiral analog. The result of the numerical integration is shown in Table 1, where values are given for the radiative decay amplitude $B(m_K^2)$ /branching ratio for four scenarios—a) simple one-loop

¹It is important to note here that when the solution Eq. 32 for $A_0(s)$ is used, the value $G_8 = G_8^{CA}/|D_0^{-1}(s = m_K^2)| \approx 6.1 \times 10^{-6} \text{GeV}^{-2}$ must be employed in order that the proper normalization to the $K_S \rightarrow \pi\pi$ decay rate be preserved.

<i>Input</i>	Im B [10^{-6}]	Re B [10^{-6}]	BR [10^{-6}]
ChPT	0.77	-0.44	2.0
Padé	0.47	-0.83	2.3
Gasser	0.49	-0.73	2.0
Gasser+V,A	0.46	-0.84	2.3

Table 1: Shown are the amplitudes and predicted branching ratios for the process $K_S \rightarrow \gamma\gamma$ using various theoretical inputs to the dispersion relation Eq. 28 as described in the text.

chiral perturbation theory; b) use of the full unitarized values and with the Padé form for the Omnès function; c) use of the full unitarized values and with a numerical representation of the Omnès function based on experiment and given by Gasser [13, 7]; d) same as c) but with vector and axial-vector exchange contributions included also in $f_0(s)$, which provides the best fit to the $\gamma\gamma \rightarrow \pi\pi$ system.

Examination of this table shows that the full unitarization procedure makes substantial changes in the predicted form of the decay amplitude with the importance of the real/imaginary pieces being interchanged for the ChPT and full unitarized calculations respectively. This is clearly seen in Figures 2 where we show the very different shapes for Im,Re $B(s)$ for these cases. Despite these differences the predicted branching ratio is remarkably robust, changing only slightly among the very different scenarios. We conclude then that the agreement between the experimental $K_S \rightarrow \gamma\gamma$ rate and that predicted via one-loop ChPT should *not* be considered as a success for the latter, since we have produced nearly identical predictions for the branching ratio from a very different set of assumptions concerning the input parameters. Truong has reached a similar conclusion using a different parametrization of the Omnès function and an approximated $\gamma\gamma \rightarrow \pi\pi$ amplitude [17].

Therefore, one must conclude that in order to distinguish between the various decay mechanisms one must examine a process which offers the chance for a rather richer experimental confrontation—the related nonleptonic radiative decay $K_L \rightarrow \pi^0\gamma\gamma$.

3 $K_L \rightarrow \pi^0 \gamma \gamma$

There has been considerable recent interest in the nonleptonic-radiative mode $K_L \rightarrow \pi^0 \gamma \gamma$. This began when one-loop chiral perturbation theory was used to generate a supposedly reliable prediction [18]

$$\text{Amp}(K_L \rightarrow \pi^0 \gamma \gamma) = \epsilon_1^\mu \epsilon_2^\nu \left(\frac{-g_{\mu\nu} k_1 \cdot k_2 + k_2^\mu k_1^\nu}{k_1 \cdot k_2} \right) \alpha C(s) \quad (33)$$

where

$$C(s) = \frac{1}{2\pi} G_8^{CA} \left[F(s, m_\pi^2)(m_\pi^2 - s) - F(s, m_K^2)(m_K^2 + m_\pi^2 - s) \right]. \quad (34)$$

Since a three-body final state is involved, what emerges is a prediction for both the overall branching ratio

$$B.R.(K_L \rightarrow \pi^0 \gamma \gamma) = 0.68 \times 10^{-6} \quad (35)$$

in addition to the shape of the decay spectrum, as shown in Figure 3. This distinctive shape is in marked contrast to that arising from a simple η, η' pole model, which gives support at lower values of $s_{\gamma\gamma}$ [19]. When experimental numbers were provided, the shape was found to be in good agreement with the ChPT prediction. However, the measured rate was nearly a factor of three larger than given in Eq. 35

$$B.R.(K_L \rightarrow \pi^0 \gamma \gamma) = \begin{cases} (1.7 \pm 0.3) \times 10^{-6} & \text{NA31[20]} \\ (1.86 \pm 0.6) \times 10^{-6} & \text{FNAL[21].} \end{cases} \quad (36)$$

Since this finding a number of authors have examined this problem. The inclusion of the $\Delta I = 3/2$ weak interaction results in a minor effect, as expected [22]. A dispersive analysis including unitarity corrections of $\mathcal{O}(p^6)$ due to the $\pi^+ \pi^-$ intermediate state has also been presented recently [23]. In this approach, the experimental results for the branching ratio as well as the spectrum in the invariant mass of the two photons can be reproduced if a somewhat sizeable contribution of vector meson exchange to the counterterm lagrangian of $\mathcal{O}(p^6)$ is assumed. A similar result has been obtained in ref. [24], however without taking into account unitarity corrections of $\mathcal{O}(p^6)$.

Here we wish to examine the contribution of higher order diagrams to the decay process, by generalizing the dispersive approach which was applied in

the previous section to $K_S \rightarrow \gamma\gamma$. By definition only the amplitude proportional to the Lorentz structure in Eq. 33 enters the calculation. However, for this amplitude, unitarity corrections due to $\pi\pi$ intermediate states will be treated to all orders in the chiral expansion. We begin by rederiving the one-loop ChPT result in this fashion. The amplitude $C(s)$ possesses a cut along the real axis from $4m_\pi^2 < s < \infty$ with a discontinuity determined via unitarity, which takes the form [25]

$$\text{Im}C(s) = \theta(s - 4m_\pi^2) \frac{1}{2} \beta(s) A_{+-0}^{CA}(s) f^{C*}(s) \quad (37)$$

where the lowest order ChPT amplitudes for $\gamma\gamma \rightarrow \pi^+\pi^-$ and $K_L \rightarrow \pi^+\pi^-\pi^0$ are given respectively by

$$\begin{aligned} f^C(s) &= f^{\text{Born}}(s), \\ \text{Amp}(K_L \rightarrow \pi^+\pi^-\pi^0) &\equiv A_{+-0}^{CA}(s) = G_8^{CA}(s - m_\pi^2). \end{aligned} \quad (38)$$

As for $K_S \rightarrow \gamma\gamma$ we now write a twice subtracted dispersion relation for the function $C(s)$, the subtraction constants being specified by the one-loop ChPT requirement that $C(s)$ vanishes at both $s = 0$ and $s = m_\pi^2$:

$$C(s) = \frac{s(s - m_\pi^2)}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im}C(s') ds'}{s'(s' - m_\pi^2)(s' - s - i\epsilon)}. \quad (39)$$

Using the lowest order chiral expression to determine $\text{Im} C(s)$ the integration can again be performed analytically yielding

$$C(s) = \frac{1}{2\pi} G_8^{CA} F(s, m_\pi^2) (s - m_\pi^2) \quad (40)$$

which reproduces exactly the (pion loop contribution to the) one-loop ChPT result, *i.e.* the first term in Eq. 34.

To provide an improved estimate, we use the dispersion relation Eq. 39, but with a more complete representation of $\text{Im} C(s)$ than just the lowest order ChPT expression (the subtraction constants are still taken from one-loop ChPT). In addition to the $\pi^+\pi^-$ intermediate state considered so far we include the discontinuity from the $\pi^0\pi^0$ intermediate state in the s-channel

to the unitarity relation ²

$$\text{Im}C(s) = \theta(s - 4m_\pi^2) \frac{1}{2} \beta(s) \left[A_{+-0}^S(s) f^{C*}(s) + \frac{1}{2!} A_{000}^S(s) f^{N*}(s) \right], \quad (41)$$

where the superscript S indicates the S-wave component of these amplitudes. Other intermediate states open up at much higher thresholds and are suppressed in the twice subtracted dispersion relation.

To evaluate the improved imaginary part of the function $C(s)$ in Eq. 41 we shall employ dispersion relations to calculate the $\gamma\gamma \rightarrow \pi\pi$ scattering amplitudes f^C , f^N and the $K \rightarrow 3\pi$ decay amplitudes A_{+-0} , A_{000} . The approach to f^C , f^N has been reviewed in section 1. We shall use the explicit results given there. As for the $K \rightarrow 3\pi$ amplitude we shall use an approximate solution to twice subtracted Khuri-Treiman equations [26]. Define the $\Delta I = \frac{1}{2}$ $K \rightarrow 3\pi$ decay amplitude as

$$\text{Amp}(K_L^0 \rightarrow \pi^a \pi^b \pi^c) = \delta^{ab} \delta^{c3} F(s_a, s_b, s_c) + \text{permutations} \quad (42)$$

where $s_i = (k - q_i)^2$. Also note that $F(s_a, s_b, s_c)$ must be symmetric in its first two arguments according to Bose statistics. The decay modes relevant to our calculation are then

$$\begin{aligned} \text{Amp}(K_L^0 \rightarrow \pi^+ \pi^- \pi^0) &\equiv A_{+-0} = F(s_a, s_b, s_c), \\ \text{Amp}(K_L^0 \rightarrow 3\pi^0) &\equiv A_{000} = F(s_a, s_b, s_c) + \text{permutations}. \end{aligned} \quad (43)$$

The discontinuity of $F(s_a, s_b, s_c)$ is provided by unitarity. Thus for $K \rightarrow 3\pi$ the unitarity condition reads in general

$$\begin{aligned} \text{Im} \langle \pi^a \pi^b \pi^c | \mathcal{H}_w^{\frac{1}{2}} | K_L^0 \rangle &= \\ \frac{1}{3!} \sum_{def} \int \frac{d^3 p_d}{(2\pi)^3 2E_d} \cdots \frac{d^3 p_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta(k - p_d - p_e - p_f) & \\ \times \langle \pi^a \pi^b \pi^c | \pi^d \pi^e \pi^f \rangle^* \langle \pi^d \pi^e \pi^f | \mathcal{H}_w^{\frac{1}{2}} | K_L^0 \rangle &. \end{aligned} \quad (44)$$

However, this equation is not amenable to an exact solution, so various approximations are necessary. We begin by approximating the 3π scattering

²This intermediate state does not contribute to $\text{Im} C$ in the chiral one-loop analysis since f^N vanishes at order p^2 . However the two neutral pions also have a $I = 0$ component, therefore final state interactions in this channel are expected to give large corrections.

amplitude by a sum of two-pion, one spectator amplitudes—

$$\langle \pi^a \pi^b \pi^c | \pi^d \pi^e \pi^f \rangle \sim \delta^{ef} \langle \pi^a \pi^b | \pi^d \pi^e \rangle + \text{permutations}. \quad (45)$$

Also, since any two-pion reaction is at low energy we include only S-wave scattering terms

$$\begin{aligned} \langle \pi^a \pi^b | \pi^d \pi^e \rangle \approx & \delta^{ab} \delta^{de} \frac{1}{3} (A^{(0)}(s) - A^{(2)}(s)) \\ & + \frac{1}{2} (\delta^{ad} \delta^{be} + \delta^{ae} \delta^{bd}) A^{(2)}(s) \end{aligned} \quad (46)$$

where

$$A^{(I)}(s) = e^{i\delta_0^I(s)} \sin \delta_0^I(s). \quad (47)$$

The unitarity relation then reduces to the simpler form

$$\begin{aligned} \text{Im}F(s, t, u) = & \int \frac{d\Omega}{4\pi} [A^{(0)*}(s) F(s, t, u) \\ & + \frac{1}{3} (A^{(0)*}(s) - A^{(2)*}(s)) (F(u, s, t) + F(t, u, s)) \\ & + \frac{1}{2} A^{(2)*}(t) (F(s, t, u) + F(u, s, t)) \\ & + \frac{1}{2} A^{(2)*}(u) (F(s, t, u) + F(t, u, s))]. \end{aligned} \quad (48)$$

The Khuri-Treiman equations are obtained by the ansatz $F(s, t, u) = U(s) + V(t) + V(u)$. We use a twice subtracted form with linear subtraction polynomials

$$\begin{aligned} U_0(s) &= a_U + b_U s, \\ V_0(s) &\equiv 0. \end{aligned} \quad (49)$$

The resulting system of equations has a simple approximate solution provided that we ignore the generally small $I = 2$ scattering term with respect to its much larger $I = 0$ counterpart:

$$\begin{aligned} U(s) &= U_0(s) + \hat{U}(s_1) \left(\frac{D_0(s_1)}{D_0(s)} - 1 \right) \frac{s - s_2}{s_1 - s_2} \\ &\quad + \hat{U}(s_2) \left(\frac{D_0(s_2)}{D_0(s)} - 1 \right) \frac{s - s_1}{s_2 - s_1} + \Phi_2, \\ V(s) &= 0. \end{aligned} \quad (50)$$

$[10^{-8}]$	<i>exp.</i>	<i>Fit (Padé)</i>	<i>Fit (Gasser)</i>
α_1	91.7 ± 0.3	$89.6 + i19.6$	$89.9 + i18.2$
β_1	-25.7 ± 0.3	$-21.8 - i13.6$	$-22.6 - i12.2$

Table 2: Shown are values for the $\Delta I = \frac{1}{2}$ component of the $K_L \rightarrow 3\pi$ decay amplitude with final state interactions generated by the Padé and Gasser forms for D_0^{-1} respectively.

where s_1, s_2 are the subtraction points, Φ_2 is a correction which can be shown to be small, D_0^{-1} is the Omnès function and [27]

$$\hat{U}(s) = U_0(s) + \frac{2}{3} \int \frac{d\Omega}{4\pi} U_0(t(s, \cos \theta)). \quad (51)$$

We still need to specify the subtraction polynomial $U_0(s)$. Since we want to input accurately the information provided by unitarity, we would like to use as much as possible the experimental information on $K_L \rightarrow 3\pi$ decays. The $K \rightarrow 3\pi$ amplitude may be expanded around the center of the Dalitz plot as (neglecting $\Delta I = 3/2$ contributions)

$$A_{+-0} = \alpha_1 - \beta_1 Y + (\zeta_1 + \xi_1) Y^2 + \frac{1}{3} (\zeta_1 - \xi_1) X^2 \quad (52)$$

with

$$Y = \frac{s_3 - s_0}{m_\pi^2}, \quad X = \frac{s_2 - s_1}{m_\pi^2}. \quad (53)$$

Assuming the coefficients α_1, β_1, \dots are real, experiment fixes α_1, β_1 at the values given in column one of Table 2. As is well known, using the current algebra expression $U_0(s) = G_8^{CA}(s - m_\pi^2)$ as a subtraction polynomial gives values for α_1 and β_1 which are too small. On the other hand, ChPT to one-loop can fit the $K \rightarrow 3\pi$ data by adjusting the counterterm coupling constants of order p^4 . In the language of dispersion relations, this means that beyond leading order ChPT not only final state interactions contribute to coefficients α_1, β_1 but also the subtraction polynomial U_0 is subject to corrections. This might also include contributions from higher resonance exchange.

In the same spirit we adopt the following phenomenological procedure. The $K \rightarrow 3\pi$ decay amplitude is calculated according to the approximate solution of Khuri-Treiman equations, Eq. 50. The subtraction constants a_U , b_U of U_0 are treated as free adjusted such that the experimentally found α_1 , β_1 are reproduced. The values needed for the case of the Padé form of D_0^{-1} are

$$a_U^{\text{fit}} = -2.3 \cdot 10^{-7}, \quad b_U^{\text{fit}} = 8.6 \cdot 10^{-6} \text{GeV}^{-2} \quad (54)$$

to be compared with the current algebra expressions $a_U^{\text{CA}} = -G_8 m_\pi^2 = -1.8 \cdot 10^{-7}$, $b_U^{\text{CA}} = G_8 = 9.1 \cdot 10^{-6} \text{GeV}^{-2}$.³ This yields the results given in the second and third columns of Table 2 for two parameterizations of the Omnès function, the Padé form, Eq. 18, and the numerical parameterization given by Gasser respectively. Interestingly, β_1 develops a rather large imaginary part as can be seen in the second and third column of Table 2.

The advantage of this approach is twofold. First we use the experimentally available information on coefficients α_1 , β_1 – the shortcomings of a too low $K \rightarrow 3\pi$ amplitude from soft pion theorems are thus avoided. Secondly, it provides us with a representation of real and imaginary part of the decay amplitude outside the physical region. This is exactly what is needed in order to make $\text{Im } C(s)$, Eq. 41, approximately real. If instead we would use the experimental expansion of A_{+-0} , Eq. 52, as an extrapolation, $\text{Im } C(s)$ would develop an unacceptable large imaginary part at rather low energies, i.e. for $\sqrt{s} \geq 450 \text{MeV}$. Since our solution to the $K \rightarrow 3\pi$ amplitude is subject to several approximations, we cannot hope that the imaginary part of $\text{Im } C(s)$ cancels completely. However, it cancels to a large extent, i.e. it never reaches ten percent of the real part in the region from two pion threshold up to $\sqrt{s} \approx 600 \text{MeV}$.

We are now ready to calculate the function $C(s)$ using the improved representation of its imaginary part. We used two parameterizations of the Omnès function D_0^{-1} , the Padé solution Eq. 18 and the numerical representation given by Gasser. In Figure 4 a) the improved imaginary part $\text{Im } C(s)$ is compared to the lowest order approximation, and in Figure 4 b) the corresponding real parts calculated from the dispersion integral are shown. In the dispersive approach, both imaginary and real part of $C(s)$

³In a similar approach, ref. [28], Truong obtains a reasonable representation of the $K \rightarrow 3\pi$ decay amplitude using current algebra constraints for subtraction constants and appending a ‘ ρ -pole’ contribution.

<i>input</i>	$BR[10^{-6}]$	$BR^{\text{abs}}[10^{-6}]$
exp	1.7 ± 0.3	—
ChPT (pion-loop)	0.59	0.39
Padé-charged	0.93	0.68
Padé-charged+neutral	1.12	0.84
Gasser-charged	0.93	0.68
Gasser-charged+neutral	1.08	0.81

Table 3: Shown are the calculated branching ratios for $K_L \rightarrow \pi^0 \gamma \gamma$ under various input assumptions. The second column indicates the contribution to the branching ratio from the absorptive part of the amplitude.

are rather enhanced, already just above the two pion threshold. Calculating $BR(K_L \rightarrow \pi^0 \gamma \gamma)$ with these inputs we find a net enhancement over the one-loop ChPT result by a factor ≈ 1.9 . The results are summarized in Table 3 where the branching ratio is given for several scenarios.

The observed enhancement is clearly seen to come from two effects:⁴

- i) The use of a corrected $K \rightarrow 3\pi$ amplitude in agreement with data gives an enhancement factor of ≈ 1.6 in the branching ratio. This has been noted earlier [25, 29]; in a very simple approach one could just scale G_8 in Eq. 38 to reproduce the experimental $K \rightarrow 3\pi$ amplitude, leading to a similar enhancement factor for $BR(K_L \rightarrow \pi^0 \gamma \gamma)$. However, a consistent calculation clearly should explain all relevant processes, $K \rightarrow 2\pi$, $K \rightarrow 3\pi$, $\gamma\gamma \rightarrow \pi\pi$, $K_S \rightarrow \gamma\gamma$ and $K_L \rightarrow \pi^0 \gamma \gamma$ by the same method. We have explained in detail above how this can be achieved.
- ii) The inclusion of the neutral two pion intermediate state in the unitarity relation brings an additional enhancement factor of ≈ 1.2 . As mentioned before, this contribution is missed in all approaches where the $\pi\pi \rightarrow \gamma\gamma$ vertex is treated only at the Born level. Although this

⁴It should be noted that our result with inclusion of charged intermediate state only is consistent with that previously calculated by Ko and Rosner [25] using a simple one-loop approximation. We do not understand the discrepancy with the dispersive calculation of Truong [17] who also includes only the charged intermediate state and finds a branching ratio of 1.3×10^{-6} .

effect is moderate, it goes in the right direction. Moreover, it is critical to include the $\pi^0\pi^0$ intermediate state in order to properly implement the constraints of unitarity.

Finally we note that using a twice subtracted dispersion relation, the contributions from the high energy region to the dispersive integral are very much suppressed. Cutting the integral at $\sqrt{s} = 600\text{MeV}$ instead of 1 GeV changes the branching ratio by less than one percent. Also, the calculated spectrum in the invariant mass of the two photons is plotted in Figure 3. It deviates insignificantly from the spectrum obtained in one loop Chiral perturbation theory. If only the charged intermediate state is included, the maximum of $\frac{1}{\Gamma}\frac{d\Gamma}{dz}$ is shifted toward higher q^2 values, contrary to the experimental trend. Inclusion of the neutral intermediate state in the unitarity relation restores the maximum to its original location.

4 Conclusions

The process $K_L \rightarrow \pi^0\gamma\gamma$ has traditionally been a difficult one to understand within the context of chiral perturbation theory. Indeed, from other successes one might have expected the one-loop chiral prediction to be accurate to $\mathcal{O}(20\%)$ or so, whereas in fact the predicted branching ratio is nearly a factor of three too small. Previous work in this area has attempted to explain this discrepancy in terms of vector meson pole contributions and/or in terms of higher loop final state interaction effects. We have here noted, however, that previous dispersion-based final state interaction calculations have included only the intermediate $\pi^+\pi^-$ intermediate state, omitting its potentially important $\pi^0\pi^0$ analog. Above we have given a mutually consistent analysis of both $K_L \rightarrow 3\pi$ and $K_L \rightarrow \pi^0\gamma\gamma$ processes using dispersion relations and have shown that by including this previously neglected $\pi^0\pi^0$ intermediate state piece the branching ratio for $K_L \rightarrow \pi^0\gamma\gamma$ is significantly enhanced, although it remains too low to fully explain the data. This should by no means be considered to be complete analysis—indeed many effects such as I=1,2 $\pi\pi$ scattering effects as well as higher order contributions to the $K_L \rightarrow 3\pi$ process have been neglected. Nevertheless, we believe that this calculation opens up interesting questions for future study about the importance of effects beyond the one loop approximation in chiral perturbation theory.

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Figure captions

- Fig. 1: Indicated are experimental data points for $\gamma\gamma \rightarrow \pi\pi$ compared to the one-loop chiral perturbative prediction (dashed) and dispersive calculations using Padé (dotted) and Gasser (solid) $D_0^{-1}(s)$ functions.
- Fig. 2: Shown are the results of one-loop chiral perturbation theory (dashed) and dispersive analysis scenario d) (solid) for $\text{Im } B(s)$ (a) and $\text{Re } B(s)$ (b) respectively.
- Fig. 3: Normalized spectra $1/\Gamma d\Gamma/dz$ in the invariant mass of the two photons ($z = q^2/M_{K^0}^2$). Plotted are chiral perturbation theory (dashed) and dispersive analysis using Gasser $D_0^{-1}(s)$ function (solid).
- Fig. 4: $\text{Im } C(s)$ (a) and $\text{Re } C(s)$ (b) in the physical decay region. Shown are the results of one-loop chiral perturbation theory (dashed) and dispersive analysis using Gasser $D_0^{-1}(s)$ function (solid). Also shown is the result of the dispersive analysis including the charged intermediate state only (dot-dashed).

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