1993

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JF Donoghue
University of Massachusetts - Amherst, donoghue@physics.umass.edu

Eugene Golowich
University of Massachusetts - Amherst, golowich@physics.umass.edu

UMHEP-390

Anatomy of a Weak Matrix Element

John F. Donoghue and Eugene Golowich

Department of Physics and Astronomy
University of Massachusetts
Amherst MA 01003 USA

Abstract

Although the weak nonleptonic amplitudes of the Standard Model are notoriously difficult to calculate, we have produced a modified weak matrix element which can be analyzed using reliable methods. This hypothetical nonleptonic matrix element is expressible in terms of the isovector vector and axialvector spectral functions $\rho_{\rm V}(s)$ and $\rho_{\rm A}(s)$, which can be determined in terms of data from tau lepton decay and e^+e^- annihilation. Chiral symmetry and the operator product expansion are used to constrain the spectral functions respectively in the low energy and the high energy limits. The magnitude of the matrix element thus determined is compared with its 'vacuum saturation' estimate, and in the future may be accessible with lattice calculations.

1 Introduction

To determine the origin of the $\Delta I = 1/2$ rule remains one of the unsolved problems of the Standard Model.[1] Despite years of effort involving theoretical techniques ranging from quark model calculations to lattice-gauge studies, a quantitative understanding of the $\Delta I = 1/2$ enhancement is still lacking. The reason for this can be seen from the structure of the $\Delta S = 1$ nonleptonic weak hamiltonian,

$$\mathcal{H}_{\rm wk}^{\Delta S=1} = \frac{g_2^2}{8} V_{\rm ed}^* V_{\rm us} \int d^4x \ iD_{\mu\nu}(x, M_{\rm W}) \ T(J_{1+i2}^{\mu\dagger}(x) J_{4+15}^{\nu}(0)) \quad , \tag{1}$$

where $iD^{\mu\nu}(x, M_{\rm W})$ is the W–boson propagator, g_2 is the gauge coupling strength for weak isospin and J_k^μ are the left–handed quark weak currents,

$$J_k^\mu = \bar{q} \frac{\lambda_k}{2} \gamma^\mu (1 + \gamma_5) q \quad , \tag{2}$$

where $q = (u \ d \ s)$. Strong interaction effects are present in Eq. (1) at all energy scales up to the W-boson mass. It is this fact which signals the difficulty of dynamically reproducing the $\Delta I = 1/2$ rule.

However, experience from both experiment and theory has taught us much about low-energy QCD. It is possible that a hybrid approach which combines data with theory might provide insight as to the source of the $\Delta I = 1/2$ effect. The calculation to follow will constitute a first step in that direction. It involves just the vector-vector part of the operator in Eq. (1) (note that we drop the KM factors)

$$\mathcal{H}_{\rm vv}^{\Delta S=1} = \frac{g_2^2}{8} \int d^4x \ iD_{\mu\nu}(x, M_{\rm W}) \ T\left(V_{1-i2}^{\mu}(x) V_{4+i5}^{\nu}(0) \right) \quad . \tag{3}$$

and its matrix elements taken between single-particle pseudoscalar meson states,

$$\begin{aligned} \mathcal{M}_{K^- \to \pi^-} &= \langle \pi^- | \mathcal{H}_{\rm vv}^{\Delta S=1} | K^- \rangle \quad , \\ \mathcal{M}_{\bar{K}^0 \to \pi^0} &= \langle \pi^0 | \mathcal{H}_{\rm vv}^{\Delta S=1} | \bar{K}^0 \rangle \quad . \end{aligned} \tag{4}$$

Our goal is to determine the matrix elements $\mathcal{M}_{\bar{K}\pi}$ as accurately as possible, despite the complications of the strong interactions,

The main theoretical tool will be chiral symmetry. It is easy to show that the single-meson matrix elements of the chiral operator of Eq. (1) vanish in the soft meson limit. By contrast, the $\mathcal{M}_{K\pi}$ of Eq. (4) have a nontrivial form in the chiral limit because they involve a *vector-vector* operator.[2] By working in the soft meson limit, we can utilize a spectral representation to study the matrix elements. Constraints on the low-energy and high-energy limits of the spectral integrals are obtainable from chiral symmetry and QCD sum rule methods, and data can be used to fill in much of the rest. It is in this sense that the 'anatomy' of these matrix elements are revealed.

Although the $\mathcal{M}_{K\pi}$ are thus a kind of theoretical laboratory from which we can learn something of value, they themselves do not reproduce the $\Delta I = 1/2$ signal. This can best be seen in terms of an effective lagrangian description. Under the usual classification scheme of $SU(3)_L \times SU(3)_R$, the operator of Eq. (3) transforms as a component of an $(\mathbf{8}, \mathbf{8})$ family of (parity–conserving) operators.[3] An $SU(3)$ invariant lagrangian which can characterize the momentum independent parts of the transition amplitudes is

$$\mathcal{L}_{ab}^{(8,8)} = g\mathrm{Tr}\ \left(\lambda_a \Sigma \lambda_b \Sigma^\dagger + \lambda_b \Sigma \lambda_a \Sigma^\dagger\right)\ , \qquad (a, b = 1, \ldots 8)\ , \tag{5}$$

where g is a constant, Σ represents the meson fields,

$$\Sigma \equiv \exp\left(i\phi_c \lambda_c / F\right)\ , \qquad (c = 1, \ldots 8)\ , \tag{6}$$

and F is the meson decay constant. For the nonleptonic weak $(\mathbf{8}, \mathbf{8})$ operator, the ratio of $\Delta I = 1/2$ and $\Delta I = 3/2$ amplitudes is fixed. Thus, given the isospin decomposition

$$\begin{aligned} \mathcal{M}_{K^- \to \pi^-} &= \mathcal{M}_{1/2} + \mathcal{M}_{3/2}\ , \\ \mathcal{M}_{\bar{K}^0 \to \pi^0} &= -\frac{1}{\sqrt{2}}\mathcal{M}_{1/2} + \sqrt{2}\mathcal{M}_{3/2}\ , \end{aligned} \tag{7}$$

it is straightforward to show

$$\frac{\mathcal{M}_{3/2}}{\mathcal{M}_{1/2}} = -\frac{2}{5}\ . \tag{8}$$

2 Derivation of a New Chiral Sum Rule

In the following, we shall express the $\bar{K} \to \pi$ transition amplitudes in the form of a spectral representation. *Throughout, all our calculations will be performed in the chiral limit of massless u, d, s quarks.* The first step in the process is to invoke the current algebra results[4]

$$\lim_{p\to 0}\langle\pi^-_{\mathbf{p}}|T\left(V^\mu_{1-i2}(x)V^\nu_{4+i5}(0)\right)|K^-_{\mathbf{p}}\rangle = \\ -\frac{1}{F^2}\langle 0|T\left(V^\mu_3(x)V^\nu_3(0)-A^\mu_3(x)A^\nu_3(0)\right)|0\rangle\ , \tag{9}$$

and

$$\lim_{p\to 0}\langle\pi^0_{\mathbf{p}}|T\left(V^\mu_{1-i2}(x)V^\nu_{4+i5}(0)\right)|\bar{K}^0_{\mathbf{p}}\rangle = \\ \frac{3}{\sqrt{2}F^2}\langle 0|T\left(V^\mu_3(x)V^\nu_3(0)-A^\mu_3(x)A^\nu_3(0)\right)|0\rangle\ , \tag{10}$$

where we take the vacuum to be an $SU(3)$ singlet. Note the dependence upon the difference of vector and axialvector terms. These quantities can be expressed in terms of spin-one spectral functions $\rho_{\rm V,A}(s)$ as defined by[5]

$$\langle 0|T\left(V^\mu_a(x)V^\nu_b(0)\right)|0\rangle = \\ i\delta_{ab}\int_0^\infty ds\ \rho_{\rm V}(s)\ (-sg^{\mu\nu}-\partial^\mu\partial^\nu)\int\frac{d^4p}{(2\pi)^4}\ \frac{e^{-ip\cdot x}}{p^2-s+i\epsilon} \tag{11}$$

and

$$\langle 0|T\left(A^\mu_a(x)A^\nu_b(0)\right)|0\rangle = -i\delta_{ab}F_\pi^2\partial^\mu\partial^\nu\int\frac{d^4p}{(2\pi)^4}\ \frac{e^{-ip\cdot x}}{p^2+i\epsilon} \\ +i\delta_{ab}\int_0^\infty ds\ \rho_{\rm A}(s)(-sg^{\mu\nu}-\partial^\mu\partial^\nu)\int\frac{d^4p}{(2\pi)^4}\ \frac{e^{-ip\cdot x}}{p^2-s+i\epsilon}\ . \tag{12}$$

Note that in the chiral limit, the spin 0 contribution is given entirely by the pion pole. It is thus possible to write the $\mathcal{M}_{\bar{K}\pi}$ in spectral form,

$$\mathcal{M}_{K^-\to\pi^-} = \frac{3i}{32\sqrt{2}\pi^2}\frac{G_\mu}{F^2}\mathcal{A} \qquad \text{and} \qquad \mathcal{M}_{\bar{K}^0\to\pi^0} = -\frac{9i}{64\pi^2}\frac{G_\mu}{F^2}\mathcal{A}\ , \tag{13}$$

where

$$\mathcal{A} = M_{\rm W}^2\int_0^\infty ds\ s^2\ln\left(\frac{s}{M_{\rm W}^2}\right)\frac{\rho_{\rm V}(s)-\rho_{\rm A}(s)}{s-M_{\rm W}^2+i\epsilon}\ . \tag{14}$$

The combination of vector and axialvector spectral functions in Eq. (14) is reminiscent of similar forms appearing in other well-known sum rules *viz.*,

$$\int_0^\infty ds\ \frac{\rho_{\rm V}(s)-\rho_{\rm A}(s)}{s} = -4\bar{L}_{10}\ , \tag{15}$$

$$\int_0^\infty ds\ (\rho_{\rm V}(s) - \rho_{\rm A}(s)) \quad = \quad F_\pi^2 \quad , \tag{16}$$

$$\int_0^\infty ds\ s\ (\rho_{\rm V}(s) - \rho_{\rm A}(s)) \quad = \quad 0 \quad , \tag{17}$$

$$\int_0^\infty ds\ s \ln\left(\frac{s}{\Lambda^2}\right)\ (\rho_{\rm V}(s) - \rho_{\rm A}(s)) \quad = \quad -\frac{16\pi^2 F_\pi^2}{3e^2}(m_{\pi^\pm}^2 - m_{\pi^0}^2) \quad . \tag{18}$$

In the first sum rule, $\bar{L}_{10}$ is related to the renormalized coefficient $L_{10}^{(r)}(\mu)$ of an $\mathcal{O}(E^4)$ operator in the effective chiral lagrangian of QCD[6],

$$\bar{L}_{10} = L_{10}^{(r)}(\mu) + \frac{144}{\pi^2}\left[\ln\left(\frac{m_\pi^2}{\mu^2}\right) + 1\right] \simeq -6.84 \times 10^{-3} \quad . \tag{19}$$

The next two relations are respectively the first and second Weinberg sum rules.[7] Finally, Eq. (18) is the formula for the $\pi^\pm$-π^0 mass splitting which was derived long ago using soft-pion methods.[8] Although containing an arbitrary energy scale Λ, this last expression is actually independent of Λ by virtue of the second Weinberg sum rule. As one would expect, Eq. (14) reduces to the structure of Eq. (18) upon dropping the prefactor of $M_{\rm W}^2$ and taking the limit $M_{\rm W}^2 \to 0$. Although it is the expression in Eq. (14) which is central to our analysis, we shall see that these other sum rules serve as crucial checks on the reliability of our determination.

3 The Spectral Functions

As mentioned earlier, the difficulty in calculating the weak nonleptonic matrix element lies in the need to understand all the physics between $s = 0$ and $s = M_{\rm W}^2$. Fortunately, with the particular combination found in Eq. (14) (*i.e.* $\rho_{\rm V} - \rho_{\rm A}$), we are able to overcome this hurdle. We have recently provided a detailed phenomenological overview[9] of $\rho_{\rm V}$-$\rho_{\rm A}$ and of the chiral sum rules in Eqs. (15-18). Briefly, the important ingredients are as follows.

At very low energy, the difference of spectral functions is uniquely determined by chiral symmetry to be[10]

$$\rho_{\rm V}(s) - \rho_{\rm A}(s) \sim \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \theta(s - 4m_\pi^2) + \mathcal{O}(p^2) \quad . \tag{20}$$

At somewhat higher energies, the vector spectral function $\rho_{\rm V}$ may be determined from both e^+e^- annihilations and decay of the tau lepton.[11] As a consistency check, the two data sets have been shown to give compatible information in the $\rho(770)$ resonance region.[12] The data reveals that as energy is increased above threshold, $\rho_{\rm V}$ is dominated first by 2π and then 4π resonances. At even higher energy, multipion production leads to a continuum component which ultimately approaches the asymptotic QCD prediction,

$$\rho_{\rm V}(s) \sim \frac{1}{8\pi^2}\left(1+\frac{\alpha_s(s)}{\pi}\right) \ . \tag{21}$$

For the axialvector spectral function $\rho_{\rm A}$, things proceed similarly, except that one must rely solely upon tau decay data. Also, above threshold it is the 3π sector which first occurs and the dominant resonance contribution is that of $a_1(1260)$. In the large energy limit, QCD predicts that $\rho_{\rm A} = \rho_{\rm V}$ to all orders in perturbation theory.[13] The *difference* between $\rho_{\rm V}$ and $\rho_{\rm A}$ arises from non-perturbative effects and may be estimated by using the operator product expansion.[14] The asymptotic energy dependence behaves as s^{-3}, and in the approximation of vacuum saturation we find

$$\rho_{\rm V}(s) - \rho_{\rm A}(s) \sim \frac{8}{9}\frac{\alpha_s\langle\sqrt{\alpha_s}\bar{q}q\rangle_0^2}{s^3} \simeq \frac{3.4\times 10^{-5}\ {\rm GeV}^6}{s^3} \ , \tag{22}$$

where we take $\alpha_s(5\ {\rm GeV}^2) \simeq 0.2$. In view of the tiny numerator, this asymptotic tail provides only a very small contribution to the sum rule integrals and thus its precise value is not very important.

A phenomenological analysis of the chiral sum rules has been performed in Ref. [9]. Both numerical and analytical approaches have been employed to convert the empirical knowledge of $\rho_{\rm V} - \rho_{\rm A}$ into statements about the sum rules. Figure 1 displays a typical solution found there for $\rho_{\rm V}$ - $\rho_{\rm A}$. It represents a fit of the $\tau \to 2\pi$, 3π, 4π decay spectra and branching ratios and $e^+e^- \to 2\pi$, 4π cross sections. Other solutions are found to have a highly similar appearance, and we refer the reader to Ref. [9] for details. As a whole, the solution set has the correct high energy and low energy behavior, as well as reproducing the correct values of the four chiral sum rules listed in Eqs. (17)-(20). Of course, the phenomenology is based on existing data with attendant uncertainties, especially with respect to the precise values of the 3π and 4π branching ratios in tau decay. The form of the curve in Fig. 1 is expected to be subject to minor refinements as future experimental results become available. However, such a determination of the spectral functions allows us to calculate the weak matrix element $\mathcal{A}$, and we ascertain its value to be

$$\mathcal{A} \ = \ -0.062 \pm 0.017\ {\rm GeV}^6 \ . \tag{23}$$

The error bars we assign about the central value is our estimate of the uncertainties associated with the current data base. We find that the most important contributions to the dispersive integral in Eq. (14) arise from the 'intermediate' energy region, $1 < s(\mathrm{GeV}^2) < 10$. This is seen to be a result of the factor of s^2 in the integrand which suppresses the effects at low s and of the vanishing of $\rho_{\rm V}$ - $\rho_{\rm A}$ as $s \to \infty$ which controls the high energy region.

We conclude this section with a model-dependent estimate of the amplitude $\mathcal{A}$. Since rigorous calculation of nonleptonic matrix elements has traditionally not been available, various approximation schemes have instead been employed, among them quark models and the vacuum saturation approach. For definiteness, we shall adopt the latter approach here. The short distance expansion

$$T\left(V^\mu_{1-i2}(x)V^\nu_{4+i5}(0)\right) = V^\mu_{1-i2}(0)V^\nu_{4+i5}(0) \ + \ \ \mathcal{O}(x) \ \ , \tag{24}$$

along with the evaluation

$$\int d^4x \ iD_{\mu\nu}(x, M_{\rm W}) = \frac{ig_{\mu\nu}}{M^2_{\rm W}} \ \ , \tag{25}$$

allows us to write

$$\mathcal{M}_{\bar{K}\pi} \simeq -\frac{iG_\mu}{\sqrt{2}}\langle\pi|\bar{d}_i\gamma^\mu u_i\bar{u}_j\gamma_\mu s_j|\bar{K}\rangle \ \ . \tag{26}$$

In the vacuum saturation approximation, Eq. (26) is determined by performing a Fierz transformation and inserting the vacuum intermediate states. Remembering that we are working in the chiral limit, we find

$$\mathcal{A}_{\rm vac} = -\frac{32\pi^2}{9}\langle\bar{u}u\rangle^2_0 \simeq -0.033 \ \mathrm{GeV}^6 \ \ . \tag{27}$$

This is about half the spectral evaluation given above.

4 Concluding Remarks

To our knowledge, this is the first time that anyone has undertaken evaluation of a weak nonleptonic matrix element using a battery of methods which, at least in principle, are both theoretically sound and practical to apply. The resulting analytic control allows us to be able to identify the most important of the underlying physics. We have found that the dominant contributions to the matrix element come from the region of intermediate energies, which is the one over which theorists have the least control. That is, at the lowest energies, rigorous predictions can be made using chiral symmetry, while at the highest energies methods of perturbative and nonperturbative QCD may be invoked. However, the intermediate energy region continues to resist the development of reliable analytic methods. Fortunately, the existence of a data base allows this range to be estimated. We have done the best that can be accomplished at this time. As the quality and quantity of data improves, we expect to be able to reduce accordingly the uncertainty in our determination.

It will be interesting to compare our evaluation of $\mathcal{M}_{\bar{K}\to\pi}$ with those of computer simulations in lattice gauge theory. It could well be that our phenomenological determination, even with the presence of the aforementioned uncertainties, is still more accurate than those of present day lattice studies. The energy range $1 < s(\mathrm{GeV}^2) < 10$ is a problematic one for lattice simulations because the energy scale is comparable to the inverse lattice size used in current computations. Lattice artifacts may then be significant. In addition, although today's lattice studies are becoming more accurate in the prediction of resonance masses and couplings, they are not yet competitive in matrix element analysis with the experimental determinations which were the prime ingredient of our evaluation.

The weak matrix element that we have studied does not immediately provide the key to solving the riddle of the $\Delta I = 1/2$ rule. However, it does provide an interesting theoretical laboratory for the development of improved calculational methods which may prove to be efficient in addressing this obdurate puzzle. Additional work on this approach is underway and will be reported on in future publications.

The research described in this paper was supported in part by the National Science Foundation.

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[1] We adopt the normalization of currents and decay constants appearing in J.F. Donoghue, E. Golowich and B.R. Holstein, **Dynamics of the Standard Model**, Cambridge University Press (1992).

[2] It would be equivalent to study the time-ordered product of *axial*-vector currents instead.

[3] Another interesting member of the $(\mathbf{8},\mathbf{8})$ operator family is

$$\mathcal{H}_{\rm vv}^{\Delta S=0} = \frac{g_2^2}{8}\int d^4x\ iD_{\mu\nu}(x,M_{\rm W})\ T\left(V_3^\mu(x)V_3^\nu(0)\right) \quad .$$

Matrix elements of this operator are related to the contribution that a hypothetical heavy vector boson of mass $M_{\rm W}$ would make to electromagnetic mass differences. Using the π^+ as an example, it resembles the corresponding electromagnetic contribution

$$\mathcal{M}_{\pi^+}^{\rm em} = \frac{(-ie)^2}{2}\int d^4x\ iD_{\mu\nu}(x,0)\ \langle\pi^+|T\left(J_{\rm em}^\mu(x)J_{\rm em}^\nu(0)\right)\ |\pi^+\rangle \quad ,$$

where $iD_{\mu\nu}(x,0)$ is now the photon propagator.

[4] Note that these formulae imply the result in Eq. (8).

[5] We work with *covariantly defined* T-products. The corresponding relations involving *non* time-ordered products are

$$\begin{aligned}
\frac{1}{2\pi}\int d^4x\ e^{iq\cdot x}\langle 0|V_a^\mu(x)V_b^\nu(0)|0\rangle &= i\delta_{ab}\ \rho_{\rm V}(q^2)\ (q^\mu q^\nu - q^2 g^{\mu\nu}) \\
\frac{1}{2\pi}\int d^4x\ e^{iq\cdot x}\langle 0|A_a^\mu(x)A_b^\nu(0)|0\rangle &= i\delta_{ab}\ [F_\pi^2\delta(q^2)q^\mu q^\nu \\
&+ \rho_{\rm A}(q^2)\ (q^\mu q^\nu - q^2 g^{\mu\nu})] \quad .
\end{aligned}$$

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[13] We remind the reader that we are working with massless u, d quarks.

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Figure Caption

Fig. 1 The spectral function $\rho_{\mathrm{V}} - \rho_{\mathrm{A}}$