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EXTERIOR POWERS OF THE REFLECTION REPRESENTATION IN SPRINGER THEORY

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ABSTRACT. Let $H^*(\mathcal{B}_e)$ be the total Springer representation of W for the nilpotent element e in a simple Lie algebra $\mathfrak{g}$. Let $\wedge^i V$ denote the exterior powers of the reflection representation V of W. The focus of this paper is on the algebra of W-invariants in

$$H^*(\mathcal{B}_e) \otimes \wedge^* V$$

and we show that it is an exterior algebra on the subspace $(H^*(\mathcal{B}_e) \otimes V)^W$ in some new cases. This was known previously for $e = 0$ by a result of Solomon [25] and was recently proved by Henderson [12] in types A, B, C when e is regular in a Levi subalgebra.

The above statement about the W-invariants implies a conjecture of Lehrer-Shoji [15] about the occurrences of $\wedge^i V$ in $H^*(\mathcal{B}_e)$ (which was stated for e is regular in a Levi subalgebra). In this paper we prove the Lehrer-Shoji conjecture in all types and its natural extension to any e (not only those with the regular condition).

In the last part of the paper we make a connection to rational Cherednik algebras which implies a result about the appearance of the Orlik-Solomon exponents in Springer theory, a connection that was established in the classical groups in [15], [27] after being observed empirically by Orlik, Solomon, and Spaltenstein in the exceptional groups.

1. INTRODUCTION

Let G be a simple algebraic group over the algebraically closed field $\mathbf{k}$ of good characteristic p. Let $\mathfrak{g}$ be its Lie algebra. Assume G is of adjoint type and connected. Let n be the rank of G and V the reflection representation of the Weyl group W, which is irreducible of dimension n.

For a nilpotent element $e \in \mathfrak{g}$, consider the l-adic cohomology $H^*(\mathcal{B}_e)$ of the Springer fiber $\mathcal{B}_e$ with coefficients in $\overline{\mathbf{Q}}_l$, an algebraic closure of the l-adic numbers, with $l \neq p$. The cohomology carries a representation of W defined by Springer [28]. We follow the definition where $H^0(\mathcal{B}_e)$ carries the trivial representation of W [13], [16].

This paper is concerned with studying the W-invariants in the bi-graded algebra

$$H^*(\mathcal{B}_e) \otimes \wedge^* V.$$

Except for two cases it seems that the invariants are themselves an exterior algebra and we conjecture:

Conjecture. *Except when $e = F_4(a_3)$ in F_4 or $e = E_8(a_7)$ in E_8, the algebra*

$$\Big(\bigoplus_{i=0}^{n} H^*(\mathcal{B}_e) \otimes \wedge^i V\Big)^W$$

is an exterior algebra on the subspace $(H^*(\mathcal{B}_e) \otimes V)^W$.

This is known to be true for $e = 0$ by Solomon [25] since in that case $\mathcal{B}_e = \mathcal{B}$ and $H^*(\mathcal{B})$ is isomorphic to the coinvariant algebra of V. For e regular in a Levi subalgebra in types A_n, B_n, C_n,

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Henderson [12] has shown that the conjecture is true. That the conjecture cannot be true for the two cases mentioned will be seen shortly. Nonetheless, it does appear in those two cases that the W-invariants are isomorphic to a quotient of the exterior algebra on $(H^*(\mathcal{B}_e)\otimes V)^W$.

For H a finite group, $\hat{H}$ is the set of irreducible finite-dimensional representations of H over $\overline{\mathbf{Q}}_l$. Let $R(H)$ be the Grothendieck group of finite-dimensional representations of H with coefficients in the ring $\mathbf{Q}(q)$ of rational functions in a variable q. Let $\langle\cdot,\cdot\rangle_H$ be the inner product on $R(H)$. We suppress the subscript if there is no ambiguity about which H we are talking about.

Let $Z_G(e)$ be the centralizer of e in G under the adjoint action and let $A(e) := Z_G(e)/Z_G^0(e)$ be the component group of e. Then $H^*(\mathcal{B}_e)$ carries a representation of $A(e)$ and the $A(e)$-action commutes with the W-action.

The cohomology of $\mathcal{B}_e$ vanishes in odd degrees (see [2], [6]). We define $Q_e \in R(W\times A(e))$ by

$$Q_e = \sum_{j\geq 0} H^{2j}(\mathcal{B}_e)q^j$$

and for $\phi\in\hat{A}(e)$ define $Q_{e,\phi}\in R(W)$ by

$$Q_{e,\phi} = \langle Q_e,\phi\rangle_{A(e)}.$$

Then

$$Q_e = \sum_{\phi\in\hat{A}(e)} Q_{e,\phi}\phi.$$

Next, we define $\{(m_1,\pi_1),(m_2,\pi_2),\ldots,(m_s,\pi_s)\}$, where $m_j\in\mathbf{Z}_{\geq 0}$ and $\pi_j\in\hat{A}(e)$, by

$$\langle Q_e, V\rangle_W = q^{m_1}\pi_1 + q^{m_2}\pi_2 + \cdots + q^{m_s}\pi_s,$$

an identity in $R(A(e))$.

It turns out that at most one π_j is nontrivial and this only occurs when $\mathfrak{g}$ is not of type A_n, B_n or C_n. We reserve π_s for this possibly nontrivial representation. The calculation of π_s will be given in Section 3.7.2 for type D_n. We note that π_s is one-dimensional when e is regular in a Levi subalgebra since in that case $A(e)$ is elementary abelian. The computation of the m_j's was carried out by Lehrer-Shoji [15] in most cases and then completed by Spaltenstein [27]. We are able to give a new proof of these computations in types A_n, B_n, C_n, but we need to rely on Spaltenstein's work in type D_n. The exceptional cases are handled by studying the tables of Green functions [2].

When e is regular in a Levi subalgebra, Lehrer and Shoji [15] conjectured that the occurrences of $\wedge^i V$ in $H^*(\mathcal{B}_e)$ are given by

$$\sum_{\phi\in\hat{A}(e)} \langle Q_{e,\phi},\wedge^i V\rangle\,\phi(1) = \prod_{j=1}^{s}(1+yq^{m_j}) \tag{1}$$

Equation 1 is equivalent to the original conjecture of Lehrer-Shoji since π_s is one-dimensional whenever e is regular in a Levi subalgebra.

The purpose of this paper is to establish the Lehrer-Shoji conjecture in all types and to extend it in two ways: (1) to the case of all nilpotent e; and (2) to incorporate the $A(e)$-action on $H^*(\mathcal{B}_e)$. Our main theorem is

Theorem 1. *Let $d=\dim(\pi_s)-1$ when e is $F_4(a_3)$ or $E_8(a_7)$ and $d=\dim(\pi_s)$, otherwise. Then the following identity holds in $R(A(e))[y]$*

$$\sum_{\phi\in\hat{A}(e)}\sum_{i=0}^{n}\langle Q_{e,\phi},\wedge^i V\rangle y^i\,\phi = (1+yq^{m_s}\pi_s + y^2q^{2m_s}\wedge^2\pi_s+\cdots+y^dq^{dm_s}\wedge^d\pi_s)\prod_{j=1}^{s-1}(1+yq^{m_j})$$

Except in the two exceptional cases, the Theorem 1 would be immediate if the Conjecture were true since there is only one nontrivial π_j. Regarding those two cases, the value of $d = \dim(\pi_s) - 1$ is the best possible since $\wedge^{\mathrm{top}}\pi_s$ equals the sign character of $A(e)$, which does not appear in the representation of $A(e)$ on $H^*(\mathcal{B}_e)$. We are not able to get very far establishing the Conjecture, but Henderson's proof [12] in B_n, C_n does extend to prove the Conjecture for all e once Theorem 1 is known.

When e is regular in a Levi subagebra $\mathfrak{l}$, $d = 1$ and so Theorem 1 becomes the Lehrer-Shoji conjecture of Equation 1 if we forget about the $A(e)$-action on $H^*(\mathcal{B}_e)$. Moreover, in that situation the m_j's are exactly the Orlik-Solomon exponents associated to the restricted hyperplane arrangement in the hyperplane arrangement for W defined by the Weyl group $W_{\mathfrak{l}}$ of $\mathfrak{l}$ [15], [27]. In the second half of the paper, we give an explanation for this coincidence using rational Cherednik algebras. Finally we remark that π_s is nontrivial exactly when $N_W(W_{\mathfrak{l}})/W_{\mathfrak{l}}$ fails to be a reflection group in its natural representation, the cases of which were determined by Howlett [14].

2. Shoji's recursive formula for Q_e

2.1. Shoji [24] [23] was able to give a recursive formula for Q_e by using the orthogonality of Green functions in Springer's original work [28]. Beynon-Spaltenstein [2] used Shoji's algorithm to compute Q_e for the exceptional groups, but outside of type A_n, there are no known closed formulas in general for Q_e.

To use Shoji's algorithm it is necessary to work over a finite field $\mathbb{F}_q$ since the algorithm involves the number of points of nilpotent orbits over $\mathbb{F}_q$. Originally, there were restrictions arising from [28] that both q and the characteristic p of $\mathbb{F}_q$ needed to be large, but these were relaxed by Lusztig in his work on character sheaves [17, Theorem 24.8] where only p good for G is needed. Lusztig also showed, although we will not need it, that the number of points of the orbits can actually be deduced from his algorithm; one only needs to know the fake degrees of the tensor product of irreducible representations of W.

2.2. Let $F : G \to G$ be a Frobenius morphism defining a split $\mathbb{F}_q$-structure on G, where $q = p^n$. Let $S = S^*(V)$ denote the symmetric algebra on V. We will view $S \in R(W)$ with V having degree 1 in q. We use the results from [23], [24] [2].

First, for $e = 0$, Q_0 is isomorphic to the harmonic polynomials in S and thus by Chevalley's result

$$S = \frac{Q_0}{\prod_{i=1}^n (1 - q^{d_i})}$$

where $\{d_1, d_2, \ldots, d_n\}$ are the fundamental degrees of W on V. Steinberg's formula for $|G^F|$ is

$$|G^F| = q^N \prod_{i=1}^n (q^{d_i} - 1)$$

where N is the number of positive roots in a root system for G. Hence as an identity in $R(W)$,

$$q^N Q_0 = (-1)^n |G^F| S. \tag{2}$$

Next, we use the orthogonality formula. For each nilpotent G-orbit $\mathcal{O}$ in $\mathfrak{g}$, we select a representative $e \in \mathcal{O}^F$ which is *split* in the language of [2] (and "distinguished" in the language of [24]). This is possible for every orbit in every $\mathfrak{g}$ except for one in E_8 [24,][2] when $q \equiv -1 \pmod 3$. Nonetheless, since this orbit $E_8(b_6)$ is distinguished (in the usual sense of nilpotent elements), the number of points of any of its three rational orbits is a monomial in q, independent of q, and the orthogonality formula will still hold (since it holds for an infinite number of q).

From now on e is assumed to be rational and split. The G^F-orbits on $\mathcal{O}^F$ are parametrized by the conjugacy classes in $A(e)$ and we denote by e_c a representative indexed by the class $c \subset A(e)$. Then define

$$Q'_{e_c} := \sum_{\phi \in \hat{A}(e)} \phi(c) Q_{e,\phi}. \tag{3}$$

Let ϵ denote the sign character of W. For any $\chi \in \hat{W}$, as in [23]

$$q^N Q_0 \otimes \chi \otimes \epsilon = \sum_e \sum_c |\mathcal{O}_{e_c}| \langle Q'_{e_c}, \chi \rangle Q'_{e_c}. \tag{4}$$

where the outer sum is over our chosen split representatives of the nilpotent G-orbits, the inner sum is over the conjugacy classes in $A(e)$, and $\mathcal{O}_{e_c}$ is the G^F-orbit through $e_c \in \mathfrak{g}^F$. Since p is good, the summations are independent of q and $|\mathcal{O}_{e_c}| = P(q)$ for some polynomial $P(x) \in \mathbf{Q}[x]$, independent of q.

Now Equation 4 can be rewritten using Equation 3 twice

$$q^N Q_0 \otimes \chi \otimes \epsilon = \sum_e \sum_c \sum_{\phi' \in \hat{A}(e)} \sum_{\phi \in \hat{A}(e)} |\mathcal{O}_{e_c}| \phi'(c) \langle Q_{e,\phi'}, \chi \rangle \phi(c) Q_{e,\phi}, \tag{5}$$

Next set $\chi = \wedge^{n-j} V$ in Equation 5. Then $\chi \otimes \epsilon = \wedge^j V$ since $\wedge^n V = \epsilon$. Then combining with Equation 2 and changing the order of the summations gives

$$(-1)^n |G^F| S \otimes \wedge^j V = \sum_e \sum_\phi [\sum_c \sum_{\phi'} \langle Q_{e,\phi'}, \wedge^{n-j} V \rangle \phi'(c) |\mathcal{O}_{e_c}| \phi(c)] Q_{e,\phi}, \tag{6}$$

an identity in $R(W)$.

2.3. Our proof of Theorem 1 in the classical groups will use results of Gyoja, Nishiyama, and Shimura [10] who studied the two-variable functions $\tilde{\tau}(\chi) \in \mathbf{Q}(q)[y]$ for each $\chi \in \hat{W}$ defined by

$$\tilde{\tau}(\chi) := \sum_{i,j} [S^i(V) \otimes \wedge^j V : \chi] q^i y^j.$$

They computed $\tilde{\tau}(\chi)$ for each $\chi \in \hat{W}$ and showed that the factorization pattern of $\tilde{\tau}(\chi)$ groups the irreducible representations of W into packets that often coincide with the grouping arising from the two-sided cells Kahzdan-Lusztig in W.

In order to use their results we multiply Equation 6 by y^j, sum up over j, and take the inner product with $\chi \in \hat{W}$, to get an identity in $R(W)[y]$:

$$(-1)^n |G^F| \tilde{\tau}(\chi) = \sum_e \sum_\phi [\sum_c \sum_{\phi'} \sum_{j=0}^{n} \langle Q_{e,\phi'}, \wedge^{n-j} V \rangle y^j \phi'(c) |\mathcal{O}_{e_c}| \phi(c)] \langle Q_{e,\phi}, \chi \rangle \tag{7}$$

Introduce the notation for the term in brackets above:

$$h_{e,\phi} := \sum_c \sum_{\phi'} \sum_{j=0}^{n} \langle Q_{e,\phi'}, \wedge^{n-j} V \rangle y^j |\mathcal{O}_{e_c}| \phi'(c) \phi(c), \tag{8}$$

an element of $\mathbf{Q}[q, y]$ which depends on e and $\phi' \in \hat{A}(e)$. Then Equation 7 becomes

$$(-1)^n |G^F| \tilde{\tau}(\chi) = \sum_e \sum_\phi h_{e,\phi} \langle Q_{e,\phi}, \chi \rangle \tag{9}$$

In Theorem 1 , we are trying to show that

$$\sum_{\phi\in\hat{A}(e)}\sum_{i=0}^{n}\langle Q_{e,\phi},\wedge^i V\rangle y^i\phi = (1+yq^{m_s}\pi_s + y^2q^{2m_s}\wedge^2\pi_s+\cdots+y^dq^{dm_s}\wedge^d\pi_s)\prod_{j=1}^{s-1}(1+yq^{m_j})$$

Replacing y by y^{-1}, multiplying through by y^n, and replacing i by $n-i$, gives

$$\sum_{\phi\in\hat{A}(e)}\sum_{i=0}^{n}\langle Q_{e,\phi},\wedge^{n-i} V\rangle y^i\phi = y^{n-d-s+1}(y^d+y^{d-1}q^{m_s}\pi_s+\cdots+q^{dm_s}\wedge^d\pi_s)\prod_{j=1}^{s-1}(y+q^{m_j}). \tag{10}$$

We will show this in the classical groups by calculating $h_{e,\phi'}$ by induction on the dimension of the orbit through e.

3. Proof of Theorem 1

We use the work of Gyoja, Nishiyama, and Shimura [10] and Spaltenstein [27] to prove the theorem in the classical types by induction on the dimension of the orbit through e. We omit the type A_n case, which could be handled in the same manner, since e is always regular in a Levi subalgebra and $A(e)$ is trivial and so the result can already be found in [12].

3.1. Notation on partitions.

Let

$$\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k > 0]$$

be a partition of m. We set $|\lambda| = m$, $l(\lambda)=k$, and

$$n(\lambda) = \sum_{i=1}^{k}(i-1)\lambda_i$$

and let λ' denote the dual or tranposed partition of λ.

Viewing a partition λ as a Young diagram in the usual way, let $x=(i,j)$ denote the coordinates of a box in the diagram where the top, left box is $(0,0)$. In other words, $x=(i,j)$ is a valid pair of coordinates if and only if $0\leq i\leq k-1$ and $0\leq j\leq \lambda_{i+1}-1$. We write $x\in\lambda$ whenever $x=(i,j)$ is a valid pair of coordinates for the partition λ.

For $x\in\lambda$, we denote the *hook length* of x by

$$h(x) = \lambda_{i+1} - j + \lambda'_{j+1} - i - 1$$

and the *content* of x by

$$c(x) = j - i.$$

3.2. Representations of W.

Recall that the irreducible representations of W in types B_n, C_n, and D_n are parametrized by pairs of partitions (α,β) such that $|\alpha|+|\beta|=n$. The representations will be denoted as $\chi^{\alpha,\beta}$. In D_n when $\alpha\neq\beta$ the representations $\chi^{\alpha,\beta}$ and $\chi^{\beta,\alpha}$ are equivalent, whereas when $\alpha=\beta$ there are two inequivalent representations $\chi_I^{\alpha,\alpha}$ and $\chi_{II}^{\alpha,\alpha}$.

Regarding the bipartition attached to the irreducible representation $\wedge^j V$, we have

Lemma 2 ([20]). *In types B_n, C_n, D_n, the irreducible representation $\wedge^j V$ of W is parametrized by the pair of partitions (α,β) with $\alpha=[n-j]$ and $\beta=[1^j]$.*

3.3. **Springer correspondence for $\wedge^j V$.** Next, we look at the nilpotent orbit attached to $\wedge^j V$ under the Springer correspondence. Recall that nilpotent orbits in type B_n are parametrized by partitions of $2n+1$ where even parts occur with even multiplicity; in type D_n, are parametrized by partitions of $2n$ where even parts occur with even multiplicity; and in type C_n, are parametrized by partitions of $2n$ where odd parts occur with even multiplicity.

For $j = 0$, $\wedge^j V$ is the trivial representation and so is attached to the regular nilpotent orbit, with partition $[2n+1], [2n], [2n-1, 1]$ in types B_n, C_n, D_n, respectively. For $j = n$, $\wedge^j V$ is the sign representation and so is attached to the zero nilpotent orbit, with partition $[1^{2n+1}], [1^{2n}], [1^{2n}]$ in types B_n, C_n, D_n, respectively. For the other cases, we have

Lemma 3. *Under the Springer correspondence, $\wedge^j V$ for $j \notin \{0, n\}$ is attached to the nilpotent orbit $\mathcal{O}_j$ with partition:*

- *Type B_n: $[2n-2j+1, 1^{2j}]$*
- *Type C_n: $[2n-2j, 2, 1^{2j-2}]$*
- *Type D_n: $[2n-2j-1, 3, 1^{2j-2}]$*

All of these orbits have a S_2 component group in types B_n and C_n. The corresponding local system under the Springer correspondence in types B_n, C_n is, however, trivial. In type D_n, the component group is S_2 for $1 < j \leq n-1$ and the corresponding local system is nontrivial.

Proof. Apply the algorithms in [5, Chapter 13.3]. We omit the details. □

Corollary 4. *Let e be a nilpotent orbit parametrized by the partition λ. Let $k = l(\lambda)$ be the number of parts of λ. Then $\wedge^j V$ does not occur in $H^*(\mathcal{B}_e)$ whenever*

- *$j > \frac{k-1}{2}$ in type B_n,*
- *$j > \frac{k}{2}$ in types C_n and D_n.*

Proof. Under the hypotheses, the partition for the nilpotent orbit $\mathcal{O}_j$ attached to $\wedge^j V$ in Lemma 3 has more parts than the partition for e. Consequently, e is not contained in the closure of $\mathcal{O}_j$. It follows from [3, Corollary 2] (see also [24, Theorem 4.4]) that the Springer representations attached to $\mathcal{O}_j$ cannot occur in $H^*(\mathcal{B}_e)$. □

Remark 3.1. The inequality in the corollary is optimal in types B_n and C_n, but it is not optimal in general in D_n. For example, in D_4 with $e = [3, 3, 1, 1]$ then $\langle Q_e, \wedge^2 V \rangle = q^3$ and $\langle Q_e, \wedge^3 V \rangle = 0$ and the result is optimal, but with $e = [3, 2, 2, 1]$ then $\langle Q_e, \wedge^2 V \rangle = 0$. This failure makes the inductive proof that we give in type D_n a bit more complicated.

3.4. **Springer correspondence, in general.** In the next lemma we give a modest constraint on the bipartitions (α, β) that can be attached to an arbitrary nilpotent element e under the Springer correspondence.

Lemma 5. *Let e be a nilpotent element in type B_n, C_n, or D_n with partition λ. Let (α, β) be the Springer representation attached to e and local system $\phi \in \hat{A}(e)$. Let $k = l(\lambda)$.*

- *Type C_n: if k is odd, then $l(\alpha) = \frac{k+1}{2}$ and if k is even, then $l(\beta) = \frac{k}{2}$;*
- *Type B_n: either $l(\alpha) = \frac{k+1}{2}$ or $l(\beta) = \frac{k-1}{2}$ (k is always odd);*
- *Type D_n: either $l(\alpha) = \frac{k}{2}$ or $l(\beta) = \frac{k}{2}$ (k is always even).*

Proof. The proof uses the algorithms in [5, Chapter 13.3]. We sketch the arguments:

Type C_n. Starting with λ we will calculate the bipartition (ξ, η) corresponding to the trivial local system on $\mathcal{O}_e$. First we make sure that the number of parts of λ is even by tacking on a 0 to λ if necessary (however, we will not adjust the definition of k). Next, list the even parts of λ in nondecreasing order as $x_1 \leq x_2 \leq \cdots \leq x_{m-1} \leq x_m$. Note that m is guaranteed to be even since the number of odd parts is even and the total number of parts of λ is now even by setting $x_1 = 0$ if necessary.

Chasing through the algorithm in [5], we see that x_i contributes $\frac{x_i}{2}$ to the bipartition: to ξ if i is even and to η if i is odd. This is a result of the fact that the odd parts in λ occur with even multiplicity.

List the odd parts of λ in nondecreasing order as $y_1 = y_2 \leq y_3 = y_4 \leq \cdots \leq y_{2l-1} = y_{2l}$. Let $y_{2i-1} = y_{2i} = y$ be consecutive numbers occurring in this sequence. If the number of even parts of λ less than y is odd, then the contribution of this pair is $\frac{y+1}{2}$ to ξ and $\frac{y-1}{2}$ to η. On the other hand, if the number of parts of λ less than y is even, then the contribution of this pair is $\frac{y-1}{2}$ to ξ and $\frac{y+1}{2}$ to η.

Now assume k is even. If the smallest part of λ, which we have denoted x_1, is even, then x_1 is nonzero and η begins will the nonzero number $\frac{x_1}{2}$. Moreover each subsequent consecutive pair of parts of λ contributes a number to ξ and a number to η. Since the contributions in this way make ξ and η nondecreasing, we see that η contains exactly $\frac{k}{2}$ (nonzero) parts. It follows that the symbol attached to e begins

$$\begin{pmatrix} 0 & & \cdots & \\ & \frac{x_1}{2}+1 & & \cdots \end{pmatrix}$$

and the second row of the symbol consists of exactly $\frac{k}{2}$ entries. Moreover, the smallest nonzero number in the symbol is $\frac{x_1}{2} + 1$ and therefore any permutation of the entries of the symbol to produce an equivalent symbol must have an initial entry in the second row of at least $\frac{x_1}{2} + 1$. We conclude that for any local system ϕ on $\mathcal{O}_e$ the corresponding bipartition (α, β) must have a β with $\frac{k}{2}$ parts.

On the other hand, if the smallest part of λ is odd, say y, then η begins with $\frac{y+1}{2}$, a nonzero number, and contains $\frac{k}{2}$ (nonzero) parts. It follows that the symbol attached to e begins

$$\begin{pmatrix} 0 & & \cdots & \\ & \frac{y+1}{2}+1 & & \cdots \end{pmatrix}$$

and the second row of the symbol contains exactly $\frac{k}{2}$ entries. Hence the smallest nonzero number in the symbol is $\frac{y+1}{2} + 1$ and as before we conclude that for any local system the corresponding bipartition (α, β) must have a β with $\frac{k}{2}$ parts.

The situation is similar if k is odd. Here, we deduce that the first row of the symbol begins with a nonzero number and contains exactly $\frac{k+1}{2}$ entries. Thus any equivalent symbol must have a first row that begins with a nonzero number and it follows that for any local system the corresponding bipartition (α, β) must have an α with $\frac{k+1}{2}$ parts.

Type B_n. Carrying out a similar analysis as in type C_n, we find that the symbol attached to e has $\frac{k+1}{2}$ entries in its first row and $\frac{k-1}{2}$ in its second row. Moreover the initial entry in the second row is nonzero. Hence any equivalent symbol provided by a nontrivial ϕ will produce a nonzero initial entry either in the first row or in the second row. We note that the precise condition that controls which of the two situations occurs depends on the value of $\phi(a_1)$ (see). If $\phi(a_1) = 1$, then a nonzero entry will definitely occur in the second row and if $\phi(a_1) = -1$, then a nonzero entry will definitely occur in the first row of the symbol. It follows that for any local system the corresponding bipartition (α, β) must have either an α with $\frac{k+1}{2}$ parts or a β with $\frac{k-1}{2}$ parts.

Type D_n. In this case, the symbol attached to e has $\frac{k}{2}$ entries in each row and the initial entry in the second row is nonzero. In fact, the rows of the symbol are considered unordered in type D_n. Hence any equivalent symbol provided by a nontrivial ϕ will produce a nonzero initial entry in one of the rows and therefore for any local system, the corresponding bipartition (α, β) must have either an α or β with $\frac{k}{2}$ parts. As noted previously, the bipartition (α, β) is unordered. □

3.5. **Explicit formula for $\tilde{\tau}(\chi)$.** The functions $\tilde{\tau}(\chi)$ from Section 2.3 were computed in [10] using results from [20].

In types B_n and C_n,

$$\tau(\chi^{\alpha,\beta}) = q^{2n(\alpha)+2n(\beta)+|\beta|} \prod_{x' \in \alpha} \frac{1 + yq^{2c(x')+1}}{1 - q^{2h(x')}} \prod_{x'' \in \beta} \frac{1 + yq^{2c(x'')-1}}{1 - q^{2h(x'')}} \tag{11}$$

In type D_n,

$$\begin{aligned} \tau(\chi^{\alpha,\beta}) = \frac{q^{2n(\alpha)+2n(\beta)}}{2\prod_{x\in\alpha\cup\beta}(1 - q^{2h(x)})} \{ & q^{|\beta|} \prod_{x' \in \alpha} (1 + yq^{2c(x')+1}) \prod_{x'' \in \beta} (1 + yq^{2c(x'')-1}) + \\ & q^{|\alpha|} \prod_{x' \in \alpha} (1 + yq^{2c(x')-1}) \prod_{x'' \in \beta} (1 + yq^{2c(x'')+1}) \} \end{aligned} \tag{12}$$

and

$$\tau(\chi_I^{\alpha,\alpha}) = \tau(\chi_{II}^{\alpha,\alpha}) = q^{4n(\alpha)+|\alpha|} \prod_{x \in \alpha} \frac{(1 + yq^{2c(x)+1})(1 + yq^{2c(x)-1})}{(1 - q^{2h(x)})^2} \tag{13}$$

The next lemma follows from Lemma 5 and the above formulas for $\tilde{\tau}(\chi^{\alpha,\beta})$.

Lemma 6. *Let $\chi_{e,\phi}$ be the (nonzero) Springer representation attached to the nilpotent element e with local system $\phi \in \hat{A}(e)$. Let λ be the partition of e and let $k = l(\lambda)$. Then $\tilde{\tau}(\chi_{e,\phi})$ is divisible by*

$$1 + q^{-c}y$$

in $\mathbf{Q}(q)[y]$ for

- *Type B_n: $c \in \{1, 3, \ldots, k-2\}$*
- *Type D_n: $c \in \{1, 3, \ldots, k-3\}$*
- *Type C_n: c odd with $c \leq k-1$ (k may be odd or even)*

Proof. In type B_n, if α has $\frac{k+1}{2}$ parts, then the values of $2c(x') + 1$ as x' runs down the first column of the Young diagram of α are $1, -1, -3, \ldots, 2-k$ and the result follows from Formula 11. The other possibility is that β has $\frac{k-1}{2}$ parts, in which case the values of $2c(x'') - 1$ as x'' runs down the first column of the Young diagram of β are $-1, -3, \ldots, 2-k$ and the result follows from Formula 11.

In type C_n if k is even, then β has $\frac{k}{2}$ parts according the lemma. The values of $2c(x'') - 1$ as x'' runs down the first column of the Young diagram of β are $-1, -3, \ldots, 1-k$ and the result follows from Formula 11. On the other hand if k is odd, then α has $\frac{k+1}{2}$ parts. The values of $2c(x') + 1$ as x' runs along the first column of the Young diagram of α are $1, -1, -3, \ldots, 2-k$ and the result follows from Formula 11.

In type D_n, then we can assume that β has $\frac{k}{2}$ parts according the lemma (the bipartition is unordered). The values of $2c(x'') - 1$ as x'' runs down the first column of the Young diagram of β are $-1, -3, \ldots, 1-k$. However, the values of $2c(x'') + 1$ as x'' runs down the first column of the Young diagram of β are $1, -1, -3, \ldots, 3-k$, which only guarantees by Formula 12 that $1 + q^{-c}y$ in the stated range divides $\tilde{\tau}(\chi_{e,\phi})$. □

3.6. **Proof in types B_n and C_n.** Let e be a nilpotent with partition λ having k parts. We will prove

$$\sum_{\phi\in\hat{A}(e)}\sum_{j=0}^{n}\langle Q_{e,\phi},\wedge^{n-j}V\rangle y^j\phi = y^{n-s}\prod_{i=1}^{s}(q^{2i-1}+y)$$

where $s=\frac{k-1}{2}$ in type B and $s=\lfloor\frac{k}{2}\rfloor$ in type C. In particular the terms in the sum on the left are zero for ϕ nontrivial. Hence, Equation 10 and thus Theorem 1 will hold.

Define

$$g_{e,\phi}=\sum_j\langle Q_{e,\phi},\wedge^{n-j}V\rangle y^j.$$

Then $g_\phi\in\mathbf{Z}[q,y]$ is of degree at most n in y. We aim to show that $g_{e,1}=y^{n-s}\prod_{i=1}^{s}(q^{2i-1}+y)$ and $g_{e,\phi}=0$ for ϕ nontrivial. Following [24], let

$$s_{\phi,\phi'}:=\sum_c|\mathcal{O}_{e_c}|\phi(c)\phi(c'),$$

so that

$$h_{e,\phi}=\sum_{\phi'}s_{\phi,\phi'}g_{e,\phi'}.$$

At this point and from now on, we insist that $\phi,\phi'\in\hat{A}(e)$ actually appear in the Springer correspondence.

We proceed by induction on the dimension of $\mathcal{O}_e$. For the zero orbit, the result is known by Solomon [25]. Let $\chi_{e,\phi}$ be a (nonzero) Springer representation for e, with local system $\phi\in\hat{A}(e)$. Plug $\chi=\chi_{e,\phi}$ into Equation 9 and use [3] to obtain

$$(-1)^n|G^F|\tilde{\tau}(\chi)=h_{e,\phi}q^{d(e)}+\sum_{e'}\sum_{\phi'}h_{e',\phi'}\langle Q_{e',\phi'},\chi\rangle \tag{14}$$

where $d(e)$ is the dimension of $\mathcal{B}_e$ and the sum on the right is now over orbit representatives e' with $e'\in\bar{\mathcal{O}}_e$ and so by induction the result is known for e'. Now the partition $\lambda_{e'}$ for e' is obtained from λ by moving boxes down in the Young diagram for λ; in particular,

$$l(\lambda_{e'})\geq l(\lambda)$$

and therefore we know that

$$\prod_{i=1}^{s}(q^{2i-1}+y)$$

divides $g_{e',1}$ in $\mathbf{Z}[q,y]$ and $g_{e',\phi'}=0$ for nontrivial $\phi'\in\hat{A}(e')$. Thus it divides all $h_{e',\phi'}$ in $\mathbf{Q}[q,y]$. By Lemma 6 this product also divides the left side of Equation 14. Hence it divides $h_{e,\phi'}$ for all $\phi'\in\hat{A}(e)$.

Now by Corollary 4 we know that $\wedge^jV$ does not occur in $Q_{e,\phi}$ if $j>s$ for any $\phi\in\hat{A}(e)$. This translates into the fact that y^{n-s} divides $g_{e,\phi}$ for each $\phi\in\hat{A}(e)$. Since the degree of $h_{e,\phi}$ in y is at most n, we conclude that for each $\phi\in\hat{A}(e)$ that

$$h_{e,\phi}=a_\phi y^{n-s}\prod_{i=1}^{s}(q^{2i-1}+y),$$

where $a_\phi \in \mathbf{Q}[q]$ [1]. In other words,

$$\sum_{\phi'} s_{\phi,\phi'} g_{\phi'} = a_\phi y^{n-s} \prod_{i=1}^{s} (q^{2i-1} + y).$$

Finally by [24, Lemma 4.11], the matrix $(s_{\phi,\phi'})$ is invertible over $\mathbf{Q}(q)$. It follows that each $g_{e,\phi}$ is equal to $y^{n-s} \prod_{i=1}^{s}(q^{2i-1} + y)$ times a polynomial in $\mathbf{Z}[q]$ since $g_{e,\phi}$ is of degree at most n in y. Examining the coefficient of y^n in $g_{e,\phi}$, which keeps track of the trivial representation of W in $Q_{e,\phi}$, we see that $g_{e,\phi} = 0$ for ϕ nontrivial since the trivial representation does not occur in $Q_{e,\phi}$ for ϕ nontrivial. Moreover, the trivial representation only occurs in degree zero in $Q_{e,1}$, so that

$$g_{e,1} = y^{n-s} \prod_{i=1}^{s} (q^{2i-1} + y).$$

This completes the proof in types B_n and C_n.

3.7. **Type D_n.**

3.7.1. *The values of m_j.* Let $\mathfrak{g}$ be of type D_n. Let e be a nilpotent element with partition λ having k parts (k must be even). The values of the m_j's were computed by Spaltenstein [27] extending the work in [15]. Let r be the number of parts of λ which are not equal to 1. Define $s = \frac{k-2}{2}$ if r is odd and $s = \frac{k}{2}$ if r is even.

Theorem 7. [27] *If r is odd,*

$$m_j = 2j - 1 \textit{ for } j \in \{1, 2, \ldots, s\}.$$

If r is even,

$$m_j = 2j - 1 \textit{ for } j \in \{1, 2, \ldots, s-1\}$$

and

$$m_s = \frac{k + r - 2}{2}.$$

3.7.2. *Computation of π_j.* The determination of the π_j's was not given in [27], but can be deduced from the work there.

If r is even and $k \neq r$, then λ contains $k - r$ parts equal to 1 and e lies in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ of type D_l with $l = n - \frac{k-r}{2}$. If λ contains an odd part different than 1, then $A(e)$ will be nontrivial and moreover the component group of e in $\mathfrak{l}$ defines an index two subgroup of $A(e)$, which we denote H.

Proposition 8. *For $j \in \{1, 2, \ldots, s-1\}$, π_j is trivial.*

If r is odd, then π_s is trivial.

If r is even, then π_s is trivial unless $k \neq r$, and λ has an odd part different from 1. If not, then π_s is the nontrivial representation of $A(e)$ which takes value 1 on H.

Proof. Following the notation of and using the results from [27] (with e in place of A and k in place of d), we have a map

$$\pi_e^* : H^*(\mathcal{P}_e) \to H^*(\mathcal{B}_e)^{W(P)}.$$

which is an isomorphism and which commutes with the action of $A(e)$ on both sides. The $W(P)$-invariants are nonzero on only three representations of W from $H^*(\mathcal{B}_e)$: the trivial representation, the reflection representation V, and a representation of dimension $n-1$ denoted by ξ. Moreover the invariants are one-dimensional in all of those cases.

[1] In fact, it lies in $\mathbf{Z}[q]$, say, by induction

There is also a map

$$i_e^*: H^*(\mathcal{P}) \to H^*(\mathcal{P}_e),$$

which is surjective when r is odd, $k+r=2n$, or $k=r$. Note that $k+r=2n$ if and only if $\lambda=[2^a,1^b]$; in particular, there are no odd parts different from 1. Since the image of i_e^* lies in the $A(e)$-invariants of $H^*(\mathcal{P}_e)$, this proves the proposition in those cases. In the remaining cases, i_e^* has a one-dimensional cokernel in degree m_s, which implies that π_j is trivial unless $j=s$. Moreover, this cokernel corresponds to V (and not ξ). Hence, if we show that $A(e)$ acts nontrivially on the two-dimensional space $H^{2m_s}(\mathcal{P}_e)$, then V corresponds to a nontrivial character of $A(e)$.

Now let $U=\ker(e)$ in the natural action of e on $\mathbf{k}^{2n}$. Then U is of dimension k and the bilinear form β on U has a radical U_0 of dimension r. Let $U'\subset U$ be a subspace of type $(\frac{k+r}{2}+1, \frac{k+r}{2}-1)$. Let $M\subset U$ be of dimension $k-r$ with $U=U_0+M$ and with β non-degenerate on M. Let $O(M)$ denote the orthogonal group defined by the restriction of β to M. Note that since $k>r$ and r is even (hence $k-r$ is even), M is nonzero and even-dimensional. Then U' contains exactly two subspaces of type $(\frac{k+r}{2}, \frac{k+r}{2})$ and these subspaces are interchanged by any determinant -1 element of $O(M)$ and fixed by $SO(M)$. It follows that the component group of $O(M)$ acts nontrivially on the two irreducible components of $\mathcal{Q}_{U'}$. Thus it acts nontrivially on $H^{2m_s}(\mathcal{Q}_U')$ and hence on $H^{2m_s}(\mathcal{Q}_U)\cong H^{2m_s}(\mathcal{P}_e)$.

To complete the proof we note that when r is even and $k\neq r$, then $A(e)$ is nontrivial if and only if λ possesses an odd part bigger than 1 (in which case it must have an even number of odd parts bigger than 1). The image of $O(M)$ lies in $Z_G(e)$ and its image in $A(e)$, which is of order two, has trivial intersection with H. It follows that the character π_s in question is the unique one which is trivial on H. □

3.7.3. *Proof of Theorem 1.* The proof proceeds as in types B_n and C_n, by induction on the dimension of $\mathcal{O}_e$, except the induction only gets us so far in general.

First, by Theorem 7, e always has $\{1,3,\dots,k-3\}$ among the values of its m_j's. Since the partition $\lambda_{e'}$ for any element e' in the closure of $\mathcal{O}_e$ has $l(\lambda_{e'})\geq k$, e' also has $\{1,3,\dots,k-3\}$ among the values of its m_j's. This allows the induction to have some force.

Imitating the B_n/C_n proof, Lemma 6 and induction imply that

$$\prod_{i=1}^{\frac{k}{2}-1}(q^{2i-1}+y)$$

divides $h_{e,\phi}$ for all $\phi\in\hat{A}(e)$ [2] . Next, by Corollary 4 we know that $\wedge^j V$ does not occur in $Q_{e,\phi}$ if $j>\frac{k}{2}$ for any $\phi\in\hat{A}(e)$. Together this means that

$$b(q,y):=y^{n-\frac{k}{2}}\prod_{i=1}^{\frac{k}{2}-1}(q^{2i-1}+y)$$

divides $h_{e,\phi}$ for all $\phi\in\hat{A}(e)$. Thus it divides $g_{e,\phi}$ for all $\phi\in\hat{A}(e)$ as in the type B_n/C_n proof.

Now b is of degree $n-1$ in y and $g_{e,1}$ is of degree n since the trivial representation occurs in the trivial isotypic component of $H^*(\mathcal{B}_e)$. Hence $g_{e,1}=b(q,y)(y+c_1)$ with $c_1\in\mathbf{Z}_{\geq 0}[q]$. On the other hand, for nontrivial ϕ, $g_{e,\phi}$ is at most degree $n-1$ in y since the trivial representation only occurs in the trivial isotypic component of $H^*(\mathcal{B}_e)$. Hence in that case, $g_{e,\phi}=b(q,y)c_\phi$ with $c_\phi\in\mathbf{Z}_{\geq 0}[q]$.

We can complete the proof by Theorem 7 and Proposition 8. Assume first that all the π_j are trivial. Then $g_{e,\phi}=0$ for ϕ nontrivial since otherwise it would be of degree exactly $n-1$ in y and therefore contribute a copy of V to the ϕ-isotypic component of $H^*(\mathcal{B}_e)$, which would contradict

[2] Again we need only consider those ϕ appearing in the Springer correspondence

Proposition 8. In this case, $g_{e,1}$ either equals $b(q,y)$ if r is odd or $b(q,y)(y+q^{m_s})$ if r is even by looking at the coefficient of y^{n-1} and using Theorem 7. This is exactly the statement of Theorem 1.

On the other hand, if π_s is nontrivial, then by looking at the coefficient of y^{n-1},

$$g_{e,\pi_s} = b(q,y)q^{m_s}$$

and

$$g_{e,1} = b(q,y)y$$

and all other $g_{e,\phi}$ are zero. This completes the proof.

3.8. **Exceptional groups.** We verified the theorem by looking at the tables derived from [2]. The verification is simplified slightly by making use of the same induction as in type D_n, which gives much (but not usually all) of $g_{e,1}$.

4. Establishing the conjecture in B_n and C_n

The fact that Theorem 1 is true for all e in types B_n and C_n allows us to verify that Henderson's hypotheses [12] for the conjecture are true. Therefore,

Theorem 4.1. *For any nilpotent element e in type B_n or C_n, the algebra*

$$(\bigoplus_{i=0}^{n} H^*(\mathcal{B}_e) \otimes \wedge^i V)^W$$

is an exterior algebra on the subspace $(H^*(\mathcal{B}_e) \otimes V)^W$.

Unfortunately at the present time we cannot do much better. We can verify the conjecture for the minimal orbit in all types and in type D_n we can observe that the proof of Theorem 7 implies that all copies of V in $H^*(\mathcal{B}_e)$ (except the one in degree m_s when r is even, $k+r \neq 2n$ and $k \neq r$) are images of copies of V in $H^*(\mathcal{B})$. But the latter does not seem to imply anything about the images of the copies of $\wedge^j V$ when $j > 1$.

5. A decomposition of the t^n-representation

5.1. **The t^n-representation.** In this section we explore a graded representation of W for each natural number t prime to the Coxeter number h of W. The representation arises in the theory of rational Cherednik algebras and is closely connected to Haiman's work on the diagonal harmonics.

To begin let t be any natural number. Consider the element $\mathcal{H}_t$ of $R(W)$ defined

$$\mathcal{H}_t = \sum_{i=0}^{n} (-1)^i q^{it} S \otimes \wedge^i V.$$

Then $\mathcal{H}_t$ can be considered as a graded virtual representation of W with q in degree 1.

By work of Gordon [9] and Berest-Etingof-Ginzburg [1], $\mathcal{H}_t$ is an actual finite-dimensional (graded) representation of W when $t = ah+1$ where $a \in \mathbf{Z}_{\geq 0}$ or t is odd in types B_n/C_n. This is known to hold whenever t is prime to h [7], [29]. We will not use the full structure of the rational Cherednik algebra here, only this fact.

Let $w \in W$. The graded trace of $w \in W$ on $\mathcal{H}_t$ is given by

$$\frac{p_w(q^t)}{p_w(q)}$$

where $p_w(x)$ denotes the characteristic polynomial of w acting on V [9], [1], or [25]. This holds for all t.

From now on, we assume that t is value for which $\mathcal{H}_t$ is an actual finite-dimensional representation. Then the ungraded trace of w, obtained by letting $q \to 1$ is give by

$$t^{s(w)}$$

where $s(w)$ is the number of eigenvalues equal to 1 in the action of w on V. Hence by [11] (see also [26]), the representation $\mathcal{H}_t$ is isomorphic as an ungraded representation to $L^\vee/tL^\vee$ where is $L^\vee$ is the lattice of coroots for W, which carries a natural action of W. We therefore call $\mathcal{H}_t$ the graded t^n representation.

In type A_n, $\mathcal{H}_t$ is related to the work of Haiman: $\mathcal{H}_t \otimes \epsilon$ when $t = h+1$ is isomorphic to the diagonal harmonics after replacing the second grading variable in the latter by q^{-1} and shifting by an appropriate power of q [9]. For more general t in type A_n, this was a conjecture of [1], now proved by Gordon-Stafford [8].

5.2. **Decomposing $\mathcal{H}_t$ into Springer representations.** Continue with the assumption on t so that $\mathcal{H}_t$ is an actual finite-dimensional representation. We will now decompose $\mathcal{H}_t$ into Springer representations, with coefficients that will turn out to be polynomials in q.

Proceeding as we did in Section 2, divide both sides of Equation 6 by $|G^F|$, multiply Equation 6 by $(-q^t)^j$, and sum up over j to obtain the identity in $R(W)$:

$$\mathcal{H}_t = \sum_e \sum_\phi f_{e,\phi}(q;t) Q_{e,\phi} \tag{15}$$

where

$$f_{e,\phi}(q;t) := (-1)^n \sum_c \sum_{\phi'} \big[\sum_{j=0}^{n} \langle Q_{e,\phi'}, \wedge^{n-j} V\rangle (-q^t)^j \phi'(c)\big] \frac{\phi(c)}{|Z_{G^F}(e_c)|}$$

Note that we have used that

$$|\mathcal{O}_{e_c}| = \frac{|G^F|}{|Z_{G^F}(e_c)|}.$$

We consider the $f_{e,\phi}(q;t)$ to be defined only when $\phi \in \hat{A}(e)$ occurs in the Springer correspondence since by the result of Shoji [24] $Q_{e,\phi}$ is nonzero if and only if this is the case.

Now we use the version of Theorem 1 in Equation 10 to simplify $f_{e,\phi}$:

$$f_{e,\phi} = q^{t(n-d-s+1)} \prod_{j=1}^{s-1} (q^t - q^{m_j}) \big(\sum_{i=0}^{d} (-1)^i q^{it+(d-i)m_s} \sum_c \frac{\wedge^{d-i} \pi_s(c) \phi(c)}{|Z_{G^F}(e_c)|} \big)$$

Let $\tilde{s} = d+s-1$ and $m = \sum_{j=0}^{s-1} m_j + dm_s$. Then factoring out some powers of q,

$$f_{e,\phi} = q^{t(n-\tilde{s})+m} \prod_{j=1}^{s-1} (q^{t-m_j} - 1) \big(\sum_{i=0}^{d} (-1)^i q^{i(t-m_s)} \sum_c \frac{\wedge^{d-i} \pi_s(c) \phi(c)}{|Z_{G^F}(e_c)|} \big) \tag{16}$$

Proposition 5.1. *$f_{e,\phi}(q;t)$ is a polynomial in q.*

Proof. By induction on the dimension of $\mathcal{O}_e$. As noted above, $f_{e,\phi}$ is only defined when ϕ occurs in the Springer correspondence, so that $\chi_{e,\phi}$ is nonzero. Take the inner product of both sides of Formula (15) with respect to $\chi = \chi_{e,\phi}$:

$$\langle \mathcal{H}_t, \chi \rangle = f_{e,\phi} q^{d(e)} + \sum_{e'} \sum_{\phi'} f_{e',\phi'} \langle Q_{e',\phi'}, \chi \rangle$$

The left-hand side is a polynomial in q by the assumption on t. On the right-hand side, we are using [3] to get the expression for the term involving e and also to know that the sum is over representatives e' of smaller orbits (or no other orbits if $e=0$) as we did in Section 2. The claim is clearly true then for $e=0$, and incidentally $f_{e,1}$ for $e=0$ is measuring the occurrences of the sign representation in $\mathcal{H}_t$. For $e\neq 0$, induction implies that the terms $f_{e',\phi'}$ are polynomial in q.

Now by [3] again, the polynomial $\langle Q_{e',\phi'},\chi\rangle$ is divisible by at least $q^{d(e)}$. Hence $q^{d(e)}$ divides $\langle \mathcal{H}_t,\chi\rangle$. It follows that $f_{e,\phi}(q;t)$ is a polynomial in q since we can divide both sides by $q^{d(e)}$ and express $f_{e,\phi}(q;t)$ as a sum of polynomials in q. □

Let $\mathfrak{l}_J$ be a Levi subalgebra of $\mathfrak{g}$ with Weyl group W_J. Define

$$W^J := N_W(W_J)/W_J.$$

Express $\mathfrak{l}_J$ as the centralizer of a toral subalgebra $\mathfrak{s}_J$. The group W^J acts naturally on $\mathfrak{s}_J$. Howlett [14] worked out the structure of W^J. We observe

Remark 5.2. W^J acts by reflections on $\mathfrak{s}_J$ if and only if π_s is trivial for e a regular nilpotent element in $\mathfrak{l}_J$.

Indeed there is connection between this observation and the next theorem, which connects the m_j to the Orlik-Solomon exponents [21].

Since our proof of Theorem 1 relied on [27] and was completely empirical in the exceptional groups, we cannot claim to improve much upon the connection to the Orlik-Solomon exponents from [15], [27]. Nonetheless, the proof certainly helps explain the appearance of these exponents.

Theorem 5.3. *When e is regular in a Levi subalgebra $\mathfrak{l}_J$, then*

$$f_{e,\phi}(1;t)=\frac{1}{|W^J|}\prod_{j=1}^{s}(t-m_j),$$

for any $\phi\in\hat{A}(e)$ appearing in the Springer correspondence. Moreover, the m_j's coincide with the Orlik-Solomon exponents for J.

For e not regular in a Levi subalgebra, $f_{e,\phi}(1;t)=0$.

Proof. Suppose e is regular in $\mathfrak{l}_J$. By Lusztig's result about induction in Springer theory [19], $s=\dim(\mathfrak{s}_J)$ and consequently s equals the rank of $Z_G(e)$.

We list a number of results for e, most of them empirical observations, which depend upon e being regular in a Levi subalgebra:

(1) $A(e)$ is elementary abelian and hence π_s is one-dimensional
(2) Each $x\in A(e)$ with $x\neq 1$ acts by outer automorphism on the reductive part of $Z_G(e)$.
(3) π_s is nontrivial if and only if for every $x\in A(e)$ with $x\neq 1$, the rank of $Z_G(e,x)$ is as most $s-2$.

Then by 2 above and the formulas for the non-split Chevalley groups [5], $|Z_{G^F}(e_c)|$ is divisible by exactly $(q-1)^s$ and no higher power of $(q-1)$ when c is trivial and by at most $(q-1)^{s-1}$ when c is nontrivial. Moreover if π_s is nontrivial then $|Z_{G^F}(e_c)|$ is divisible by at most $(q-1)^{s-2}$ when c is nontrivial.

First, assume π_s is trivial. Then

$$f_{e,\phi}=q^{t(n-s)+m}\prod_{j=1}^{s}(q^{t-m_j}-1)\Big(\sum_c\frac{\phi(c)}{|Z_{G^F}(e_c)|}\Big)$$

When $c=1$, the highest power of $(q-1)$ dividing the denominator in the sum is s, but when $c\neq 1$, the highest power is at most $s-1$. Hence in the limit $q\to 1$, the terms for $c\neq 1$ go to zero. This shows that $f_{e,\phi}(1;t)$ is independent of ϕ since its value is determined only at $c=1$ and all ϕ are one-dimensional. Moreover, in the limit $f_{e,\phi}=d_J\prod_{j=1}^{s}(t-m_j)$ where d_J is related to the number of points of $(\mathcal{O}_e)^F$. By [18] and then the fact that the Weyl group of $Z_G(e)$ coincides with W^J [4], this constant d_J is $\frac{1}{|W^J|}$.

Next, assume π_s is nontrivial. Then

$$f_{e,\phi} = q^{t(n-s)+m} \prod_{j=1}^{s-1} (q^{t-m_j} - 1) \sum_c (q^{t-m_s} - \pi_s(c)) \frac{\phi(c)}{|Z_{G^F}(e_c)|}$$

When $c = 1$, the highest power of $(q-1)$ dividing the denominator in the sum is s. But when $c \neq 1$, the highest power is at most $s-2$. Hence regardless of the value of $\pi_s(c) \in \{\pm 1\}$, the numerator is divisible by a higher power of $(q-1)$ and so the terms for $c \neq 1$ go to zero when $q \to 1$. Again $f_{e,\phi}(1;t)$ is independent of ϕ since its value is determined only at $c = 1$ and all ϕ are one-dimensional. And the same analysis shows that the limit is $\frac{1}{|W^J|}\prod_{j=1}^{s}(t-m_j)$.

Now when e is not regular in a Levi subalgebra, we can use the induction theorem in [19] to guarantee that $\tilde{s} = s + d$ is larger than the rank of $Z_G(e)$. Indeed, let $e \in \mathfrak{l}_J$ be distinguished, but not regular. Then the Springer fiber $\mathcal{B}_e^L$ for e with respect to $\mathfrak{l}_J$ contains a copy of the reflection representation of W_J in $H^2(\mathcal{B}_e^L)$. It follows from Frobenius reciprocity that the multiplicity $\tilde{s}$ of V in $H^*(\mathcal{B}_e) \cong \mathrm{Ind}_{W_L}^{W} \mathcal{B}_e^L$ is strictly greater than $r = \dim(\mathfrak{s}_J)$ where again $\mathfrak{l}_J$ is the centralizer of $\mathfrak{s}_J$. Hence when π_s is trivial, $f_{e,\phi}(1;t) = 0$. When π_s is nontrivial, we have computed empirically that for each c, the power of $q-1$ dividing the numerator exceeds the power dividing the denominator in Equation 16. Hence, $f_{e,\phi}(1;t) = 0$ in those cases too.

Consequently, we have established that the ungraded representation on $\mathcal{H}_t$ is isomorphic to

$$\sum_J \frac{1}{|W^J|} \prod_{j=1}^{s} (t - m_j^J)\, \mathrm{Ind}_{W_J}^{W}(1),$$

where the sum is over a set of representatives J of the W-orbits on the subsets of a set of simple roots for $\mathfrak{g}$. Here, we have used that $\sum Q_{e,\phi} \cong H^*(\mathcal{B}_e)$ as a W-representation since the $\phi \in \hat{A}(e)$ are one-dimensional and $H^*(\mathcal{B}_e) \cong \mathrm{Ind}_{W_J}^{W}(1)$ by [19] when e is regular in $\mathfrak{l}_J$.

Now we can finish by [26, Lemma 4.2] where it is shown that any decomposition of a representation of W into a sum of representations of the form $\mathrm{Ind}_{W_J}^{W}(1)$ is unique. Hence by [26, Proposition 4.7] the coefficients (which are polynomial in t) in the decomposition must have roots that coincide with the Orlik-Solomon exponents. This also gives another way to see that the leading coefficient of $f_{e,\phi}(1;t)$ is $\frac{1}{|W^J|}$ when e is regular in $\mathfrak{l}_J$. □

Remark 5.4. This decomposition of the t^n-representation into "Green functions" arising from the orthogonality formulas should be compared to the one obtained from fixed-point varieties in the affine flag manifold in [26]. A connection along these line was already noticed in [1].

The explicit computation of $f_{e,\phi}(q;t)$ will appear in a sequel paper with Reiner [22].

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