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Weyl group multiple Dirichlet series constructed from quadratic characters

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WEYL GROUP MULTIPLE DIRICHLET SERIES CONSTRUCTED FROM QUADRATIC CHARACTERS

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ABSTRACT. We construct multiple Dirichlet series in several complex variables whose coefficients involve quadratic residue symbols. The series are shown to have an analytic continuation and satisfy a certain group of functional equations. These are the first examples of an infinite collection of unstable *Weyl group multiple Dirichlet series* in greater than two variables having the properties predicted in [2].

1. INTRODUCTION

Let Φ be an irreducible root system of rank r with Weyl group W, and let K be a global field containing the n^{th} roots of unity. In [2] is described a heuristic method to associate to this data a multiple Dirichlet series Z in r complex variables with coefficients given by n^{th} order Gauss sums. Moreover, Z is expected to have an analytic continuation to $\mathbb{C}^r$ and to satisfy a group of functional equations isomorphic to W. These *Weyl group multiple Dirichlet series* generalize several constructions of multiple Dirichlet series that have previously appeared in the literature. We present some examples and outline their connections with analytic number theory and automorphic forms at the end of this introduction. The paper [2] suggests a method for the unified treatment of all of these examples.

Brubaker, Bump, and Friedberg [3] have given a precise definition of Z in the *stable case*; by definition, this means n is sufficiently large for a fixed Φ. In [3] the authors show that for such n, the Weyl group multiple Dirichlet series admit meromorphic continuation and have the expected group of functional equations. They also prove that the coefficients of the stable series satisfy a certain *twisted multiplicativity* (cf. (1.1)) that reduces their computation to the case of the P-power coefficients, for P a prime in the ring of integers of the field K. After multiplying by appropriate normalizing zeta factors, the authors show that the P-parts form a Dirichlet polynomial whose non-zero coefficients are naturally parametrized by the elements of the Weyl group W.

In the unstable range, when n is small relative to Φ, one still expects to be able to construct multiple Dirichlet series with the same properties as

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in the stable range. However, simple examples show that the coefficients of the P-parts are no longer parametrized solely by the elements of W. We expect that terms corresponding to the elements of W will be present, but will form only part of the P-part polynomial. Some progress—including a beautiful conjectural description of the P-parts for type A unstable series via Gelfand–Tsetlin patterns—is given in Brubaker, Bump, Friedberg, Hoffstein [4]. There one can also find a proof that the multiple Dirichlet series associated to $\Phi = A_2$ are the Whittaker coefficients of Eisenstein series on the metaplectic cover of GL_3. We refer the reader to [4] for further discussion of the connection between multiple Dirichlet series and Whittaker coefficients of metaplectic Eisenstein series.

In this paper, we focus on the case $n = 2$ and Φ simply-laced of rank r. This assumption on Φ is no loss of generality, since when $n = 2$ the non-simply-laced cases described in [2] can be obtained by setting variables to be equal in a series associated to a simply-laced root system. These series are unstable for $r \geq 3$, and the results of [2, 3] cannot be applied to derive the desired properties of Z. For such series the quadratic Gauss sums are essentially quadratic residue symbols, and the associated Weyl group multiple Dirichlet series can be expressed as sums of quadratic Dirichlet L-functions. Our main result is that these quadratic Weyl group multiple Dirichlet series have a meromorphic continuation to $\mathbb{C}^r$ and satisfy a group of functional equations isomorphic to the Weyl group W. We refer to Section 4 for the exact definition of Z and to Theorems 5.4 and 5.5 for a precise statement of these results.

For an example, let $\Phi = A_r$ and $K = \mathbb{Q}$. Then the series Z has the form

$$\sum \frac{a(m_1, m_2, \dots, m_r)}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r}},$$

where the sum is over all positive integers m_i. If $m_1 m_2 \cdots m_r$ is odd and squarefree, we have

$$a(m_1, m_2, \dots, m_r) = \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_3}\right) \cdots \left(\frac{m_{r-1}}{m_r}\right).$$

The coefficients satisfy the following twisted multiplicativity property:

$$a(m_1 m_1', \dots, m_r m_r') = a(m_1, \dots, m_r) a(m_1', \dots, m_r') \prod_{j=1}^{r-1} \left(\frac{m_j}{m_{j+1}'}\right)\left(\frac{m_j'}{m_{j+1}}\right), \tag{1.1}$$

when $(m_1 \cdots m_r, m_1' \cdots m_r') = 1$. The identity (1.1) reduces the description of the coefficients to that of the coefficients

$$a(p^{k_1}, \dots, p^{k_r}), \tag{1.2}$$

where p ranges over all primes. For a fixed prime p, the coefficients (1.2) can be organized into a generating function

$$\sum_{k_1,\ldots,k_r\geq 0} \frac{a(p^{k_1},\ldots,p^{k_r})}{p^{k_1 s_1}\cdots p^{k_r s_r}}. \tag{1.3}$$

One of our main results is an explicit description of this generating function, given in Theorem 3.4; it turns out that (1.3) is a *rational* function of the p^{-s_i} that is itself invariant under a certain Weyl group action.

We conclude this introduction with a few examples of quadratic multiple Dirichlet series that have previously appeared in the literature. We refer the reader to the survey papers [2,13] for a more comprehensive list of examples.

The first example (in more than one variable) was found by Siegel [22]:

$$Z(s,w) = \sum_{\substack{d,m\geq 1\\ d,m \text{ odd}}}\sum \frac{\chi_d(\hat{m})}{m^s d^w} a(d,m), \tag{1.4}$$

where $\hat{m}$ denotes the part of m relatively prime to the squarefree part of d and χ_d is the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. The multiplicative factor $a(d,m)$ is defined by

$$a(d,m) = \prod_{\substack{p \text{ prime}\\ p^k\|d,\ p^l\|m}} a(p^k,p^l),$$

and

$$a(p^k,p^l) = \begin{cases} \min(p^{k/2},p^{l/2}) & \text{if } \min(k,l) \text{ is even,}\\ 0 & \text{otherwise.}\end{cases} \tag{1.5}$$

Siegel obtained this series as the Mellin transform of a half-integral weight Eisenstein series for the congruence subgroup $\Gamma_0(4)$. (Actually, Siegel's series is a linear combination of series of this form.) As Siegel noted, this integral representation implies two functional equations for $Z(s,w)$, one coming from the functional equation of the Eisenstein series, and one coming from the Mellin transform, via the automorphicity of the Eisenstein series. These functional equations take the form

$$Z(s,w) \mapsto Z(1-s, w+s-1/2) \quad \text{and} \quad Z(s,w)\mapsto Z(s, 3/2-s-w).$$

These two functional equations commute with one another, and thus generate a group isomorphic to the Klein four group.

In fact, it turns out that Siegel's series satisfies a group of *twelve* functional equations. In our notation, Siegel's series is the quadratic series associated to the root system of type A_2. This means that (1.4) actually possesses a group of functional equations G isomorphic to the direct product of the Weyl group of type A_2 together with order 2 group of symmetries of the Dynkin diagram of A_2. These extra functional equations—which are not at all obvious from Siegel's presentation of his series—were first noted in unpublished work of Bump and Hoffstein, who recognized this Mellin transform of the metaplectic GL_2 Eisenstein series as the Fourier–Whittaker coefficient of a minimal parabolic metaplectic Eisenstein series on the double cover of GL_3.[1] The full group of functional equations, as well as the meromorphic continuation of $Z(s,w)$, was worked out in detail by Fisher–Friedberg [15], using methods totally separate from the work of Bump–Hoffstein. For an application of $Z(s,w)$ to a mean value result for sums of quadratic Dirichlet L-functions, see Goldfeld–Hoffstein [18] as well as [15].

For a rank 3 example, take the Rankin–Selberg convolution of two half-integral weight Eisenstein series for $\Gamma_0(4)$. This yields the quadratic A_3 series, which has the form

$$\sum\sum\sum_{\substack{d,n_1,n_2>0\\ d,n,n_2 \text{ odd}}} \frac{\chi_d(\hat{n}_1)\chi_d(\hat{n}_2)}{n_1^{s_1}n_2^{s_2}d^w} a(n_1,n_2,d). \tag{1.6}$$

Here $a(n_1,n_2,d)$ is a multiplicative weighting factor first explicitly written down by Fisher and Friedberg [16] (see also our Example 3.7). It is expected that the A_3 series is a Whittaker coefficient of a minimal parabolic Eisenstein series on the double cover of GL_4. Applications of the A_3 series include mean value results for sums of squares of quadratic Dirichlet L-functions.

More examples of higher rank have also appeared in the literature and have been applied to analytic number theory. The quadratic D_4 series was treated by Diaconu, Goldfeld and Hoffstein [14], who used it to prove mean value results for sums of cubes of quadratic Dirichlet L-functions. This was first proved by Soundararajan [23] by other methods. The results of [14] and [23] are stated over $\mathbb{Q}$, but the methods of multiple Dirichlet series work over any global field. One of us (GC) recently used the quadratic A_5 series to establish a mean value result for central values of zeta functions of biquadratic number fields [11]. The results of this paper simultaneously unify and generalize all of these earlier constructions.

Finally, we remark that one may also construct double Dirichlet series roughly of the form

$$\sum\sum_{\substack{d,n\geq 0\\ d,n \text{ odd}}} \frac{\chi_d(\hat{n})}{n^s d^w} b_g(n), \tag{1.7}$$

where the $b_g(n)$ are Fourier coefficients of a Hecke cuspform g on GL_2 or GL_3. These have been studied in the papers [8, 19] (for g on GL_2), and [10, 12] (for g on GL_3). Though we do not directly address such series in this paper, our methods may easily be adapted to establish the analytic continuation and functional equations of (1.7).

We briefly indicate how to define (1.7) precisely when g is a GL_3 form. This is the heart of the problem, since once the series has been correctly

[1] A more general connection between double Dirichlet series and Whittaker coefficients of a metaplectic GL_3 Eisenstein series is proven in [4].

defined, it is easy to mimic the procedure of Section 5 to establish the functional equations and analytic continuation. To precisely define (1.7), it once again suffices to specify its p-part. Let $\alpha_1, \alpha_2, \alpha_3$ be the Satake parameters of g at an unramified prime p. Let $f(x_1, x_2, x_3, x_4)$ be the rational function associated to the root system of type D_4 given by Theorem 3.4. (We take x_4 to be the variable corresponding to the central node.) Then the generating series giving the precise form of the p-part of the series (1.7) is $f(\alpha_1 x, \alpha_2 x, \alpha_3 x, y)$, where $x = p^{-s}, y = p^{-w}$.

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2. Preliminaries

Let K be a number field with ring of integers $\mathcal{O}$. Let S_f be a finite set of non-archimedean places such that S_f contains all places dividing 2 and the ring of S_f-integers $\mathcal{O}_{S_f}$ has class number 1. Let S_∞ be the set of archimedean places, and let $S = S_f \cup S_\infty$.

Let $\left(\frac{a}{*}\right)$ be the quadratic residue symbol attached to the extension $K(\sqrt{a})$ of K, extended as in [15]; we review the definition below. A slightly different but essentially equivalent formalism appears in the papers [2–4]. We find the setup of [15] simpler in the quadratic case.

For each place v, let K_v denote the completion of K at v. For v nonarchimedean, let P_v be the corresponding ideal of $\mathcal{O}$, and let $q_v = |P_v|$ be its norm. Let C be the product $\prod_{v \in S_f} P_v^{n_v}$ where n_v is defined to be $\max\{\mathrm{ord}_v(4), 1\}$. Let H_C be the narrow ray class group modulo C, and let $R_C = H_C \otimes \mathbb{Z}/2\mathbb{Z}$. Write the finite group R_C as a direct product of cyclic groups, choose a generator for each, and let $\mathcal{E}_0$ be a set of ideals of $\mathcal{O}$ prime to S that represent these generators. For each $E_0 \in \mathcal{E}_0$, choose $m_{E_0} \in K^\times$ such that $E_0 \mathcal{O}_{S_f} = m_{E_0} \mathcal{O}_{S_f}$. Let $\mathcal{E}$ be a full set of representatives for R_C of the form $\prod_{E_0 \in \mathcal{E}_0} E_0^{n_{E_0}}$, with $n_{E_0} \in \mathbb{Z}$. If $E = \prod_{E_0 \in \mathcal{E}_0} E_0^{n_{E_0}}$ is such a representative, then let $m_E = \prod_{E_0 \in \mathcal{E}_0} m_{E_0}^{n_{E_0}}$. Note that $E\mathcal{O}_{S_f} = m_E \mathcal{O}_{S_f}$ for all $E \in \mathcal{E}$. For convenience we assume that $\mathcal{O} \in \mathcal{E}$ and $m_{\mathcal{O}} = 1$.

Let $\mathcal{J}(S)$ be the group of fractional ideals of $\mathcal{O}$ coprime to S_f. Let $I, J \in \mathcal{J}(S)$ be coprime. Write $I = (m)EG^2$ with $E \in \mathcal{E}$, $m \in K^\times$, $m \equiv 1 \bmod C$, and $G \in \mathcal{J}(S)$ such that $(G, J) = 1$. Then, following [15], the quadratic residue symbol $\left(\frac{mm_E}{J}\right)$ is defined, and if $I = (m')E'G'^2$ is another such decomposition, then $E' = E$ and $\left(\frac{m'm_E}{J}\right) = \left(\frac{mm_E}{J}\right)$. In view of this define the quadratic residue symbol $\left(\frac{I}{J}\right)$ to be $\left(\frac{mm_E}{J}\right)$. For $I = I_0 I_1^2$ with I_0 squarefree, we denote by χ_I the character $\chi_I(J) = \chi_{I_0}(J) = \left(\frac{I_0}{J}\right)$.

Further, in the expression $\chi_I(\hat{J})$, we let $\hat{J}$ represent the part of J coprime to I_0. This character χ_I depends on the choices above, but we suppress this from the notation.

Proposition 2.1 (Reciprocity)**.** *[15] Let $I,\ J \in \mathcal{J}(S)$ be coprime, and $\alpha(I,J) = \chi_I(J)\chi_J(I)^{-1}$. Then $\alpha(I,J)$ depends only on the images of I and J in R_C.*

Proof. See Neukirch [21], Theorem 8.3 of Chapter 6. □

Let $\mathcal{I}(S)$ be the set of integral ideals prime to S_f. Let $L^S(s,\chi_J)$ be the L-function of the character χ_J, with the places in S removed. We let $L_S(s,\chi_J)$ be the product over the places in S. Thus

$$L(s,\chi_J) = L^S(s,\chi_J)L_S(s,\chi_J).$$

If ξ is any idèle class character then the completed L-function $L(s,\xi)$ satisfies a functional equation

$$L(s,\xi) = \epsilon(s,\xi)L(1-s,\xi^{-1}), \tag{2.1}$$

where $\epsilon(s,\xi)$ is the epsilon factor of ξ.

Proposition 2.2. *Let $E, J \in \mathcal{O}(S)$ be squarefree with associated characters χ_E, χ_J of conductors $\mathfrak{f}_E, \mathfrak{f}_J$ respectively. Suppose that $\chi_J = \chi_E\chi_I$ with $I \in K^\times$, $I \equiv 1 \bmod C$. Let ψ be another character unramified outside S. Then*

$$\epsilon(s,\chi_J\psi) = \epsilon(1/2,\chi_I)\psi(\mathfrak{f}_J/\mathfrak{f}_E)\left(|\mathfrak{f}_J/\mathfrak{f}_E|\right)^{1/2-s}\epsilon(s,\chi_E\psi). \tag{2.2}$$

Here $\epsilon(1/2,\chi_I)$ is given by a (normalized) Gauss sum, as in Tate's thesis. When χ_I is a quadratic character, we have $\epsilon(1/2,\chi_I) = 1$.

We remark that the Γ-factors of the L-function appear in the contribution of the archimedean places $L(s,\chi_J)$. When the base field K is totally real, these Γ-factors will depend on χ_J, but only on the narrow ray class of J. For example, when $K = \mathbb{Q}$ and d a fundamental discriminant, the Γ-factor of $L(s,\chi_d)$ is $\Gamma(\frac{s}{2})$ if $d>0$ and $\Gamma(\frac{s+1}{2})$ if $d<0$.

3. A Weyl group action on rational functions

Let Φ be an irreducible simply laced root system of rank r with Weyl group W. Choose an ordering of the roots and let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into positive and negative roots. Let

$$\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

be the set of simple roots and let σ_i be the Weyl group element corresponding to the reflection through the hyperplane perpendicular to α_i. We say that i and j are *adjacent* if $i \neq j$ and $(\sigma_i\sigma_j)^3 = 1$. The Weyl group W is generated by the simple reflections $\sigma_1, \sigma_2, \ldots, \sigma_r$, which satisfy the relations

$$(\sigma_i\sigma_j)^{r(i,j)} = 1 \text{ with } r(i,j) = \begin{cases} 3 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 1 & \text{if } i = j, \text{ and} \\ 2 & \text{otherwise,} \end{cases} \tag{3.1}$$

for $1 \leq i, j \leq r$. The action of the generators σ_i on the roots is

$$\sigma_i \alpha_j = \begin{cases} \alpha_i + \alpha_j & \text{if } i \text{ and } j \text{ are adjacent,} \\ -\alpha_j & \text{if } i = j, \text{ and} \\ \alpha_j & \text{otherwise.} \end{cases} \tag{3.2}$$

Though it will play no role in this section, we will assume that the indices are ordered so that for each j, the i adjacent to j are either all less than j or all greater than j.

Let l denote the length function on W with respect to the generators $\sigma_1, \sigma_2, \ldots, \sigma_r$, and define

$$\mathrm{sgn}(w) = (-1)^{l(w)}.$$

Let Λ_Φ be the lattice generated by the roots. Any $\alpha \in \Lambda_\Phi$ has a unique representation as an integral linear combination of the simple roots:

$$\alpha = k_1 \alpha_1 + k_2 \alpha_2 + \cdots + k_r \alpha_r. \tag{3.3}$$

We call the set $\mathrm{Supp}(\alpha)$ of j such that $k_j \neq 0$ in (3.3) the *support* of α. We put

$$d(\alpha) = k_1 + k_2 + \cdots + k_r.$$

Introduce a partial ordering on Λ_Φ by defining $\alpha \succeq 0$ if each $k_i \geq 0$ in (3.3). Given $\alpha, \beta \in \Lambda_\Phi$, define $\alpha \succeq \beta$ if $\alpha - \beta \succeq 0$.

Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

be half the sum of the positive roots. For each w in the Weyl group set

$$\Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}.$$

We gather some simple properties of W we will need later.

Lemma 3.1. *Let $w \in W$.*

(a) *The cardinality of $\Phi(w)$ is the length $l(w)$ of w.*

(b) *We have*

$$\rho - w\rho = \sum_{\alpha \in \Phi(w^{-1})} \alpha. \tag{3.4}$$

(c) *Let $\sigma_i \in W$ be a generator such that $l(\sigma_i w) = l(w) + 1$. Then*

$$\Phi(\sigma_i w) = \Phi(w) \cup \{w^{-1}\alpha_i\}.$$

(d) *Let $\sigma_i \in W$ be a generator such that $l(w\sigma_i) = l(w) + 1$. Then*

$$\Phi(w\sigma_i) = \sigma_i\left(\Phi(w)\right) \cup \{\alpha_i\}.$$

(e) *The set of simple reflections σ_i occurring in any reduced expression for w is uniquely determined by w.*

(f) *Let $J = \mathrm{Supp}(\rho - w\rho)$. Then w lies in the subgroup $\langle \sigma_j \mid j \in J \rangle$.*

Proof. Statements (a)–(e) can be easily found in many standard references, e.g. [20]. We were unable to locate a precise reference for (f), and so for the convenience of the reader provide a proof. We will prove (f) under the assumption that W is a simply-laced Weyl group. Note that by (b) the set $\mathrm{Supp}(\rho - w\rho)$ makes sense for any $w \in W$. We proceed by induction on $l(w)$.

First assume $l(w) = 1$, so that $w = \sigma_i$, a simple reflection. Then $\rho - \sigma_i\rho = \alpha_i$. Hence the result is true in this case.

Now assume the result is true for lengths up to $l(w)$. By (e) it suffices to check the truth of the statement on any reduced expression for w. Let $\sigma_i u$ be a reduced expression for w, so that $l(u) = l(w) - 1$. Let $J = \mathrm{Supp}(\rho - u\rho)$ and write $\rho - u\rho = \sum_{j\in J} k_j\alpha_j$, where $k_j > 0$. By (b) and (d), we have

$$\rho - w\rho = \sum_{\alpha\in\Phi(w^{-1})} \alpha = \alpha_i + \sum_{\alpha\in\sigma_i\Phi(u^{-1})} \alpha = \alpha_i + \sum_{j\in J} k_j\sigma_i(\alpha_j). \tag{3.5}$$

Write the last expression as

$$\alpha_i + \sum_{j=1,\ldots,r} k'_j\alpha_j = \sum_{j=1,\ldots,r} k''_j\alpha_j. \tag{3.6}$$

We claim $k''_j \neq 0$ if $j \in J$. Indeed, assume $j \neq i$. If j is not adjacent to i, then $\sigma_i(\alpha_j) = \alpha_j$. On the other hand if j is adjacent to i, then $\sigma_i(\alpha_j) = \alpha_i + \alpha_j$. Hence if $j \neq i$ we must have $k''_j = k'_j \geq k_j$.

Now supppose $j = i$. Then the only problem is that we might have $k'_i = -1$, which would lead to an expression for $\rho - w\rho$ not involving α_i. However, by (3.5) and (3.6) we have

$$\sum_{\alpha\in\sigma_i\Phi(u^{-1})} \alpha = \sum_{j=1,\ldots,r} k'_j\alpha_j, \tag{3.7}$$

and the left of (3.7) is a sum over positive roots. Thus $k''_i = 1 + k'_i > 0$. This completes the proof. $\square$

Let $F = \mathbb{C}(\mathbf{x}) = \mathbb{C}(x_1, x_2, \ldots, x_r)$ be the field of rational functions in the variables $x_1, x_2, \ldots, x_r$. For any $\alpha \in \Lambda_\Phi$, let $\mathbf{x}^\alpha \in F$ be the monomial $x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r}$, where the exponents k_i are determined as in (3.3). Our immediate goal is to define an action of the Weyl group W on F. It will turn out that to construct a multiple Dirichlet series with group of functional equations isomorphic to the group W, it suffices to construct a rational function f invariant under this W-action and satisfying certain limiting conditions, see Section 5 and Proposition 5.1.

We define this W-action in stages. First, for $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ define $\sigma_i\mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} x_i x_j\sqrt{q} & \text{if } i \text{ and } j \text{ are adjacent,} \\ 1/(qx_j) & \text{if } i = j, \text{ and} \\ x_j & \text{otherwise.} \end{cases} \tag{3.8}$$

It is easy to see that

$$\begin{array}{lll} & \sigma_i^2\mathbf{x} = \mathbf{x} & \text{for all } i, \\ (3.9) & \sigma_i\sigma_j\sigma_i\mathbf{x} = \sigma_j\sigma_i\sigma_j\mathbf{x} & \text{if } i \text{ and } j \text{ are adjacent,} \\ & \sigma_i\sigma_j\mathbf{x} = \sigma_j\sigma_i\mathbf{x} & \text{otherwise.} \end{array}$$

Next, define $\epsilon_i\mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} -x_j & \text{if } i \text{ and } j \text{ are adjacent,} \\ x_j & \text{otherwise.} \end{cases} \tag{3.10}$$

Clearly $\epsilon_i^2\mathbf{x} = \mathbf{x}$ and $\epsilon_i\epsilon_j\mathbf{x} = \epsilon_j\epsilon_i\mathbf{x}$, and we have

$$\sigma_i\epsilon_j\mathbf{x} = \begin{cases} \epsilon_i\epsilon_j\sigma_i\mathbf{x} & \text{if } i \text{ and } j \text{ are adjacent,} \\ \epsilon_j\sigma_i\mathbf{x} & \text{otherwise.} \end{cases} \tag{3.11}$$

For $f \in F$ define

$$f_i^+(\mathbf{x}) = \frac{f(\mathbf{x}) + f(\epsilon_i\mathbf{x})}{2} \quad \text{and} \quad f_i^-(\mathbf{x}) = \frac{f(\mathbf{x}) - f(\epsilon_i\mathbf{x})}{2}. \tag{3.12}$$

Finally we can define the action of W on F for a generator $\sigma_i \in W$:

$$(f|\sigma_i)(\mathbf{x}) = -\frac{1 - qx_i}{qx_i(1-x_i)} f_i^+(\sigma_i\mathbf{x}) + \frac{1}{x_i\sqrt{q}} f_i^-(\sigma_i\mathbf{x}) \tag{3.13}$$

Lemma 3.2. *The definition* (3.13) *extends to give an action of W on F.*

Proof. The proof amounts to verifying that the relations (3.1) are respected by (3.13). These are straightforward computations that involve identities in rational functions that are independent of f and the global structure of the root system Φ. We will show in detail that $f|\sigma_i^2 = f$, and will explain what computations are involved in proving $f|\sigma_i\sigma_j\sigma_i = f|\sigma_j\sigma_i\sigma_j$ when i is adjacent to j. The final relation, that $f|\sigma_i\sigma_j = f|\sigma_j\sigma_i$ when i and j are not adjacent, is proved by the same technique and will be left to the reader.

Define

$$\begin{aligned} c_i(\mathbf{x}) &= \frac{1}{2}\left(\frac{qx_i - 1}{qx_i(1-x_i)} + \frac{1}{\sqrt{q}x_i}\right), \quad \text{and} \\ d_i(\mathbf{x}) &= \frac{1}{2}\left(\frac{qx_i - 1}{qx_i(1-x_i)} - \frac{1}{\sqrt{q}x_i}\right) \end{aligned}$$

for $i = 1, 2, \dots, r$. We can rewrite (3.13) as

$$(f|\sigma_i)(\mathbf{x}) = c_i(\mathbf{x})f(\sigma_i\mathbf{x}) + d_i(\mathbf{x})f(\epsilon_i\sigma_i\mathbf{x}). \tag{3.14}$$

It is then easy to compute

$$\begin{aligned} (f|\sigma_i^2)(\mathbf{x}) = (c_i(\mathbf{x})c_i(\sigma_i\mathbf{x}) + d_i(\mathbf{x})d_i(\sigma_i\mathbf{x}))f(\mathbf{x}) \\ + (c_i(\mathbf{x})d_i(\sigma_i\mathbf{x}) + d_i(\mathbf{x})c_i(\sigma_i\mathbf{x}))f(\epsilon_i\mathbf{x}). \end{aligned}$$

Hence for $f|\sigma_i^2 = f$ we need

$$c_i(\mathbf{x})c_i(\sigma_i\mathbf{x}) + d_i(\mathbf{x})d_i(\sigma_i\mathbf{x}) = 1, \tag{3.15a}$$

$$c_i(\mathbf{x})d_i(\sigma_i\mathbf{x}) + d_i(\mathbf{x})c_i(\sigma_i\mathbf{x}) = 0. \tag{3.15b}$$

This is quickly seen as follows. Let

$$A = \frac{qx_i - 1}{qx_i(1-x_i)}, \quad B = \frac{1}{\sqrt{q}x_i}.$$

Then $c_i(\mathbf{x}) = (A+B)/2$ and $d_i(\mathbf{x}) = (A-B)/2$. One can check that $c_i(\sigma_i\mathbf{x}) = (A^{-1}+B^{-1})/2$ and $d_i(\sigma_i\mathbf{x}) = (A^{-1}-B^{-1})/2$, so (3.15a)–(3.15b) follow easily.

Now we suppose i is adjacent to j, and consider $F_1 = f|\sigma_i\sigma_j\sigma_i$ and $F_2 = f|\sigma_j\sigma_i\sigma_j$. Repeatedly applying (3.14) and the relations (3.9) and (3.11), we can write both F_1 and F_2 as linear combinations of the four functions

$$f(\sigma_i\sigma_j\sigma_i\mathbf{x}),\ f(\epsilon_i\sigma_i\sigma_j\sigma_i\mathbf{x}),\ f(\epsilon_j\sigma_i\sigma_j\sigma_i\mathbf{x}), \text{ and } f(\epsilon_i\epsilon_j\sigma_i\sigma_j\sigma_i\mathbf{x}).$$

Comparing coefficients of these linear combinations gives four identities in rational functions that must be satisfied for F_1 to equal F_2. For instance, the identity needed for equality of the coefficients of $f(\sigma_i\sigma_j\sigma_i\mathbf{x})$ in F_1, F_2 is

$$\begin{aligned} c_i(\mathbf{x})c_j(\sigma_i\mathbf{x})c_i(\sigma_j\sigma_i\mathbf{x}) + d_i(\mathbf{x})d_2(\epsilon_i\sigma_i\mathbf{x})d_i(\epsilon_i\sigma_j\sigma_i\mathbf{x}) = \\ c_j(\mathbf{x})c_i(\sigma_j\mathbf{x})c_j(\sigma_i\sigma_j\mathbf{x}) + d_j(\mathbf{x})d_2(\epsilon_j\sigma_j\mathbf{x})d_j(\epsilon_j\sigma_i\sigma_j\mathbf{x}). \end{aligned}$$

Such identities are easily verified with the aid of a computer algebra system. This completes the proof. □

Lemma 3.3. *Let $g, h \in F$ and $w \in W$.*

(a) $(g+h)|w = g|w + h|w$

(b) *If $g(\mathbf{x}) = g_\alpha(\mathbf{x}) = \mathbf{x}^\alpha$ is a monomial, then*

$$g(w\mathbf{x}) = q^{d(w\alpha-\alpha)/2}\mathbf{x}^{w\alpha}.$$

(c) *If g is an even function of all the x_j, then*

$$(gh|w)(\mathbf{x}) = g(w\mathbf{x}) \cdot (h|w)(\mathbf{x}).$$

Proof. Each part of the Lemma can be proven by first establishing the result for the generators σ_i, and then verifying that if the result is true for $w_1, w_2 \in W$, then it is true for the product w_1w_2. Part (a) is obvious. For part (b), we have $g(\sigma_i\mathbf{x}) = q^{d(\sigma_i\alpha-\alpha)/2}\mathbf{x}^{\sigma_i\alpha}$ by (3.2) and (3.8). Assume (b) holds for $w_1, w_2 \in W$. Then we have

$$\begin{aligned} w_1w_2(\mathbf{x}^\alpha) &= q^{d(w_2\alpha-\alpha)/2}w_1(\mathbf{x}^{w_2\alpha}) \\ &= q^{d(w_2\alpha-\alpha)/2}q^{d(w_1w_2\alpha-w_2\alpha)/2}\mathbf{x}^{w_1w_2\alpha} \\ &= q^{d((w_1w_2)\alpha-\alpha)/2}\mathbf{x}^{(w_1w_2)\alpha}, \end{aligned}$$

as required. For part (c), first note that if g is an even function of x_j for each index j adjacent to i, then

$$(gh|\sigma_i)(\mathbf{x}) = g(\sigma_i\mathbf{x}) \cdot (h|\sigma_i)(\mathbf{x}).$$

Part (b) implies that if $g(\mathbf{x})$ is even in any variable, then $g(\sigma_i\mathbf{x})$ is even in the same variable. The proof of (c) is now easily completed. □

We now state the main result of this section.

Theorem 3.4. *There exists a rational function $f \in F$ that is W-invariant under the $|$ operation induced by* (3.13) *and satisfies*

(1) *for each $i = 1, 2, \ldots, r$, the function f satisfies the following limiting condition: if $x_j = 0$ for every j adjacent to i, then*

$$f(\mathbf{x})(1 - x_i) \text{ is independent of } x_i. \tag{3.16}$$

(2) $f(0, 0, \ldots, 0) = 1$.

Remark 3.5. We expect that the rational function satisfying the conditions of Theorem 3.4 is unique. We have verified the uniqueness by a laborious induction for the root systems A_n $(n \leq 5)$ and D_4.

Example 3.6. For the root system A_2, the rational function f satisfying the conditions of Theorem 3.4 is

$$f_{A_2} = f_{A_2}(x_1, x_2) = \frac{1 - x_1 x_2}{(1 - x_1)(1 - x_2)(1 - q x_1^2 x_2^2)}. \tag{3.17}$$

The Taylor series coefficients of f_{A_2} coincide with the q-part of Siegel's series (1.4). That is, if we write

$$f(x_1, x_2) = \sum_{k,l \geq 0} a_{kl}(q) x_1^k x_2^l,$$

then

$$a_{kl}(q) = \begin{cases} \min(q^{k/2}, q^{l/2}) & \text{if } \min(k, l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

This should be compared with (1.5).

Example 3.7. For the root system A_3, with central node corresponding to x_2, the rational function is $f_{A_3} = f_{A_3}(x_1, x_2, x_3) =$

$$\frac{1 - x_1 x_2 - x_2 x_3 + x_1 x_2 x_3 + q x_1 x_2^2 x_3 - q x_1^2 x_2^2 x_3 - q x_1 x_2^2 x_3^2 + q x_1^2 x_2^3 x_3^2}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - q x_1^2 x_2^2)(1 - q x_2^2 x_3^2)(1 - q^2 x_1^2 x_2^2 x_3^2)}.$$

This can be expressed in terms of the A_2 rational function f_{A_2} from Example 3.6. Indeed, for $|x_i| < 1/q$, we have

$$f_{A_3}(x_1, x_2, x_3) = \frac{1}{1 - q x_1 x_2^2 x_3} \int f_{A_2}(x_1, t) f_{A_2}(x_2 t^{-1}, x_3) \frac{dt}{t}, \tag{3.18}$$

where the integral is taken over the circle $|t| = 1/q$.

The identity (3.18) originates in the representation of the A_3 multiple Dirichlet series (1.6) as a Rankin-Selberg convolution of two metaplectic Eisenstein series on the double cover of GL_2 (cf. Section 1). The factor $(1 - q x_1 x_2^2 x_3)^{-1}$ can be interpreted as the q-part of the normalizing zeta factor arising in the convolution, cf. [5, Section 1.1].

The relation between the above examples and the results of [2–4] is discussed in Remark 3.12 at the end of this section.

Since W is finite, it is easy to construct functions in F that are W-invariant by averaging over the group. The difficulty lies in finding the proper function to average so that the condition (3.16) is satisfied.

To this end, define

$$\Delta(\mathbf{x}) = \prod_{\alpha\in\Phi^+} (1 - q^{d(\alpha)}\mathbf{x}^{2\alpha}),$$

and let

$$j(w,\mathbf{x}) = \Delta(\mathbf{x})/\Delta(w\mathbf{x}).$$

Then j satisfies the one-cocycle relation

$$j(ww',\mathbf{x}) = j(w,w'\mathbf{x})j(w',\mathbf{x}). \tag{3.19}$$

Lemma 3.8. *We have*

$$j(\sigma_i,\mathbf{x}) = -qx_i^2$$

for each simple reflection σ_i. Moreover, let $w \in W$ and let $\alpha = \rho - w^{-1}\rho$. Then we have

$$j(w,\mathbf{x}) = sgn(w)q^{d(\alpha)}\mathbf{x}^{2\alpha}.$$

Proof. The second statement follows from the first and the cocycle relation (3.19). For the first, write

$$\Delta(\mathbf{x}) = (1 - q^{d(\alpha_i)}\mathbf{x}^{2\alpha_i}) \prod_{\substack{\alpha\in\Phi^+\\ \alpha\neq\alpha_i}} (1 - q^{d(\alpha)}\mathbf{x}^{2\alpha}).$$

Using Lemma 3.3,

$$\begin{aligned}\Delta(\sigma_i\mathbf{x}) &= (1 - q^{d(\alpha_i)}q^{d(\sigma_i\alpha_i-\alpha_i)}\mathbf{x}^{2\sigma_i\alpha_i}) \prod_{\substack{\alpha\in\Phi^+\\ \alpha\neq\alpha_i}} (1 - q^{d(\alpha)}q^{d(\sigma_i\alpha-\alpha)}\mathbf{x}^{2\sigma_i\alpha})\\ &= (1 - q^{-d(\alpha_i)}\mathbf{x}^{-2\alpha_i}) \prod_{\substack{\alpha\in\Phi^+\\ \alpha\neq\alpha_i}} (1 - q^{d(\sigma_i\alpha)}\mathbf{x}^{2\sigma_i\alpha})\end{aligned}$$

since $\sigma_i\alpha_i = -\alpha_i$. But by Lemma 3.1 the positive roots in $\Phi^+\backslash\{\alpha_i\}$ are permuted by σ_i. Therefore

$$\Delta(\sigma_i\mathbf{x}) = -\frac{1}{qx_i^2}\Delta(\mathbf{x}),$$

as claimed. □

We are now ready to construct the function whose existence is claimed in Theorem 3.4. Define

$$f_0(\mathbf{x}) = \sum_{w\in W} j(w,\mathbf{x})(1|w)(\mathbf{x}), \tag{3.20}$$

and put

$$f(\mathbf{x}) = f_0(\mathbf{x})\Delta(\mathbf{x})^{-1}. \tag{3.21}$$

We claim $f(\mathbf{x})$ satisfies the conditions of Theorem 3.4.

The invariance of f is clear. To verify the limiting condition (3.16) we need the following lemma:

Lemma 3.9. *Let w be an element of the Weyl group W.*

(a) $\mathbf{x}^{\rho-w\rho}(1|w)(\mathbf{x})$ *is regular at the origin.*
(b) $\mathbf{x}^{\rho-\sigma_i w\rho}\left(\frac{1}{x_i}|w\right)(\mathbf{x})$ *is regular at the origin for* $i=1,2,\ldots,r$.

Proof. The proof of the lemma is by induction on the length of w. If w is the identity element, (a) and (b) above are trivial. Suppose (a) and (b) are true for $w_0 \in W$ and that i is such that $l(\sigma_i w_0) = l(w_0)+1$. Then

$$(1|\sigma_i w_0)(\mathbf{x}) = \left.\left(\frac{qx_i-1}{qx_i(1-x_i)}\right)\right| w_0 \tag{3.22}$$

$$= \left.\left[g_1(x_i)+g_2(x_i)\tfrac{1}{x_i}\right]\right| w_0 \tag{3.23}$$

where

$$g_1(x_i)=\frac{q-1}{q(1-x_i^2)} \quad \text{and} \quad g_2(x_i)=\frac{qx_i^2-1}{q(1-x_i^2)}$$

are both even functions of x_i. Therefore, by Lemma 3.3 (c),

$$(1|\sigma_i w_0)(\mathbf{x}) = g_1(w_0\mathbf{x})(1|w_0)(\mathbf{x}) + g_2(w_0\mathbf{x})(\tfrac{1}{x_i}|w_0)(\mathbf{x}).$$

Since $g_i(w\mathbf{x})$ is regular at the origin for $i=1,2$, to finish the proof of (a) we must show that

$$\mathbf{x}^{\rho-\sigma_i w_0\rho}(1|w_0)(\mathbf{x}) \quad \text{and} \quad \mathbf{x}^{\rho-\sigma_i w_0\rho}(\tfrac{1}{x_i}|w_0)(\mathbf{x})$$

are both regular at the origin. The second statement term is regular by virtue of the inductive hypothesis. As for the first, by induction it suffices to show that $\rho-\sigma_i w_0\rho \succeq \rho - w_0\rho$, or equivalently, by Lemma 3.1 (d), that

$$\sum_{\alpha\in\Phi(w_0^{-1})} \alpha \preceq \sum_{\alpha\in\Phi(w_0^{-1})} \sigma_i(\alpha)+\alpha_i. \tag{3.24}$$

In fact we claim that for each $\alpha \in \Phi(w_0^{-1})$ either

$$\alpha \preceq \sigma_i\alpha \text{ or } \sigma_i\alpha \in \Phi(w_0^{-1}). \tag{3.25}$$

Indeed, we know that $\alpha-\sigma_i\alpha$ must be an integral multiple of α_i, say $\alpha-\sigma_i\alpha = n\alpha_i$. If $n\le 0$ then the first alternative in (3.25) holds. If $n>0$, then

$$w_0^{-1}\sigma_i(\alpha) = w_0^{-1}(\sigma_i\alpha-\alpha+\alpha) = \\ w_0^{-1}(\alpha)+w_0^{-1}(\sigma_i\alpha-\alpha) = w_0^{-1}\alpha - nw_0^{-1}(\alpha_i) \in \Phi^-.$$

Now α is in $\Phi(w_0^{-1})$ and α_i is not. Therefore $w_0^{-1}\sigma_i(\alpha)$ is in Φ^- and $\sigma_i(\alpha) \in \Phi(w_0^{-1})$. The proof of (b) is similar. □

Proof of Theorem 3.4. Let f be defined as in (3.21). To complete the proof of Theorem 3.4, we verify that f satisfies the limiting condition (3.16).

Fix an index i with neighbors $j_1,\ldots,j_k$. Let W_0 be the subgroup of W generated by the σ_j with $j\ne i$ and $j\ne j_1,\ldots j_k$. If we set $x_{j_1}=\cdots=x_{j_r}=0$ in

$$f(\mathbf{x}) = \Delta(\mathbf{x})^{-1}\sum_{w\in W} j(w,\mathbf{x})(1|w)(\mathbf{x}), \tag{3.26}$$

then Lemmas 3.1(e), 3.8, and 3.9(a) imply that every summand in (3.26) vanishes except for those with w in the group generated by σ_i and W_0. Since σ_i is in the centralizer of W_0, (3.26) becomes

$$\Delta(\mathbf{x})^{-1} \sum_{w \in W_0} [1 + j(\sigma_i, \mathbf{x})(1|\sigma_i)]|w.$$

The term in the brackets equals

$$1 + \frac{qx_i^2 - x_i}{1 - x_i} = \frac{1 - qx_i^2}{1 - x_i}$$

by (3.22). Since each $w \in W_0$ is composed of reflections σ_j for j neither neighboring nor equal to i, the term $\frac{1-qx_i^2}{1-x_i}$ can be pulled outside the summation, and leaves behind a factor of $1/(1 - x_i)$ after $1 - qx_i^2$ cancels with the same term in Δ. This completes the proof of Theorem 3.4. $\square$

For use in the following sections, we establish some further properties of the invariant function f. Write $f(\mathbf{x}) = f(\mathbf{x}; q)$ as a power series in the x_i :

$$f(\mathbf{x}; q) = \sum_{k_1, \ldots, k_r \geq 0} a(k_1, \ldots, k_r; q) x_1^{k_1} \cdots x_r^{k_r}. \tag{3.27}$$

We will often write $f(\mathbf{x})$ or $a(k_1, \ldots, k_r)$ when the dependence on q is not relevant. The main fact about the q-dependence relevant for us is the following:

Proposition 3.10. *For Φ fixed, there exists constants $C_1, C_2 > 0$ such that $a(k_1, \ldots, k_r; q) < C_1 q^{C_2|k|}$, where $|k| := k_1 + \cdots + k_r$.*

Proof. From the definition of f, it is clear that its numerator is polynomial in q and that its denominator is a finite product of terms of the form $(1 - q^{l_0} x_1^{l_1} \cdots x_r^{l_r})$ for some positive integers l_i. Expanding this out in a geometric series gives us the polynomial bound in q. $\square$

The reason for the introduction of the Weyl group action (3.13) and the relevance to L-functions will be made more clear in the next section. We conclude this section by explaining a consequence of the W-invariance of the function f. Take the power series expansion of f in $r - 1$ of the variables x_i. Thus the coefficients of this expansion will be functions of the one remaining variable, x_{j_0}, say. The invariance of f under σ_{j_0} will force these coefficients to satisfy certain functional equations. We make this explicit.

Proposition 3.11. *Fix q and an index j_0. Let*

$$\hat{k} = (k_1, \ldots, k_{j_0-1}, k_{j_0+1}, \ldots, k_r)$$

be an $(r-1)$-tuple of nonnegative integers. Define

$$T(x_{j_0}; \hat{k}) = \sum_{k_{j_0}=0}^{\infty} a(k_1, \ldots, k_{j_0-1}, k_{j_0}, k_{j_0+1}, \ldots, k_r) x_{j_0}^{k_{j_0}}.$$

Let $n(\hat{k}) = \sum_{j \; adj(j,j_0)} k_j$.

(a) *If* $n(\hat{k}) = 2\gamma$ *is even, then*

$$(1-x)T(x_{j_0};\hat{k}) = (1 - 1/(qx_{j_0}))(x_{j_0}\sqrt{q})^{2\gamma} T\left(\frac{1}{qx_{j_0}};\hat{k}\right).$$

(b) *If* $n(\hat{k}) = 2\gamma + 1$ *is odd, then*

$$T(x_{j_0};\hat{k}) = (x_{j_0}\sqrt{q})^{2\gamma} T\left(\frac{1}{qx_{j_0}};\hat{k}\right).$$

(c) *Let* C_1, C_2 *be the constants of Proposition 3.10. For* $|x_{j_0}| < q^{-C_2}$, *we have*

$$|T(x_{j_0},\hat{k})| < C_1 q^{-C_2|\hat{k}|},$$

where $|\hat{k}| = \sum_{j \neq j_0} k_j$.

Proof. From (3.13) and the invariance of f under σ_{j_0}, we know that $(1 - x_{j_0})f^+_{j_0}(\mathbf{x}) = (1 - \frac{1}{qx_{j_0}}))f^+_{j_0}(\sigma_{j_0}\mathbf{x})$. Comparing the coefficients of

$$f^+_{j_0}(\mathbf{x}) = \sum_{\hat{k} \,:\, n(\hat{k}) \text{ even}} T(x_{j_0};\hat{k}) \prod_{j \neq j_0} x_j^{k_j}$$

and

$$f^+_{j_0}(\sigma_{j_0}\mathbf{x}) = \sum_{\hat{k} \,:\, n(\hat{k}) \text{ even}} T\left(\frac{1}{qx_{j_0}};\hat{k}\right)\left(\prod_{j \neq j_0} x_j^{k_j}\right)\left(\prod_{j : j, j_0 \text{ adj.}} (x_{j_0}\sqrt{q})^{k_j}\right)$$

yields (a). The proof of (b) follows after a similar comparison of $f^-_{j_0}(\mathbf{x})$ and $f^-_{j_0}(\sigma_{j_0}\mathbf{x})$.

□

Remark 3.12. The rational functions of Theorem 3.4 will be used to define the p-parts of the multiple Dirichlet series of the following section. (Here p is a prime of norm q.) An alternative description of the p-parts of multiple Dirichlet series is given in the papers [2–4]. The first two of these papers deal with *stable* Weyl group multiple Dirichlet series constructed from n^{th} order characters and Gauss sums. As noted in the introduction, the series studied in this paper (the $n = 2$ case) fall outside the stable range provided $\Phi \neq A_2$.

To conclude this section, we describe the precise connection between the p-part polynomial of [2–4] and invariant rational function f constructed above. Our function f consists of both the p-part polynomial and the p-part of the normalizing zeta factors of [2–4]. In Eq. (30) of [3], the normalizing zeta factor of the quadratic multiple Dirichlet series associated to the root system Φ of rank r is defined to be

$$\prod_{\alpha\in\Phi^+}\zeta(2\langle\alpha,\mathbf{s}\rangle - d(\alpha)+1). \tag{3.28}$$

Here, $\mathbf{s}$ is an r-tuple of complex numbers and

$$\langle\alpha,\mathbf{s}\rangle = \alpha_1 s_1 + \cdots + \alpha_r s_r.$$

(Note: the formula of [3] is related to ours by the change of variable $s_i \mapsto 2s_i - 1/2$.) Thus, setting $x_i = q^{-s_i}$, this product of zeta functions has p-part

$$D(\mathbf{x}) = \prod_{\alpha\in\Phi^+}(1 - q^{d(\alpha)-1}\mathbf{x}^{2\alpha}). \tag{3.29}$$

Then $f(\mathbf{x})D(\mathbf{x})$ is a polynomial in the x_i. After making the change of variable $x_i \mapsto x_i\sqrt{q}$, this is the p-part polynomial of [2–4].

Let us compare our Examples 3.6 and 3.7 with the formulas of [2–4]. We begin with the A_2 series, (3.17). Multiply $f_{A_2}(x,y)$ by $(1-x^2)(1-y^2)(1-qx^2y^2)$. The result $N(x,y;A_2)$ is a sum of 6 terms which correspond to the 6 elements of the Weyl group W. Make the change of variable $x\to x\sqrt{q}, y\to y\sqrt{q}$ in $N(x,y;A_2)$ to get

$$1+\sqrt{q}x+\sqrt{q}y - q^{3/2}x^2y - q^{3/2}xy^2 + q^2x^2y^2.$$

Then the coefficient of $x^{k_1}y^{k_2}$ is precisely the coefficient $H(p^{k_1},p^{k_2})$ given in (13) of [2], after replacing $g(1,p)$ by $\sqrt{q}$ and $g(p,p^2)$ by $-q$. Thus, in this stable example, our result is identical to the result of [2].

Turning to Example 3.7, multiplying f_{A_3} by

$$(1-x_1^2)(1-x_2^2)(1-x_3^2)(1-qx_1^2x_2^2)(1-qx_2^2x_3^2)(1-q^2x_1^2x_2^2x_3^2)$$

yields a sum of 26 terms. After changes of variables as in the paragraph above, 24 of these terms correspond to the 24 elements of the Weyl group of A_3 under the association (6) of [3]. However, (6) of [3] is intended to be applicable only in the stable case; the missing 2 terms are a manifestation of the instability of this example.

To investigate the connection between f and the Weyl group, consider the rational function $f_0 = f_0(\mathbf{x};q)$ from (3.20). Expand f_0 as a power series in the variables x_i:

$$f_0(\mathbf{x};q) = \sum_{k_1,\ldots,k_r\geq 0} a(k_1,\ldots,k_r;q)x_1^{k_1}\cdots x_r^{k_r}. \tag{3.30}$$

It is not difficult to see that (3.30) contains terms in bijection with the Weyl group. Indeed, consider the function $f_0(\mathbf{x};1)$ obtained by formally setting $q=1$ and applying the definition (3.20). If $q=1$, then the W-action (3.13) simplifies considerably, and one readily computes

$$f_0(\mathbf{x};1) = \sum_{w\in W}(-1)^{l(w)+d(\rho-w\rho)}\mathbf{x}^{\rho-w\rho}.$$

Since ρ lies in the interior of the Weyl chamber, it follows that the monomials $\mathbf{x}^{\rho - w\rho}$ are all distinct. This proves that (3.30) contains terms in bijection with W. This also shows that, as functions in q, the coefficents of the unstable terms vanish when $q = 1$.

Finally, we note that Brubaker, Bump, Friedberg and Hoffstein [4] have given a conjecture for the p-parts, applicable for all n when $\Phi = A_r$. In this conjecture the terms of the numerator are parametrized not by Weyl group elements, but rather by Gelfand–Tsetlin patterns of rank r with top row $(r, r-1, \dots, 2, 1)$. These terms include terms parametrized by the Weyl group; as monomials they coincide with the $\mathbf{x}^{\rho - w\rho}$ from above. Moreover, the additional unstable terms in their conjectural p-parts satisfy a remarkable geometric property. Let P be the convex polytope obtained by taking the convex hull of the points $\rho - w\rho, w \in W$ in the vector space $\Lambda_\Phi \otimes \mathbb{R}$. Then the unstable terms are supported on monomials $\mathbf{x}^\alpha$ with $\alpha \in \Lambda_\Phi$ and lying in P.

The authors of [4] provide much convincing evidence for their conjecture, including verification that for $n = 2$, the conjecture agrees with our results for $A_r, r \leq 5$. Unfortunately, our methods do not readily provide a means to attack their conjecture as it appears difficult to extract the coefficients of the numerator of the rational function of Theorem 3.4 from the definition (3.21), and because lots of cancellation occurs during the averaging process. The connection between our construction and that of [4] is currently under investigation by the authors in joint work with Bump and Friedberg.

4. Definition of the quadratic Weyl group multiple Dirichlet series

We continue to let Φ denote an irreducible simply-laced root system of rank r. We recall our convention on the ordering of the indices: for each j, the i which are adjacent to j are either all less than j or all greater than j.

Let

$$\Psi = (\psi_1, \psi_2, \dots, \psi_r)$$

be a collection of r idèle class characters unramified outside of S. Given a collection $\mathbf{I} = (I_1, \dots, I_r)$ of ideals in $\mathcal{I}(S)$ we denote by $\Psi(\mathbf{I})$ the product

$$\prod_i \psi_i(C_i).$$

and by $H(\mathbf{I})$ the coefficient $H(I_1, I_2, \dots, I_r)$ defined below.

Definition 4.1. The coefficient $H(I_1, I_2 \dots, I_r)$ is defined by the following two conditions:

(1) Suppose $\mathbf{I} = (P^{k_1}, \dots, P^{k_r})$, where P is a fixed prime ideal of norm q. Then

$$H(P^{k_1}, \dots, P^{k_r}) = a(k_1, \dots, k_r; q).$$

(2) Given ideals $I_j, I'_j \in \mathcal{I}(S)$ with $(I_1 I_2 \cdots I_r, I'_1 I'_2 \cdots I'_r) = 1$ we have

$$\frac{H(I_1 I'_1, \ldots, I_r I'_r)}{H(I_1, \ldots, I_r) H(I'_1, \ldots, I'_r)} = \prod_{\substack{i,j \text{ adj.} \\ i<j}} \left(\frac{I_i}{I'_j}\right) \left(\frac{I'_i}{I_j}\right)$$

Note that the second condition and Proposition 3.10 imply the bound

$$|H(I_1, \ldots, I_r)| \ll |I_1 \cdots I_r|^C \tag{4.1}$$

for some constant C. If the ideals $I_1, \ldots, I_r$ is are pairwise relatively prime, then $H(\mathbf{I})$ has an especially simple form:

Lemma 4.2. *If the ideals $I_1, \ldots, I_r \in \mathcal{I}(s)$ are pairwise relatively prime, then*

$$H(I_1, \ldots, I_r) = \prod_{\substack{i,j \text{ adj.} \\ i<j}} \left(\frac{I_i}{I_j}\right).$$

Proof. We have

$$H(I_1, I_2, \ldots, I_r) = H(I_1, 1, \ldots, 1) H(1, I_2, \ldots, I_r) \prod_{i,1 \text{ adj.}} \left(\frac{C_1}{C_i}\right).$$

Now use the fact that $H(C, 1 \ldots, 1) = 1$ and induct. □

We may finally define the family of multiple Dirichlet series that is the main subject of this paper. For an r-tuple $\mathbf{s} = (s_1, \ldots, s_r)$ of complex numbers, define

$$Z_S(\mathbf{s}, \Psi) = \sum_{\mathbf{I}=(I_1,\ldots,I_r) \in \mathcal{I}(S)^r} \frac{\Psi(\mathbf{I}) H(\mathbf{I})}{\prod_j |I_j|^{s_j}} \tag{4.2}$$

By the (4.1) we see that the sum defining $Z_S(\mathbf{s}, \Psi)$ will converge absolutely for $\mathrm{Re}(s_j)$ sufficiently large, $1 \le j \le r$.

We will find it convenient to extend this definition to allow linear combinations of idèle class characters in place of Ψ. If

$$\Xi = \sum b_\Psi \Psi$$

for some collection of complex numbers b_Ψ, we define

$$Z_S(\mathbf{s}, \Xi) = \sum b_\Psi Z_S(\mathbf{s}, \Psi).$$

In the particular applications we have in mind, the r-tuple Ξ will consist of combinations of idèle class characters and characteristic functions δ_E for classes E in R_C.

Remark 4.3. The coefficient function H is similar to but slightly different from the function of the same name in [2–4]. To compare the two, denote the function in [2–4] by H_{WMD}. As explained in Remark 3.12, the coefficient generating function $f(\mathbf{x};q)$ contains both the P-part polynomial of [2–4] and the normalizing zeta factor (3.28). Therefore, we expect the equality

$$D(\mathbf{x})\sum_{k_1,\dots,k_r} H(P^{k_1},\dots,P^{k_r})x_1^{k_1}\cdots x_r^{k_r} = \sum_{k_1,\dots,k_r} H_{\mathrm{WMD}}(P^{k_1},\dots,P^{k_r})y_1^{k_1}\cdots y_r^{k_r} \tag{4.3}$$

where $x_i\sqrt{q}=y_i$ and D is the denominator given in (3.29). The coefficients on the right hand side are to be understood to mean those defined in [2] when $\Phi = A_2$ and to mean those conjectured in [4] when $\Phi = A_r$ for $r\geq 3, n=2$. As mentioned in Remark 3.12, we have checked equality of (4.3) for $r\leq 5$, $n=2$.

5. Functional equations and analytic continuation

In this section we show that the family of multiple Dirichlet series $Z_S(\mathbf{s},\Psi)$ as Ψ ranges over r-tuples of quadratic idèle class characters unramified outside of S satisfies a group of functional equations isomorphic to W, the Weyl group of the root system Φ. Summing over the j_0^{th} index in the series (4.2) defining $Z_S(\mathbf{s},\Psi)$ will produce an L-function having a functional equation as $s_{j_0}\mapsto 1-s_{j_0}$. This functional equation will induce a functional equation in the multiple Dirichlet series relating the values at $\mathbf{s}=(s_1,\dots,s_r)$ to the values at $\sigma_{j_0}\mathbf{s}=(s'_1,\dots,s'_r)$, where

$$s'_j = \begin{cases} s_j+s_{j_0}-1/2 & \text{if } j \text{ and } j_0 \text{ are adjacent,}\\ 1-s_{j_0} & \text{if } j=j_0, \text{ and}\\ s_j & \text{otherwise.}\end{cases} \tag{5.1}$$

These functional equations are involutions generating the group of functional equations of $Z_S(\mathbf{s},\Psi)$. Note that if we set $x_j = q^{-s_j}$, then this action corresponds to the action (3.8) of W on $\mathbf{x}=(x_1,\dots,x_r)$ by the variable change $x_j=q^{-s_j}$.

We now exhibit the functional equations in detail. Fix an index j_0. Then summing (4.2) over this index first produces

$$\sum_{\substack{j=1,\dots,r\\ j\neq j_0}}\sum_{I_j\in\mathcal{I}(S)}\frac{\prod_{j\neq j_0}\psi_j(I_j)}{\prod_{j\neq j_0}|I_j|^{s_j}}\cdot\sum_{I_{j_0}\in\mathcal{I}(S)}\frac{H(I_1,\dots,I_{j_0-1},I_{j_0},I_{j_0+1},\dots,I_r)}{|I_{j_0}|^{s_{j_0}}}\psi_{j_0}(I_{j_0}). \tag{5.2}$$

Our goal is to express the innermost sum as the product of a partial L-series with a Dirichlet polynomial, and to exhibit the precise functional equation that it satisfies.

Let $N=\prod_{j\neq j_0} I_j$ and let $M=\prod_{j:j,j_0\,\text{adj.}} I_j$. We will assume that $j_0>j$ for all indices j adjacent to j_0. Setting $\psi=\psi_{j_0}$ and $s=s_{j_0}$, we begin by removing the ideals relatively prime to N from the inner sum above:

$$\begin{aligned}
&\sum_{I_{j_0}\in\mathcal{I}(S)} \frac{H(I_1,\dots,I_{j_0},\dots,I_r)}{|I_{j_0}|^s}\psi(I_{j_0})\\
&= \sum_{I|N^\infty}\sum_{(J,N)=1} \frac{H(I_1,\dots,IJ,\dots,I_r)}{|IJ|^s}\psi(IJ)\\
&= \sum_{I|N^\infty} \frac{H(I_1,\dots,I,\dots,I_r)}{|I|^s}\psi(I)\left[\sum_{(J,N)=1}\frac{\psi(J)}{|J|^s}\prod_{j:j,j_0\,\text{adj.}}\left(\frac{I_j}{J}\right)\right]\\
&= L_{S_N}(s,\psi\chi_M)\sum_{I|N^\infty}\frac{H(I_1,\dots,I,\dots,I_r)}{|I|^s}\psi(I),
\end{aligned}$$

where S_N is the set of places in S together with the places dividing N. (If we had chosen j_0 such that all j adjacent to j_0 had been greater than j_0 the only difference would be that the partial L-function in front would instead be associated to the character $\psi\psi_M\chi_M$ where ψ_M is the (unramified outside S) idèle class character given by $J\mapsto\alpha(I,J)$ which depends only on the class of M in R_C.) The sum over $I|N^\infty$ decomposes as a product over the primes dividing N. Let P be a prime divisor of N. Let β_j be the order of P in I_j and let $I_j^{(P)}$ denote the part of I_j relatively prime to P. Thus $I_j = I_j^{(P)}P^{\beta_j}$. Then

$$\begin{aligned}
(5.3)\qquad &\sum_{I|N^\infty}\frac{H(I_1,\dots,I,\dots,I_r)}{|I|^s}\psi(I)\\
&= \sum_{\substack{I|N^\infty\\(I,P)=1}}\sum_{k=0}^{\infty}\frac{H(I_1^{(P)}P^{\beta_1},\dots,IP^k,\dots,I_r^{(P)}P^{\beta_r})}{|I|^s|P|^{ks}}\psi(IP^k).
\end{aligned}$$

Using the twisted multiplicativity, the term $H(I_1^{(P)}P^{\beta_1},\dots,IP^k,\dots,I_r^{(P)}P^{\beta_r})$ in the numerator can be pulled apart to yield

$$\begin{aligned}
(5.4)\quad &H(I_1^{(P)},\dots,I,\dots,I_r^{(P)})H(P^{\beta_1},\dots,P^k,\dots,P^{\beta_r})\\
&\times\left[\prod_{\substack{i<j,\text{adj.}\\ i,j\neq j_0}}\left(\frac{I_i^{(P)}}{P^{\beta_j}}\right)\left(\frac{P^{\beta_i}}{I_j^{(P)}}\right)\right]\left[\prod_{j:j,j_0\,\text{adj.}}\left(\frac{I_j^{(P)}}{P^k}\right)\left(\frac{P^{\beta_j}}{I}\right)\right].
\end{aligned}$$

The first bracketed product of characters is a constant which can be pulled outside the summation and will be ignored. Summing over k we get

$$\sum_{k=0}^{\infty}\frac{H(P^{\beta_1},\dots,P^k,\dots,P^{\beta_r})}{|P|^{ks}}\psi(P^k)\prod_{j:j,j_0\,\text{adj.}}\left(\frac{I_j^{(P)}}{P^k}\right).$$

Thus, up to a constant of absolute value 1, (5.3) is

$$\prod_{\substack{P|N \\ P^{\beta_j}||I_j}} \sum_{k=0}^{\infty} \frac{H(P^{\beta_1}, \ldots, P^k, \ldots, P^{\beta_r})}{|P|^{ks}} \psi(P^k) \left(\frac{M^{(P)}}{P^k}\right). \tag{5.5}$$

Write $M = M_0 M_1^2 M_2^2$ with M_0 squarefree and $(M_0 M_1, M_2) = 1$. Therefore, M_1 consists of primes which divide M to odd power and M_2 of primes dividing M to even order. In further evaluating the product (5.5), we distinguish three cases: P relatively prime to M, P divides M to odd order, and P divides M to even order.

Case 1: P relatively prime to M. This means that for all the neighbors j of $j_0, P^{\beta_j} = 1$. By the limiting condition of Theorem 3.4 we conclude that

$$\sum_{k=0}^{\infty} \frac{H(P^{\beta_1}, \ldots, P^k, \ldots, P^{\beta_r})}{|P|^{ks}} \psi(P^k) \chi_M(P^k)$$

is a constant (independent of s) multiple of

$$(1 - \psi(P)\chi_M(P)|P|^{-s})^{-1},$$

the P-part of the L-function $L(s, \psi\chi_M)$. The constant is given by the $k = 0$ term:

$$H(P^{\beta_1}, \ldots, P^k, \ldots, P^{\beta_r}) << |P|^{C(\beta_1 + \cdots + \beta_r)}. \tag{5.6}$$

Case 2: P divides M to odd order. Let the order of P in M be $2\gamma + 1$. Let $\epsilon = \pm 1$ be $\psi(P) \left(\frac{M^{(P)}}{P}\right)$. Thus the P-part of (5.5) is

$$H_P(s) := \sum_{k=0}^{\infty} \frac{H(P^{\beta_1}, \ldots, P^k, \ldots, P^{\beta_r})}{|P|^{ks}} \epsilon^k.$$

By virtue of the functional equation satisfied by f_i^- (Proposition 3.11), $H_P(s)$ satisfies

$$H_P(s) = |P|^{\gamma(1-2s)} H_P(1-s).$$

Taking the product over all P dividing M to odd order, we have

$$\prod_{\mathrm{ord}_P(M) odd} H_P(s) = |M_1|^{1-2s} \prod_{\mathrm{ord}_P(M) odd} H_P(1-s). \tag{5.7}$$

Case 3: P divides M to even order. Let the order of P in M be 2γ. In this case, $\chi_{M^{(P)}} = \chi_M$ since χ_M depends only on the squarefree part of M. The P-part of (5.5) is

$$H_P(s) := \sum_{k=0}^{\infty} \frac{H(P^{\beta_1}, \ldots, P^k, \ldots, P^{\beta_r})}{|P|^{ks}} \psi\chi_M(P^k).$$

Again by Proposition 3.11, this can be written as

$$(1 - \psi\chi_M(P)|P|^{-s})^{-1} H_P(s),$$

where $H_P(S)$ satisfies

$$H_P(s) = |P|^{2\gamma(1/2-s)} H_P(1-s).$$

Taking the product over all P dividing M to even order, we have

$$\prod_{\mathrm{ord}_P(M) even} H_P(s) = |M_2|^{1-2s} \prod_{\mathrm{ord}_P(M) even} H_P(1-s). \tag{5.8}$$

Putting together the 3 cases above, we get an expression for (5.2) in terms of an L-function.

Proposition 5.1. *Fix ideals $I_j \in \mathcal{I}(S)$ for $j \neq j_0$. Then*

$$\sum_{I_{j_0} \in \mathcal{I}(S)} \frac{H(I_1, \ldots, I_{j_0}, \ldots, I_r)}{|I_{j_0}|^s} \psi(I_{j_0}) = L^S(s, \psi\chi_M) Q(s)$$

where $Q(s)$ is a finite Euler product depending on the ideals $I_1, \ldots I_{j_0-1}, I_{j_0+1}, \ldots, I_r$ and the character ψ which satisfies

$$Q(s) = |M_1 M_2|^{1-2s} Q(1-s). \tag{5.9}$$

For $s > C_2$, there exists C_3 such that

$$|Q(s)| < |N|^{C_3}.$$

Here, C_2 is the constant from Proposition 3.11.

Proof. The only unproven part of the Proposition is the claim about the size of $Q(s)$. It follows from Proposition 3.11 that

$$|Q(s)| < C_1^{\omega(N)} |N|^{C_2}$$

where $\omega(N)$ is the number of prime divisors of N. Hence we may take $C_3 = C_2 + \log_2 C_1$. □

Let

$$\hat{L}^S(s, \psi\chi_M) := \sum_{I_{j_0} \in \mathcal{I}(S)} \frac{H(I_1, \ldots, I_{j_0}, \ldots, I_r)}{|I_{j_0}|^s} \psi(I_{j_0}).$$

Note that (5.9) forces $Q(s)$ to be a Dirichlet polynomial. Therefore, $Q(s)$ is an entire function of s. This implies that $\hat{L}^S(s, \psi\chi_M)$, has an analytic continuation to $s \in \mathbb{C}$, with at most a simple pole at $s = 1$. This simple pole will exist if and only if $\psi\chi_M$ is the trivial character. Moreover, $\hat{L}^S(s, \psi\chi_M)$ will satisfy a functional equation as $s \mapsto 1 - s$.

Proposition 5.2. *There is a factor $A(s, \psi, E)$ depending only on ψ and the class $E \in \mathcal{E}$ of M such that*

$$L_S(s, \psi\chi_M) \hat{L}^S(s, \psi\chi_M) = A(s, \psi, E) M^{1/2-s} L_S(1-s, \psi\chi_M) \hat{L}^S(1-s, \psi\chi_M).$$

In fact, $L_S(s, \psi\chi_M)$ also depends only on ψ and the class of M in $\mathcal{E}$. The function $A(s, \psi, E)$ is of the form $A_0^{1/2-s}$ where $A_0 = \frac{|\mathfrak{f}_M|}{|M_0 \mathfrak{f}_E|}$.

This is immediate from (5.9), the functional equation (2.1) of the L-function $L(s, \psi\chi_M)$ and the description of the epsilon factor in Proposition 2.2. We emphasize that the fact that $L_v(s, \psi\chi_M)$ depends only on ψ and the class of M in $\mathcal{E}$ is true for both the archimedean and nonarchimedean places $v \in S$—see the remark after Propostion 2.2.

In the usual way, we can use the functional equation of the preceding proposition to obtain a convexity estimate for $\hat{L}^S$:

$$\hat{L}^S(s, \psi\chi_M) << |N|^{C_3}|M|^{2C_2+1}$$

for $\mathrm{Re}(s) > -C_2$, with the implicit constant depending on the set S and $\mathrm{Im}(s)$. This final estimate allows us to analytically continue $Z_S(\mathbf{s}, \Psi)$ slightly beyond the initial domain of absolute convergence.

Proposition 5.3. *For each j_0, the multiple Dirichlet series $Z_S(\mathbf{s}, \Psi)$ has an analytic continuation to the the domain*

$$\Omega = \{(s_1, \ldots, s_r) \in \mathbb{C}^r : Re(s_{j_0}) > -C_2, Re(s_j) > C_3+2C_2+2, \ \text{for} \ \ j \neq j_0\}.$$

The actual constants C_1, C_2, C_3 are unimportant. The point is that the base of the tube domain described in the previous proposition is the complement of a compact subset of the base of the orthant

$$X = \{\mathrm{Re}(s_j) > 0 \text{ for } j = 1, 2, \ldots, r\}.$$

Let E be a class in $\mathcal{E}$. Let $\delta_{j_0,E}$ be the function on $\mathcal{I}(S)^r$ defined by

$$\delta_{j_0,E}(I_1, \ldots, I_r) = \begin{cases} 1 & \text{if} \displaystyle\prod_{j: j, j_0 \text{ adj.}} I_j \sim E \\ 0 & \text{otherwise.} \end{cases}$$

Write

$$Z_S(\mathbf{s}, \Psi) = \sum_{E \in \mathcal{E}} Z_S(s, \Psi\delta_{j_0,E}).$$

Then

$$Z_S(\mathbf{s}, \Psi\delta_{j_0,E}) = \sum_{\substack{j=1,\ldots,r \\ j \neq j_0}} \sum_{I_j \in \mathcal{I}(S)} \frac{\prod_{j \neq j_0} \psi_j(I_j)}{\prod_{j \neq j_0} |I_j|^{s_j}} \hat{L}^S(s, \psi\chi_M).$$

Write $B(s, \psi, \mathcal{E}) = L_S(s, \psi, \chi_M)$ and multiple $Z_S(\mathbf{s}, \Psi\delta_{j_0,E})$ by this factor. Then, using Proposition 5.2, we have the functional equation

$$B(s, \psi, E)Z_S(\mathbf{s}, \Psi\delta_{j_0,E}) = A(s, \psi, E)B(1-s, \psi, E)Z_S(\sigma_{j_0}\mathbf{s}, \Psi\delta_{j_0,E}).$$

Recall that the action of W on $\mathbf{s}$ was given in (5.1). Summing over E we get the functional equation for $Z_S(\mathbf{s}, \Psi)$.

Theorem 5.4. *For each $j_0 = 1, 2, \ldots, r$,*

$$Z_S(\mathbf{s}, \Psi) = \sum_{E \in \mathcal{E}} A(s, \psi, E) \frac{B(1-s, \psi, E)}{B(s, \psi, E)} Z_S(\sigma_{j_0}\mathbf{s}, \Psi\delta_{j_0,E}).$$

Let $\vec{Z}_S(\mathbf{s})$ be the vector consisting of the $Z_S(\mathbf{s}, \Psi)$ as Ψ ranges over r-tuples of quadratic idèle class characters unramified outside of S. Writing an arbitrary element $w \in W$ in terms of the simple reflections, we may express Theorem 5.4 as

$$\vec{Z}_S(\mathbf{s}) = \Phi(\mathbf{s}; w)\vec{Z}_S(w\mathbf{s}) \tag{5.10}$$

for some matrix $\Phi(\mathbf{s}; w)$.

Theorem 5.5. *The function $Z_S(\mathbf{s}, \Psi)$ has an analytic continuation to $\mathbb{C}^r$. The collection of these functions as Ψ ranges over r-tuples of quadratic idèle class characters unramified outside of S satisfies a group of functional equations isomorphic to W. This action of W is given by Theorem 5.4 and (5.10). Finally, $Z_S(\mathbf{s}, \Psi)$ is analytic outside the hyperplanes $(w\mathbf{s})_j = 1$, for $w \in W, 1 \leq j \leq r$. Here $(w\mathbf{s})_j$ denotes the j^{th} component of $w\mathbf{s}$.*

Proof. The argument is identical to that given in the proof of Theorem 5.9 of [3], and we do not repeat the details here. However, for the convenience of the reader, we give a sketch. Using the functional equations (5.10), we may extend the domain of analyticity of $Z_S(\mathbf{s}, \Psi)$ to translates of Ω by the group W. The union of the translates forms a tube domain in $\mathbb{C}^r$ whose base is the complement of a compact subset of $\mathbb{R}^r$. We may then apply Bochner's theorem [1] to extend $Z_S(\mathbf{s}, \Psi)$ to an analytic function on all of $\mathbb{C}_r$. □

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