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Envelopes and osculates of Willmore surfaces

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ENVELOPES AND OSCULATES OF WILLMORE SURFACES

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1. INTRODUCTION

Special surfaces in 3 and 4–space allow transformations preserving their special properties: classically these are the Bäcklund transformations of surfaces of constant curvature and the Darboux transformations of isothermic surfaces and, more recently, also the Bäcklund and Darboux transformations of Willmore surfaces [BFL$^+$02]. These transformations allow to construct more complicated examples of these special surfaces from simple known examples.

On the other hand, a complex holomorphic curve in $\mathbb{CP}^n$ gives rise to new holomorphic curves: the higher osculating curves and envelopes. Since every conformal surface in 3 or 4–space is a (quaternionic) holomorphic map $f : M \to \mathbb{HP}^1 = S^4$ of a Riemann surface M, we expect that the geometric constructions of envelopes and osculating curves for (quaternionic) holomorphic curves in $\mathbb{HP}^n$ should relate to the classical transformation theory of special surfaces. It is in this spirit that we study osculating and enveloping constructions for holomorphic curves $f : M \to \mathbb{HP}^n$ and their effects on conformally parameterized surfaces with special emphasis on Willmore surfaces.

One of the obstacles in defining osculating curves for a holomorphic curve $f : M \to \mathbb{HP}^n$ lies in the fact that the osculating flag

$$L \subset V_1 \subset \cdots \subset V_{n-1} \subset V = M \times \mathbb{H}^{n+1}$$

built of the successive derivatives of f is only continuous [FLPP01] along the Weierstrass points $D \subset M$ of f. This motivates the study of a more restricted class of holomorphic curves $f : M \to \mathbb{HP}^n$, the *Frenet curves*. For these curves the osculating flag is smooth on M and so is the so–called *canonical complex structure S* on the trivial $\underline{\mathbb{H}}^{n+1}$–bundle V over M. This complex structure is the analog of the mean curvature sphere congruence along a conformal immersion into 3 or 4–space. In particular, S stabilizes the osculating flag V_k and renders all V_k holomorphic as maps into the appropriate quaternionic Grassmannians.

We show that the class of Frenet curves is closed under osculating and enveloping constructions: the *k-th osculating curve f_k* of a Frenet curve f is obtained by intersecting V_k with a $\mathbb{HP}^{n-k} \subset \mathbb{HP}^n$ and is again Frenet. The *envelope* of a Frenet curve $\tilde{f} : M \to \mathbb{HP}^{n-1}$ is a holomorphic curve $f : M \to \mathbb{HP}^n$ whose first osculating curve f_1 equals $\tilde{f}$. Provided that the space of tangents $H^0(K\tilde{L})$ has a nowhere vanishing section ω there is an envelope f of $\tilde{f}$ satisfying $df = \omega$ which is again Frenet. Since the components $f = [f_0 : \cdots : f_n] : M \to \mathbb{HP}^n$ of a holomorphic curve are conformal maps into 4-space, the osculating and enveloping constructions build families of new conformal maps

of a Riemann surface M from a given one by differentiation, integration and algebraic manipulations.

Finally, we define the notion of a Willmore curve into $\mathbb{HP}^n$ and discuss their osculating and enveloping constructions: Willmore curves are Frenet curves $f : M \to \mathbb{HP}^n$ which are critical for the Willmore energy with respect to variations $f_t : M_t \to \mathbb{HP}^n$ by Frenet curves, including variations of the complex structure on M. For $n = 1$ these are the usual Willmore surfaces in 3 and 4–space [Wil93]. The Euler-Lagrange equation expresses the fact that the canonical complex structure S on V is harmonic. It turns out that, at least for Willmore spheres, the geometric constructions of envelopes and osculates preserve the Willmore property. This allows us to construct families of Willmore spheres in 3 and 4–space from a given Willmore sphere by differentiation, integration and algebraic manipulations. Even if the Willmore sphere one starts with is a twistor projection of a rational curve into $\mathbb{CP}^{2n+1}$, the osculating and enveloping spheres will generally not come from rational curves. In particular, we obtain minimal spheres with planar ends in $\mathbb{R}^4$ via these constructions.

For higher genus Willmore surfaces the osculating and enveloping constructions need not preserve the Willmore property. Rather, they give examples of the larger class of *constrained* Willmore surfaces, i.e., holomorphic curves critical for the Willmore energy under variations fixing the Riemann surface M. Of course, in the case $M = S^2$ constrained Willmore is the same as Willmore.

2. Osculates and envelopes of Frenet curves in $\mathbb{HP}^n$

The successive higher derivatives of a holomorphic curve in $\mathbb{CP}^n$ form a holomorphic flag, the Frenet flag. The intersection of the kth osculating flag with a complementary $\mathbb{CP}^{n-k}$ gives [GH94, Ch. 2.4] a new holomorphic curve in $\mathbb{CP}^{n-k}$. The analogous construction for a holomorphic curve in $\mathbb{HP}^n$ requires the existence of a smooth osculating flag. From previous work, [FLPP01], it is known that a holomorphic curve in $\mathbb{HP}^n$ has a smooth Frenet flag away from its Weierstrass points into which the flag generally extends only continuously.

We will briefly recall notions and results of quaternionic holomorphic geometry, for more details see [FLPP01, Sec. 2.5, 4.1, and 4.2] and [BFL$^+$02, Ch. 5 and 6].

Recall that a map f into $\mathbb{HP}^n$ is the same as a quaternionic line subbundle L of the trivial $\mathbb{H}^{n+1}$–bundle $V = \underline{\mathbb{H}}^{n+1}$ over M, namely $L_p = f(p)$ for $p \in M$. A smooth map $f : M \to \mathbb{HP}^n$ of a Riemann surface M is *holomorphic*, [FLPP01], if there exists a complex structure $J \in \Gamma(\text{End } L)$, $J^2 = -1$, such that

$$(2.1) \qquad\qquad *\delta_L = \delta_L J.$$

Here $\delta_L = \pi_L \nabla|_L \in \Omega^1(\text{Hom}(L, V/L))$ is the derivative of f where $\pi_L : V \to V/L$ is the canonical projection, ∇ is the trivial connection on V, and $*\omega(X) = \omega(J_M X)$ for a 1-form ω on M.

We say that a holomorphic curve $f : M \to \mathbb{HP}^n$ admits a *Frenet flag* if there exists a smooth flag $V_0 = L \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$ of quaternionic subbundles of rank $V_k = k + 1$ such that

$$(2.2) \qquad\qquad \nabla\Gamma(V_k) \subset \Omega^1(V_{k+1}),$$

and a smooth complex structure $S \in \Gamma(\mathrm{End}(V)), S^2 = -1$, stabilizing the flag, with

$$(2.3) \qquad *\delta_k = S\delta_k = \delta_k S.$$

As above, $\delta_k = \pi_{V_k} \nabla|_{V_k} \in \Omega^1(\mathrm{Hom}(V_k/V_{k-1}, V_{k+1}/V_k))$ are the derivatives of the V_k, where $\pi_{V_k} : V \to V/V_k$ are the canonical projections. Note that such an osculating flag is necessarily unique, whereas the complex structure S is not. Moreover, $*\delta_0 = \delta_0 S$ implies by (2.1) that $S|_L = J$.

If $f : M \to \mathbb{HP}^n$ is a holomorphic curve then it is shown in [FLPP01, Lemma 4.1] that f admits a Frenet flag away from a discrete set of points, the *Weierstrass points* $D \subset M$. These consist of the zeros of the flag derivatives δ_k as in the case of complex curves [GH94, Ch. 2.4]. Moreover, over $M \setminus D$ there is a unique complex structure of V, the *canonical complex structure*, satisfying (2.3) and additionally the following condition: if

$$(2.4) \qquad \nabla S = 2(*Q - *A)$$

is the type decomposition of the derivative of a complex structure S, i.e., $Q \in \Gamma(\bar{K} \mathrm{End}(V))$ and $A \in \Gamma(K \mathrm{End}(V))$, then

$$(2.5) \qquad Q|_{V_{n-1}} = 0 \quad \text{or, equivalently,} \quad AV \subset L.$$

Whereas the Frenet flag generally extends continuously [FLPP01, Lemma 4.10] across the Weierstrass points D, the canonical complex structure S may become singular as the following example shows [Pet04]: a holomorphic curve $f : M \to \mathbb{HP}^1$ is a branched conformal immersion into $S^4 = \mathbb{HP}^1$ whose branch points are the Weierstrass points $D \subset M$. The Frenet flag $L \subset V$ is clearly smooth on M. The canonical complex structure S on V is the mean curvature sphere congruence along f at the immersed points $p \in M$: the set of eigenlines for S_p on $\mathbb{H}^2$ is a round 2–sphere in S^4 which touches f at p by (2.3), and condition (2.5) says [BFL$^+$02, Thm. 2] that this 2–sphere is the mean curvature sphere of f at $p \in M$. If $f : M \to \mathbb{HP}^1$ is the twistor projection of a complex holomorphic curve $h : M \to \mathbb{CP}^3$ then the mean curvature sphere is given by the tangent line $W_1 \subset V$ of h, namely $S|_{W_1} = i$ and $S|_{W_1 j} = -i$. But the tangent $W_1 \subset V$ of h can become quaternionic, i.e., $W_1 = W_1 j$ at some $p \in M$. In this case the mean curvature sphere S degenerates to a point at $p \in M$ and thus the complex structure S cannot be extended into $p \in M$. To avoid these difficulties, we will only consider holomorphic curves $f : M \to \mathbb{HP}^n$ which have a smooth canonical complex structure. For conformal maps $f : M \to \mathbb{HP}^1$ this means that the mean curvature sphere congruence extends smoothly across the branch points.

Lemma 2.1. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with smooth canonical complex structure S. Then the Frenet flag of f extends smoothly across the Weierstrass points.*

Proof. Since

$$*\delta_0 = S\delta_0 = \delta_0 S$$

the derivative $\delta_0 \in H^0(K \mathrm{Hom}_+(L, V/L))$ of f is a complex holomorphic bundle map [Les02]. Here and throughout the paper $\mathrm{Hom}_\pm$ denote the complex linear respectively complex antilinear homomorphisms. Thus $\mathrm{Im}\,\delta$ defines a smooth quaternionic line subbundle in V/L whose lift under the canonical projection $\pi : V \to V/L$ gives the first osculating bundle $V_1 \subset V$. Proceeding inductively, we extend all the osculating bundles V_k smoothly across the Weierstrass points. $\qquad \square$

Definition 2.2. A holomorphic curve $f : M \to \mathbb{HP}^n$ is called a *Frenet curve* if the canonical complex structure, and hence also the Frenet flag, extends smoothly across the Weierstrass points.

To construct a first osculating or tangent curve of a Frenet curve $f : M \to \mathbb{HP}^n$, we choose a hyperplane $H \subset \mathbb{H}^{n+1}$ and intersect the first Frenet flag V_1 with H. If f does not intersect the hyperplane H, we obtain a smooth curve $\tilde{f} : M \to \mathbb{HP}^{n-1} = \mathbb{P}(H)$. By transversality there are hyperplanes not intersecting the curve.

Definition 2.3. The intersection of the first flag V_1 of a Frenet curve $f : M \to \mathbb{HP}^n$ with a hyperplane $H \subset \mathbb{H}^{n+1}$ is called the *first osculating* or *tangent curve* of f with respect to H. Conversely, an *envelope* of f is a holomorphic curve $\tilde{f} : M \to \mathbb{HP}^{n+1}$ whose tangent curve equals f.

We now show that these constructions preserve Frenet curves.

Theorem 2.4. *A tangent curve of a Frenet curve is Frenet.*

Proof. Let $f : M \to \mathbb{HP}^n$ be a Frenet curve with canonical complex structure S and $H \subset \mathbb{H}^{n+1}$ a hyperplane not intersecting f. Then $V = L \oplus H$ and $\tilde{L} = H \cap V_1$ is the tangent curve of f so that $V_1 = L \oplus \tilde{L}$. Via these splittings we identify $H = V/L$ and $\tilde{L} = V_1/L$. The flag $\tilde{V}_k = V_{k+1} \cap H$ is the Frenet flag of $\tilde{L}$ since $\tilde{\delta}_k = \delta_{k+1}$ and $\tilde{S} = S_{V/L}$. Moreover, $\tilde{S}$ is the canonical complex structure of $\tilde{L}$ since

$$2 * \tilde{Q}|_{\tilde{V}_{n-2}} = (\tilde{\nabla}\tilde{S})''|_{\tilde{V}_{n-2}} = \pi(\nabla S)''|_{V_{n-1} \cap H} = 2\pi * Q|_{V_{n-1} \cap H} = 0\,,$$

where $\tilde{\nabla}$ is the connection on $H = V/L$ given by $\tilde{\nabla} = \pi\nabla|_H$. $\square$

Applying the theorem successively we obtain

Corollary 2.5. *Let $f : M \to \mathbb{HP}^n$ be a Frenet curve. Then V splits into a direct sum $V = \oplus_{i=0}^n L_i$ of Frenet curves $f_i : M \to \mathbb{HP}^{n-i}$ where $f_0 = f$ and f_n is a point in $\mathbb{HP}^1$.*

Remark 2.6. For a Frenet curve $f : M \to \mathbb{HP}^n$ the kth osculating curve $f_k : M \to \mathbb{HP}^{n-k}$, where $L_k = H_k \cap V_k$, depends only on the choice of the complementary $n + 1 - k$–plane H_k. Therefore one obtains a $k(n + 1 - k)$–dimensional family of Frenet curves in $\mathbb{HP}^{n-k}$. Since the $n - 1^{\text{st}}$ osculating curve is a map into $\mathbb{HP}^1$, we get a $2(n - 1)$–dimensional family of branched conformal immersions $f_{n-1} : M \to \mathbb{HP}^1$ via the tangent construction.

To construct an envelope f to a given tangent curve $\tilde{f}$, we have to prescribe the possible tangents which f should envelop. These are the holomorphic $\tilde{L}$–valued 1–forms $\omega \in H^0(K\tilde{L})$, where a section of $K\tilde{L}$ is holomorphic if and only if it is d^∇–closed, [FLPP01, Sec. 2.3]. Generally, the construction of the complex structure of L from the complex structure of $\tilde{L}$ requires ω to have no zeros. Locally, there always exist holomorphic 1–forms ω without zeros.

Theorem 2.7. *Let $\tilde{f} : M \to \mathbb{HP}^{n-1}$ be a Frenet curve and $\omega \in H^0(K\tilde{L})$ a holomorphic 1–form without zeros. Then there exists a Frenet curve $f : M \to \mathbb{HP}^n$ with monodromy whose tangent curve is $\tilde{f}$.*

Proof. Since $d^\nabla\omega = 0$ there exists a section $\tilde{\psi} \in \Gamma(\tilde{V})$ of the trivial $\mathbb{H}^n$–bundle $\tilde{V}$ with translational monodromy such that

$$\nabla\tilde{\psi} = \omega.$$

Then $\psi = \tilde{\psi} \oplus 1$ is a nowhere vanishing section of the trivial $\mathbb{H}^{n+1}$–bundle $V = \tilde{V} \oplus \underline{\mathbb{H}}$. The line subbundle $L = \psi \mathbb{H} \subset V$ corresponds to a smooth map $f : M \to \mathbb{HP}^n$ with loxodromic monodromy. Since $\omega \in H^0(K\tilde{L})$ is nowhere zero, we define $N : M \to S^2 \subset \mathbb{H}$ by

$$*\omega = \tilde{S}\omega = \omega N$$

where $\tilde{S}$ is the canonical complex structure of $\tilde{V}$. Via the splitting $V = L \oplus \tilde{V}$ the complex structure $J\psi = \psi N$ on L defines the complex structure

$$\hat{S} = J \oplus \tilde{S}$$

on V. By construction $\hat{S}$ stabilizes the flag $L \subset V_1 \subset \ldots \subset V_{n-1} \subset V$ with $V_k = L \oplus \tilde{V}_{k-1}$ where $\tilde{V}_k$ is the Frenet flag of $\tilde{f}$. Identifying $\tilde{V} = V/L$ we get $\tilde{\delta}_{k-1} = \delta_k$ which implies

$$*\delta_k = \hat{S}\delta_k = \delta_k \hat{S}$$

for $k = 1, \ldots, n$. Therefore, to see that V_k is the Frenet flag of L, it suffices to calculate

$$*\delta_0 \psi = *\omega = \tilde{S}\omega = \hat{S}\delta_0\psi$$

and

$$*\delta_0 \psi = *\omega = \omega N = \delta_0 \psi N = \delta_0 J\psi = \delta_0 \hat{S}\psi \, ,$$

where we used $\delta_0 \psi = \pi_L \nabla \psi = \omega$.

It remains to show that the canonical complex structure S of f extends smoothly into the Weierstrass points. Since $\hat{S}$ already stabilizes the flag V_k the canonical complex structure S decomposes in the splitting $V = L \oplus \tilde{V}$ into

$$(2.6) \qquad S = \begin{pmatrix} J & B \\ 0 & \tilde{S} \end{pmatrix} ,$$

where $B : \tilde{V} \to L$ is smooth away from the Weierstrass points of f. Furthermore, the trivial connection ∇ on V decomposes into

$$(2.7) \qquad \nabla = \begin{pmatrix} \nabla^L & 0 \\ \delta_0 & \tilde{\nabla} \end{pmatrix} ,$$

where $\delta_0 \in \Gamma(K \operatorname{Hom}_+(L, \tilde{L}))$ is the derivative of L. To check that S is smooth on M it suffices to show that B is smooth on M. But the $(1,0)$–part of ∇S is given by

$$A = \frac{1}{4}(*\nabla S + S\nabla S) = \begin{pmatrix} A_L & \tilde{\eta} \\ 0 & \tilde{A} + \frac{1}{2} * \delta_0 B \end{pmatrix} ,$$

where $-2*A_L = (\nabla^L J)'$, $-2*\tilde{A} = (\tilde{\nabla}\tilde{S})'$ and $\tilde{\eta} \in \Omega^1(\operatorname{Hom}(\tilde{V}, L))$. Since S is the canonical complex structure of f it follows from (2.5) that

$$(2.8) \qquad \tilde{A} + \frac{1}{2} * \delta_0 B = 0$$

away from the Weierstrass points. By assumption $\delta_0 \psi = \omega$ is nowhere vanishing so that (2.8) defines B smoothly also in the Weierstrass points. $\qquad \square$

Remark 2.8. One can always recover the original curve from any of its tangent curves: given a Frenet curve f and a hyperplane $H \subset \mathbb{H}^{n+1}$ then there is, up to scale, a unique nowhere vanishing section $\psi \in \Gamma(L)$ with $\nabla \psi = \delta \psi$, namely projections of $\varphi_0 \in V = L \oplus H$ onto L. Let $\tilde{f}$ be the tangent curve of f with respect to H and let $\omega = \delta \psi \in H^0(K\tilde{L})$. Choosing the appropriate constant of integration we recover f.

Remark 2.9. The enveloping construction is closely related to the Bäcklund transformation for Willmore surfaces [BFL$^+$02] and its generalization [LP05] to holomorphic curves f : $M \to \mathbb{HP}^n$. Given such a curve f the Bäcklund transformation [LP05] constructs new holomorphic curves $f^\sharp : M \to \mathbb{HP}^k$ by integration of k many holomorphic 1–forms in $H^0(KL)$ for $1 \leq k \leq \dim H^0(KL)$. In the simplest case of $k = 1$, the conformal immersion $f^\sharp : M \to \mathbb{HP}^1$ given in affine coordinates by $df^\sharp = < \alpha, \omega >$ is a Bäcklund transformation of f with respect to $\alpha \in (\mathbb{H}^{n+1})^*$. Here $\omega \in H^0(KL)$ and $\alpha|_L \in \Gamma(L^{-1})$ are assumed to be nowhere vanishing sections.

But an envelope $\hat{f} : M \to \mathbb{HP}^{n+1}$ of f also arises from integrating $\omega \in H^0(KL)$. In fact, we see that the nowhere vanishing section $\psi \in \Gamma(\hat{L})$ constructed in the proof of Theorem 2.7 satisfies
$$d < \hat{\alpha}, \psi >=< \hat{\alpha}, \nabla \psi >=< \alpha, \omega >= df^\sharp,$$
where $\hat{\alpha}$ is an extension of α to a form in $(\mathbb{H}^{n+2})^*$. In particular, this shows that the Bäcklund transform $f^\sharp$ is given by the projection of the envelope $\hat{f}$ onto a suitable $\mathbb{HP}^1 \subset \mathbb{HP}^{n+1}$.

3. OSCULATES AND ENVELOPES OF WILLMORE SPHERES

It is a classical fact that the mean curvature sphere congruence of a Willmore surface in 3-space is harmonic. The analog of the mean curvature sphere in the setting of Frenet curves $f : M \to \mathbb{HP}^n$ is the canonical complex structure. We will show that a Frenet curve is Willmore if and only if the canonical complex structure is harmonic. Moreover, the constructions of the previous sections preserve Willmore spheres in $\mathbb{HP}^n$.

As an application, we can construct Willmore spheres in S^4 and minimal spheres with planar ends in $\mathbb{R}^4$ from rational curves in $\mathbb{CP}^{2n+1}$: the twistor projection [FLPP01, Lemma 2.7] of a rational curve in $\mathbb{CP}^{2n+1}$ is a Willmore curve in $\mathbb{HP}^n$. Applying the tangent construction gives Willmore spheres in $\mathbb{HP}^1 = S^4$ which themselves are generally not twistor. Therefore, stereographic projection from an appropriately chosen point on the surface yields a minimal surface in $\mathbb{R}^4$ with planar ends [Mon00].

Definition 3.1. A Frenet curve $f : M \to \mathbb{HP}^n$ is called *Willmore curve* if f is critical for the *Willmore energy*

(3.1)
$$W = 2 \int_M < A \wedge *A >,$$

under compactly supported variations by Frenet curves, where we also allow the conformal structure on M to vary.

Here A is the $(1,0)$–part, (2.4), of the derivative ∇S of the canonical complex structure S. For an endomorphism B we let $< B >:= \frac{1}{4} \operatorname{tr}_\mathbb{R} B$ be the real trace of B. Note that $\operatorname{End}_\pm$ are perpendicular with respect to this trace inner product. $S^2 = -1$ implies that A anticommutes with S so that $A \in \Gamma(K \operatorname{End}_-(V))$.

In case $f : M \to \mathbb{R}^4$ is an immersion, and thus a Frenet curve into $\mathbb{HP}^1$, the above definition gives the usual Willmore energy

$$W(f) = \frac{1}{2} \int_M |H|^2 - K - K^\perp,$$

where H is the mean curvature of f, K the Gaussian curvature and $K^\perp$ the curvature of the normal bundle of f all computed with respect to the induced metric on M. The critical points of this functional are called Willmore surfaces and have a long history attached to them: [Bla29], [Wil68], [Wei78], [LY82], [Bry84], [Eji88], [Sim93], [Mon00]. The next ★ theorem, linking the Willmore condition with harmonicity, is the natural generalization of the corresponding theorems in 3 and 4–space.

Theorem 3.2. *Denote by $\mathcal{Z} = \{S \in \mathrm{End}(\mathbb{H}^{n+1}) \mid S^2 = -1\}$ the space of complex structures on $\mathbb{H}^{n+1}$. A Frenet curve $f : M \to \mathbb{HP}^n$ is Willmore if and only if the canonical complex structure $S : M \to \mathcal{Z}$ is harmonic, that is to say, $d^\nabla * A = 0$.*

Proof. To compute the Euler–Lagrange equation of the Willmore energy, we adopt the following perspective: rather then varying the holomorphic curve, we vary the background connection by gauge transformations. Since $\mathrm{GL}(n+1, \mathbb{H})$ acts transitively on the space of flags in $\mathbb{H}^{n+1}$ together with complex structures stabilizing the flag, these two points of view are equivalent. In particular, $L \subset V_t$ is a Frenet curve over M with canonical complex structure S where we denote by V_t the bundle V equipped with the trivial connection $\nabla_t = B_t \nabla B_t^{-1}$. Calculating the infinitesimal variation of W we get

$$(3.2) \qquad \dot{W} = \int_M 4 < \dot{A} \wedge *A > +2 < A \wedge \dot{*}A > .$$

From equation (2.4) we have $4A_t = *_t \nabla_t S + S \nabla_t S$ and therefore

$$4\dot{A} = \dot{*}\nabla S + 2 * \omega_- S + 2S \omega_- S = \dot{*}\nabla S + 4\omega'_- ,$$

where $\omega = \dot{\nabla} = -\nabla \dot{B} \in \Omega^1(\mathrm{End}(V))$ is the infinitesimal variation of ∇ and the subscript $\pm$ denotes the projection into $\mathrm{End}_\pm$. Since S is the canonical complex structure, we have (2.4) and (2.5) so that

$$
\begin{aligned}
< \dot{*}\nabla S \wedge *A > &= 2 < \dot{*} * Q \wedge *A > -2 < \dot{*} * A \wedge *A > \\
&= -2 < \dot{*} * A \wedge *A >= 2 < * \dot{*} A \wedge *A >= 2 < \dot{*} A \wedge A > \\
&= -2 < A \wedge \dot{*} A >,
\end{aligned}
$$

where we also have used that $\dot{*}$ and $*$ anti-commute. Combining these formulas we get

$$\dot{W} = 4 \int_M < \omega'_- \wedge *A > .$$

Now observe that $* \omega''_- = \omega''_- S$ which implies that $\omega''_- \wedge *A = 0$ by type. Finally, using Stokes Theorem and recalling that $\mathrm{End}_\pm$ are perpendicular, we obtain

$$\dot{W} = 4 \int_M < \omega \wedge *A >= 4 \int_M < \dot{B} d^\nabla * A >$$

for any $\dot{B} \in \Gamma(\mathrm{End}(V))$ with compact support in M. In particular, if f is Willmore, then $d^\nabla * A = 0$. By the standard arguments [BFL$^+$02, Prop. 5] this is the Euler–Lagrange equation for $S : M \to \mathcal{Z}$ being harmonic. $\qquad \square$

Remark 3.3. Any holomorphic curve $f : M \to \mathbb{HP}^n$ has a Willmore energy, namely the Willmore energy of the (quaternionic) holomorphic line bundle L^{-1} which coincides with the functional (3.1) for Frenet curves [FLPP01, Def. 2.5]. Thus, a natural definition for a Willmore curve would be a holomorphic curve $f : M \to \mathbb{HP}^n$ critical with respect to compactly supported variation by holomorphic curves $f_t : M_t \to \mathbb{HP}^n$ including variations M_t of the conformal structure on M. The arguments in Theorem 3.2 then imply that the canonical complex structure S of f is harmonic away from the discrete Weierstrass

points $D \subset M$ of f. Unfortunately, this does not guarantee that S extends into the Weierstrass points: the example in the beginning of Section 2 of twistor projections into $\mathbb{HP}^n$ of complex holomorphic curves in $\mathbb{CP}^{2n+1}$ shows that S can become singular on D. But if we knew a priori that S is continuous on D, then the recent regularity result [Hél04] for harmonic maps into pseudo–Riemannian manifolds will show that S is indeed smooth in D and hence f by Lemma 2.1 a Frenet curve. In other words, we could replace the assumption of Definition 3.1 that a Willmore curve is Frenet by the condition that the canonical complex structure S extends continuously into the Weierstrass points and still have Theorem 3.2 valid.

So far we have seen that the osculating and enveloping constructions preserve Frenet curves and thus conformality. The situation becomes much more subtle when considering Willmore curves: it turns out that the osculates and envelopes of Willmore curves are examples of constrained Willmore curves. These are Frenet curves critical for the Willmore energy under compactly supported variations by Frenet curves preserving the conformal structure of M. The theory of constrained Willmore surfaces in 3–space has only recently been given firm foundation [BPP04] and its generalization to Frenet curves in $\mathbb{HP}^n$ is little understood at present. Therefore, we restrict ourselves from now on to Willmore spheres in $\mathbb{HP}^n$ in which case the conformal constraint is void.

Theorem 3.4. *The tangent curve of a Willmore sphere $f : S^2 \to \mathbb{HP}^n$ is Willmore.*

Proof. Let $\tilde{f} : S^2 \to P(H)$ be the tangent curve of f with respect to the hyperplane $H \subset \mathbb{H}^{n+1}$, i.e., $\tilde{L} = H \cap V_1$. Using the splitting $V = L \oplus H$, we decompose the canonical complex structure S of f into

$$S = \begin{pmatrix} J & B \\ 0 & \tilde{S} \end{pmatrix}.$$

Here $J \in \Gamma(\operatorname{End}(L))$ is the complex structure on L and $\tilde{f}$ is a Frenet curve with canonical complex structure $\tilde{S}$. Moreover, the $(0,1)$–part of ∇S calculates to

$$(3.3) \qquad\qquad Q = \frac{1}{4}(S\nabla S - *\nabla S) = \begin{pmatrix} 0 & \eta \\ 0 & \tilde{Q} \end{pmatrix}$$

with $\eta \in \Omega^1(\operatorname{Hom}(H, L))$ and $2*\tilde{Q} = (\tilde{\nabla}\tilde{S})''$. In order to show that $\tilde{f}$ is Willmore, we have to calculate $d^{\tilde{\nabla}} * \tilde{A} = 0$ which by (2.4) is equivalent to $d^{\tilde{\nabla}} * \tilde{Q} = 0$. Since f is Willmore

$$0 = d^{\nabla} * Q = \begin{pmatrix} 0 & d^{\tilde{\nabla}, \nabla^L} * \eta \\ 0 & d^{\tilde{\nabla}} * \tilde{Q} + \delta \wedge *\eta \end{pmatrix}.$$

Flatness of ∇ implies (2.7) that δ is closed and thus there exists $C \in \Gamma(\operatorname{Hom}(L, \tilde{L}))$ with

$$\nabla C = \delta.$$

Now, $C * \eta \in \Omega^1(\operatorname{End}(H))$ is a 1–form with values in

$$\tilde{\mathcal{R}} = \{R \in \operatorname{End}(H) \mid R\tilde{V}_{n-2} = 0, RH \subset \tilde{L}\}$$

which satisfies

$$(3.4) \qquad\qquad d^{\tilde{\nabla}}(C * \eta) = \delta \wedge *\eta + C d^{\nabla^L, \tilde{\nabla}} * \eta = -d^{\tilde{\nabla}} * \tilde{Q}.$$

Corollary 4.5 below now implies that $\tilde{f}$ is Willmore. $\qquad\qquad\square$

As an application of this theorem, we construct minimal spheres with planar ends in $\mathbb{R}^4$:

Corollary 3.5. *Let $h : S^2 \to \mathbb{CP}^{2n+1}$ be a rational curve whose n^{th} osculating space W_n does not contain a quaternionic subspace, i.e, $W_n \oplus W_n j = \mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$. Then h gives rise to a $2(n-1)$–dimensional family of minimal spheres in $\mathbb{R}^4$ with planar ends via twistor projection and osculating construction.*

Proof. Under our assumption on the complex holomorphic curve h it is shown in [FLPP01, Lemma 2.7] that the twistor projection $f : M \to \mathbb{HP}^n$ of h has the smooth canonical complex structure S given by $S|_W = i$ and is Willmore. For a generic choice of complementary hyperplane $H \subset \mathbb{H}^{n+1}$ the tangent curve will not be twistor. Proceeding successively, we obtain a $2(n-1)$–dimensional family of Willmore spheres in $\mathbb{HP}^1$ which are not twistor projections from $\mathbb{CP}^3$. Therefore, by stereographic projections [Mon00], [BFL$^+$02], we obtain a $2(n-1)$–dimensional family of minimal spheres with planar ends in $\mathbb{R}^4$. $\quad\square$
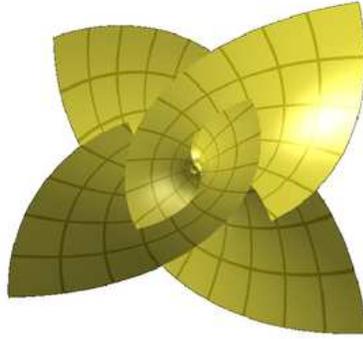

FIGURE 1. Minimal sphere in $\mathbb{R}^4$ with planar ends obtained by tangent construction on a twistor projection of a holomorphic curve in $\mathbb{CP}^5$, [Hel02]

Remark 3.6. It is shown in [BP] that any minimal sphere in $\mathbb{R}^4$ with planar ends arises from a rational curve into $\mathbb{CP}^{2n+1}$ via twistor projections and osculating constructions.

As a further application we obtain a splitting of $V = \underline{\mathbb{H}}^{n+1}$ into Willmore spheres which is analogous to the splitting in Corollary 2.5 for Frenet curves:

Corollary 3.7. *Let $f : S^2 \to \mathbb{HP}^n$ be a Willmore sphere. Then V splits into a direct sum $V = \oplus_{i=0}^n L_i$ of Willmore spheres $f_i : S^2 \to \mathbb{HP}^{n-i}$ where $f_0 = f$ and f_n is a point in $\mathbb{HP}^1$.*

According to Theorem 2.7 the construction of a Frenet curve $f : S^2 \to \mathbb{HP}^n$ with a given first osculating curve $\tilde{f} : S^2 \to \mathbb{HP}^{n-1}$ requires the prescription of tangents $\omega \in H^0(K\tilde{L})$. If $\tilde{f}$ is Willmore there are natural choices for such ω. Since $d^{\tilde{\nabla}} * \tilde{A} = 0$ we can view $\tilde{A} \in H^0(K\operatorname{Hom}_+(\overline{\tilde{V}}, \tilde{V}))$ as a complex holomorphic bundle map. Therefore its kernel defines a smooth codimension 1 subbundle $W \subset \tilde{V}$ provided $\tilde{A} \not\equiv 0$, i.e., $\tilde{f}$ is not a twistor projection. By transversality, we can choose a non–zero $b \in \tilde{V}$ such that $\tilde{V} = W \oplus b\mathbb{H}$. Then

$$\omega = *\tilde{A}b \in H^0(K\tilde{L})$$

is a holomorphic section which vanishes at the zeros of $\tilde{A}$. Theorem 2.7 requires ω to have no zeros for the construction of the enveloping Frenet curve f. Nevertheless, the regularity of Willmore surfaces enables us to extend the construction across the zeros of these specially chosen ω.

Theorem 3.8. *Every Willmore sphere* $\tilde{f} : S^2 \to \mathbb{HP}^{n-1}$, *which is not a twistor projection of a holomorphic curve in* $\mathbb{CP}^{2n-1}$, *is a tangent curve of a Willmore sphere* $f : S^2 \to \mathbb{HP}^n$.

Proof. We first prove that f, as constructed in Theorem 2.7, is Frenet. Recall that $f : S^2 \to \mathbb{HP}^n$ is given as $L = \psi\mathbb{H}$ where $\delta\psi = \omega = *\tilde{A}b$. The bundle $W \subset \tilde{V}$ defined by $\ker\tilde{A}$ is stable under the canonical complex structure $\tilde{S}$ of $\tilde{f}$. Let $\beta \in \Gamma(\tilde{V}^*)$ with $< \beta, W >= 0$ and $< \beta, b >= 1$. Then

$$J\psi = -\psi < \beta, \tilde{S}b >$$

defines a complex structure on L and f is admits a Frenet flag (2.2), (2.3) with respect to the complex structure $J \oplus \tilde{S}$ by the proof of Theorem 2.7.

We now show that the canonical complex structure extends smoothly across the Weierstrass points of f. Away from these points the canonical complex structure of f can be expressed by

$$S = \begin{pmatrix} J & B \\ 0 & \tilde{S} \end{pmatrix}$$

in the splitting $V = L \oplus \tilde{V}$. Then

$$A = \frac{1}{4}(*\nabla S + S\nabla S) = \begin{pmatrix} A_L & \tilde{\eta} \\ 0 & \tilde{A} + \frac{1}{2} * \delta B \end{pmatrix}$$

and since $\operatorname{Im} A \subset L$ by (2.5), we get

$$\tilde{A} + \frac{1}{2} * \delta B = 0$$

away from the Weierstrass points of f. But the smooth bundle map $\psi\beta \in \Gamma(\operatorname{Hom}(\tilde{V}, L))$ satisfies

$$\tilde{A} = \tilde{A}b\beta = - *\delta\psi\beta$$

and therefore $B = 2\psi\beta$ is smooth across the Weierstrass points. Thus, the canonical complex structure extends smoothly across the Weierstrass points of f.

It remains to show that f is Willmore. Using (3.3) and the fact that $\tilde{f}$ is Willmore, i.e., $d^{\tilde{\nabla}} * \tilde{Q} = 0$, we see that

$$d^{\nabla}(*Q - \operatorname{pr}_L *Q) = \begin{pmatrix} 0 & 0 \\ 0 & d^{\tilde{\nabla}} * \tilde{Q} \end{pmatrix} = 0.$$

As in the proof of the previous theorem, we denote by

$$\mathcal{R} = \{R \in \operatorname{End}(V) \mid RV_{n-1} = 0, RV \subset L\},$$

where V_{n-1} is the $n - 1^{\mathrm{st}}$ osculating bundle of f. Then $\operatorname{pr}_L *Q$ takes values in $\mathcal{R}$ and Corollary 4.5 below implies that f is Willmore. $\square$

Remark 3.9. From Remark 2.9 and Theorem 3.8 we see that the enveloping construction for Willmore spheres coincides up to a suitable projection with the Bäcklund transformation for Willmore spheres.
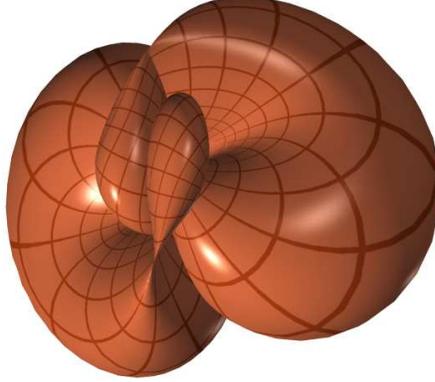
FIGURE 2. Constrained Willmore torus in $\mathbb{R}^4$ obtained by tangent construction on a twistor projection of an elliptic curve in $\mathbb{CP}^5$, [Hel02]

4. TECHNICAL LEMMAS

We conclude our paper by providing various technical lemmas and the corollary used in the proofs of Theorem 3.4 and Theorem 3.8. Given a Frenet curve $f : M \to \mathbb{HP}^n$ with osculating flag V_k, we let

$$\mathcal{R} = \{R \in \mathrm{End}(V) \mid RV_{n-1} = 0, \ RV \subset L\} = \mathrm{Hom}(V/V_{n-1}, L) \,.$$

Lemma 4.1. *If $f : M \to \mathbb{HP}^n$ is a Frenet curve with canonical complex structure S then*

$$d^\nabla * Q \in \Omega^2(\mathcal{R})$$

where $2 * Q = (\nabla S)''$.

Proof. Recall that by (2.4) we have $d^\nabla * A = d^\nabla * Q$. Since $A \in \Gamma(K \mathrm{Hom}_-(V, L))$ we obtain

$$\pi_L d^\nabla * Q = \pi_L d^\nabla * A = \delta_0 \wedge *A = 0 \,,$$

where $\pi_L : V \to V/L$ and $\delta_0 = \pi_L \nabla|_L \in \Gamma(K \mathrm{Hom}_+(L, V/L))$ is the derivative of f. This shows that $d^\nabla * Q$ is L–valued.

On the other hand, for a section $\psi \in \Gamma(V_{n-1})$, we compute

$$(d^\nabla * Q)\psi = - * Q \wedge \nabla \psi = - * Q \wedge \delta_{n-1} \psi = 0 \,.$$

Here we used that $QV_{n-1} = 0$, $\delta_{n-1} = \pi_{V_{n-1}} \nabla|_{V_{n-1}} \in \Gamma(K \mathrm{Hom}_+(V_{n-1}, V/V_{n-1}))$ and $*Q = QS$. In other words, $d^\nabla * Q$ vanishes on V_{n-1}, and therefore $d^\nabla * Q \in \Omega^2(\mathcal{R})$. $\qquad\square$

Lemma 4.2. *Let $f : M \to \mathbb{HP}^n$ be a Frenet curve with complex structure S. Then $\eta \in \Omega^1(\mathcal{R})$ and $d^\nabla \eta \in \Omega^2(\mathcal{R})$ imply $\eta \in \Gamma(K\mathcal{R}_+)$.*

Proof. Since η is L–valued,

$$0 = \pi_L d^\nabla \eta = \delta_0 \wedge \eta$$

implies $\eta \in \Gamma(K\mathcal{R})$. Furthermore, for $\psi \in \Gamma(V_{n-1})$

$$0 = (d^\nabla \eta)\psi = -\eta \wedge \delta_{n-1} \psi$$

shows that $*\eta = \eta S$ and thus $\eta \in \Gamma(K\mathcal{R}_+)$. $\qquad\square$

Let $f : M \to \mathbb{HP}^n$ be a Frenet curve with complex structure S on the trivial $\mathbb{H}^{n+1}$–bundle V. From (2.4) it follows that the S–anticommuting part of ∇ is given by $A + Q$. The S–commuting part of ∇ is a complex connection $\nabla_+ = \partial + \bar{\partial}$ whose $(0,1)$–part $\bar{\partial}$ defines a complex holomorphic structure on V. From (2.3) it follows that

$$0 = (\delta_k)''_+ = \pi_{V_k} \bar{\partial}|_{V_k}$$

so that $V_k \subset V$ is $\bar{\partial}$–stable. Therefore $\bar{\partial}$ induces a complex holomorphic structure on the complex line bundle $K\mathcal{R}_+$.

Lemma 4.3. *Let $f : M \to \mathbb{HP}^n$ be a Frenet curve with canonical complex structure S and $\eta \in \Omega^1(\mathcal{R})$ with $d^\nabla \eta = d^\nabla * Q$. Then $\eta \in H^0(K\mathcal{R}_+)$ is a complex holomorphic section.*

Proof. From Lemma 4.1 and Lemma 4.2 it follows that $\eta \in \Gamma(K\mathcal{R}_+)$. Since $\nabla = \nabla_+ + A + Q$ we obtain

$$d^\nabla \eta = d^{\nabla_+} \eta + [A \wedge \eta] + [Q \wedge \eta] \,.$$

But $\eta \in \Gamma(K\mathcal{R}_+)$, and A and Q anticommute with S so that

$$(d^\nabla \eta)_+ = \bar{\partial} \eta \,.$$

On the other hand,

$$d^\nabla * Q = d^{\nabla_+} * Q + [A \wedge *Q] + [Q \wedge *Q] = d^{\nabla_+} * Q \,,$$

where we used that $[A \wedge *Q] = 0$ and $[Q \wedge *Q] = 0$ by type and symmetry considerations. This shows $(d^\nabla * Q)_+ = 0$ and therefore

$$\bar{\partial} \eta = (d^\nabla \eta)_+ = (d^\nabla * Q)_+ = 0 \,,$$

which proves that $\eta \in H^0(K\mathcal{R}_+)$. $\qquad\square$

Lemma 4.4. *If $f : M \to \mathbb{HP}^n$ is a Frenet curve with complex structure S then*

$$\delta_{n-1} \circ \ldots \circ \delta_0 \in H^0(K^n \operatorname{Hom}_+(L, V/V_{n-1}))$$

is complex holomorphic and hence the degree of the complex line bundle $K\mathcal{R}_+$ is

$$\deg K\mathcal{R}_+ = (n+1) \deg K - \operatorname{ord}(\delta_{n-1} \circ \ldots \circ \delta_0).$$

In particular, if $M = S^2$ then $\deg K\mathcal{R}_+ < 0$, so that $K\mathcal{R}_+$ has no global holomorphic sections.

Proof. From (2.3) and [Les02] it follows that the derivative δ_k of the kth osculating flag V_k is a complex holomorphic section of the complex line bundle $K \operatorname{Hom}_+(V_k/V_{k-1}, V_{k+1}/V_k)$. Therefore

$$\delta_{n-1} \circ \ldots \circ \delta_0 \in H^0(K^n \operatorname{Hom}_+(L, V/V_{n-1}))$$

and the degree of the complex line bundle $K^n \operatorname{Hom}_+(L, V/V_{n-1})$ is given by

$$\operatorname{ord}(\delta_{n-1} \circ \ldots \circ \delta_0) = \deg(K^n \operatorname{Hom}_+(L, V/V_{n-1})) \,.$$

Since the degree of a complex quaternionic line bundle is the degree of the underlying complex line bundle, [FLPP01], we have

$$\deg(K^n \operatorname{Hom}_+(L, V/V_{n-1})) = n \deg K + \deg(V/V_{n-1}) - \deg L \,.$$

Recalling that $\mathcal{R}_+ = \operatorname{Hom}_+(V/V_{n-1}, L)$, we finally obtain

$$\deg K\mathcal{R}_+ = \deg K + \deg L - \deg(V/V_{n-1}) = (n+1) \deg K - \operatorname{ord}(\delta_{n-1} \circ \ldots \circ \delta_0).$$

$\qquad\square$

Combining the last two lemmas and Theorem 3.2 yields the following

Corollary 4.5. *Let $f : S^2 \to \mathbb{HP}^n$ be a Frenet curve with canonical complex structure S satisfying*

$$d^\nabla(*Q + \eta) = 0$$

for some $\eta \in \Omega^1(\mathcal{R})$. Then f is Willmore, i.e., $d^\nabla * Q = 0$.

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